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AXIOMS AND MODELS FOR ORIGAMI GEOMETRY

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Abstract. The elementary fold operations of origami have been traditionally called "axioms," yet they are more accurately described as construction postulates asserting the existence of specific folds. This article proposes a formal axiom system for origami geometry that distinguishes three components: axioms of the ambient geometry, axioms governing the fold as a reflection, and construction postulates. The axioms of the first two groups determine the complete catalogue of possible elementary fold operations, as shown in previous work; the construction postulates represent a selection from that catalogue, and different selections yield systems of different constructive power. The framework is illustrated by the trivial one-dimensional case and developed in full for the standard two-dimensional case. We characterize the models of the axiom system as Desarguesian ordered planes equipped with a nondegenerate symmetric bilinear form, and show that the construction postulates correspond to a nested hierarchy of coordinate subfields: the Thalian, Pythagorean, Euclidean, and origami numbers.

1. Introduction

The elementary fold operations of origami, introduced by Justin [11] and rediscovered by Huzita [10] and others, are commonly known as the "axioms" of origami. As has been noted by several authors (e.g., [8, 13]), however, the designation is not fully appropriate: the operations are not axioms in the sense of foundational logic, since some of them may be derived from others, and some may fail to be realizable depending on the configuration of given points and lines.

In mathematical logic, the term *axiom* refers to a statement assumed to be true within a formal system, from which theorems are derived by rules of inference. Classical examples include the Peano axioms for the natural

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numbers and the Zermelo–Fraenkel axioms for set theory. In geometry, the tradition initiated by Hilbert [9] provides a model: the axiom system specifies undefined primitives (points, lines, incidence), axioms governing their properties, and from these all of Euclidean geometry is derived.

The fold operations of origami are more analogous to Euclid's *postulates*—specifically, to the compass-and-straightedge postulates that assert the *existence* of certain constructions. The origami fold operations similarly assert that certain folds *can be performed*, subject to conditions on the given points and lines.

A proper axiom system for origami geometry should therefore distinguish between three components:

(1) **Axioms of the ambient geometry**—the properties of the space in which folds take place.
(2) **Axioms of the fold**—the properties of the fold operation as a reflection.
(3) **Construction postulates**—existence assertions for fold lines satisfying specified incidence constraints.

A key observation, developed in Section 7, is that the axioms of the first two groups *determine* the complete catalogue of possible elementary fold operations. The structure of the ambient geometry and the nature of reflection jointly constrain what incidence combinations can be imposed on a fold line with finitely many solutions; the enumeration of all such combinations is a theorem proved in [13]. The construction postulates then represent a *selection* from this determined catalogue, and different selections yield systems of different constructive power—just as, in classical geometry, choosing straightedge alone, straightedge and compass, or straightedge, compass and angle trisector yields progressively more powerful systems within the same Euclidean plane.

Recent work by Beklemishev, Dmitrieva and Makowsky [6] has addressed the axiomatization of origami from the perspective of mathematical logic, using Wu's axioms for orthogonal geometry [19] as a foundation and recasting the Huzita–Justin operations as first-order logical axioms. Their analysis establishes bi-interpretations between origami-geometric theories and theories of fields, and proves that the first-order theory of origami is undecidable.

The present article takes a complementary, geometrically motivated approach. Rather than taking orthogonality as a primitive, the axiom system is built on the concept of *reflection*, which is the geometric essence of a fold. The construction postulates incorporate the precise *existence conditions* derived in previous studies [13, 15]. The framework is illustrated first by the trivial one-dimensional case (Section 2) and then developed in full for the standard two-dimensional case (Sections 3–9).

A main contribution of the article concerns the *models* of the axiom system (Section 10). We show that axioms A1–A4 (ambient geometry) and F1–F4 (fold as reflection) characterize Desarguesian ordered planes equipped with a nondegenerate symmetric bilinear form (Theorem 10.1). The first construction postulate—the existence of perpendicular bisectors—then forces the form to be universal, which over ordered fields is equivalent to the standard inner product. This places origami geometry within the tradition of Bachmann's reflection-based foundations [5], and makes the axiom-vs-postulate distinction algebraically substantive: the axioms determine the *ambient geometry*, while the construction postulates determine the *coordinate field*. Different selections of postulates yield the Thalian, Pythagorean, Euclidean, and origami number fields in a nested hierarchy [1]. Extensions to 3D folding, where the framework applies with natural modifications [14, 17, 4], will be treated in a companion article.

2. Warm-up: the one-dimensional case

Before treating the standard 2D case, it is instructive to consider origami in one dimension. The framework, though trivial, illustrates the general structure.

2.1. **Setting.** The "medium" is a Euclidean line $\mathbb{R}$. The geometric objects are *points* only (there are no lines in a 1D geometry). A fold is a reflection of the line across a point: given a fold point $c \in \mathbb{R}$, the reflection maps each point x to $2c - x$.

2.2. **Axioms.** The ambient axioms are those of a 1D Euclidean geometry (a complete ordered field). The fold axiom asserts that reflection across a point is a well-defined involution that fixes the fold point and maps every other point to a distinct image equidistant from the fold point on the opposite side.

2.3. **Incidence constraints and classification.** There is only one type of geometric object (points), and so there is only one non-trivial incidence constraint: given two distinct points P and Q, find a fold point c such that the reflection maps P to Q. The solution is unique: $c = (P + Q)/2$, the midpoint of P and Q. This single constraint already consumes all degrees of freedom of the fold point (one degree of freedom in 1D). Therefore there is exactly one elementary fold operation.

2.4. **Construction postulate and constructive power.** The sole construction postulate is: *given two distinct points P and Q, there exists a fold point that maps P to Q.* Starting from two initial points (say 0 and 1), iterated application of this postulate generates all midpoints, then midpoints of midpoints, and so on. The set of 1D origami-constructible points is thus the ring of *dyadic rationals* $\mathbb{Z}\left[\frac{1}{2}\right] = \{\, m/2^n \mid m \in \mathbb{Z},\, n \geq 0 \,\}$.

Remark 2.1. *The 1D case already exhibits the general pattern: the axioms determine the catalogue (one operation), the postulate is a selection (here the only possible selection), and the constructive power is characterized by an algebraic structure (the dyadic rationals).*

3. The two-dimensional case: language and primitives

We now turn to the standard setting of origami geometry. The formal language $\mathcal{L}_{\text{origami}}$ is a two-sorted first-order language with equality.

Definition 3.1 (Sorts)**.** *The language has two sorts:*

- **Pt***, the sort of* points*, with variables $P, Q, R, \ldots$*
- **Ln***, the sort of* lines*, with variables $m, n, \ell, \ldots$*

Definition 3.2 (Predicates)**.** *The language has the following primitive predicates:*

(1) $\mathrm{Inc}(P, m)$ — *"point P lies on line m," of type* $\mathbf{Pt} \times \mathbf{Ln}$.
(2) $\mathrm{Sym}(P, m, Q)$ — *"point Q is the reflection of point P across line m," of type* $\mathbf{Pt} \times \mathbf{Ln} \times \mathbf{Pt}$.

Remark 3.1. *We use* $\mathrm{Sym}(P, m, Q)$ *as a primitive rather than the more common predicates of parallelism, orthogonality, or betweenness. This choice reflects the foundational role of reflection in origami: a fold* is *a reflection. Orthogonality, midpoints and distance can all be defined in terms of* Inc *and* Sym *(see Definitions 5.1 and 5.2 below).*

Remark 3.2. *We write $P \in m$ for* $\mathrm{Inc}(P, m)$*, and $\mathcal{F}_m(P) = Q$ for* $\mathrm{Sym}(P, m, Q)$*. The image of a line n under reflection in m is a derived notion:*

$$\mathcal{F}_m(n) = n' \quad \overset{def}{\Longleftrightarrow} \quad \forall P \left(P \in n \to \mathcal{F}_m(P) \in n'\right) \\ \wedge \ \forall Q \left(Q \in n' \to \mathcal{F}_m(Q) \in n\right).$$

4. Axioms of the ambient geometry

The first group of axioms establishes the Euclidean plane in which folds take place. The medium is assumed to be an infinite Euclidean plane [7, 13].

4.1. **Incidence axioms.** These follow Hilbert [9].

Axiom A1 (Line determination). *For any two distinct points P and Q, there exists a unique line m such that $P \in m$ and $Q \in m$.*

Axiom A2 (Non-degeneracy of lines). *Every line contains at least two distinct points.*

Axiom A3 (Non-collinearity). *There exist three points not all on a common line.*

4.2. **Parallel axiom. Axiom A4** (Playfair's axiom). *For any line m and any point P not on m, there exists a unique line through P that does not intersect m.*

Remark 4.1. *Parallelism is definable from incidence: $m \parallel n$ iff $m = n$ or $\forall P\, \neg(P \in m \wedge P \in n)$. Axiom A4 restricts the geometry to the Euclidean case; origami on the sphere* [12] *or the hyperbolic plane* [3] *would require modifications.*

5. Axioms of the fold operation

The second group of axioms governs the reflection mapping Sym, which models the act of folding.

Axiom F1 (Existence and uniqueness of reflection). *For every line m and every point P, there exists a unique point Q such that* $\mathrm{Sym}(P, m, Q)$.

Remark 5.1. *This ensures that the fold is a well-defined function* $\mathcal{F}_m \colon \mathbf{Pt} \to \mathbf{Pt}$ *for each line m.*

Axiom F2 (Involution). $\mathrm{Sym}(P, m, Q) \to \mathrm{Sym}(Q, m, P)$.

Axiom F3 (Fixed-point characterization). $\mathrm{Sym}(P, m, P) \leftrightarrow P \in m$.

Before stating the next axiom, we introduce defined notions.

Definition 5.1 (Perpendicularity). *Two lines m and n are* perpendicular *(written $m \perp n$) if there exists a point P on n, not on m, whose reflection across m also lies on n:*

$$m \perp n \quad \overset{def}{\Longleftrightarrow} \quad \exists P \left(P \in n \wedge P \notin m \wedge \mathcal{F}_m(P) \in n\right).$$

Remark 5.2. *When $\mathcal{F}_m(P) \in n$ and $P \in n$ with $P \notin m$, the fold line m maps one ray of n (from the intersection point) to the other, which is the geometric content of perpendicularity. The relation is symmetric: if $m \perp n$, then $n \perp m$. This follows from F2 and F4, since reflecting the intersection point across n maps the two rays of m to each other.*

Definition 5.2 (Midpoint). *If $P \neq Q$ and $\mathcal{F}_m(P) = Q$, the* midpoint *of P and Q is the intersection of line m with the line through P and Q. That is, the unique point M such that $M \in m$ and $M \in \ell_{PQ}$, where ℓ_{PQ} is the line determined by P and Q (Axiom A1).*

Axiom F4 (Perpendicular bisector). *If $\mathcal{F}_m(P) = Q$ and $P \neq Q$, then m is perpendicular to the line through P and Q, and m passes through their midpoint.*

Remark 5.3. *The axiom system A1–A4, F1–F4 does* not *assert that perpendicular bisectors always exist. That is, for a given pair of distinct points P, Q, there may or may not be a line m with $\mathcal{F}_m(P) = Q$. The existence of such a line is the content of the first construction postulate (C1, Section 8). This separation is deliberate: the axioms describe the* properties *of reflections, while the postulates assert the* existence *of specific fold lines.*

5.1. **Derived properties.** From axioms A1–A4 and F1–F4, a number of properties follow as theorems.

Definition 5.3 (Congruence). *Two segments $\overline{PQ}$ and $\overline{P'Q'}$ are* congruent *(written $\overline{PQ} \cong \overline{P'Q'}$) if there exists a finite composition of reflections mapping P to P' and Q to Q'. This defines an equivalence relation on pairs of points; see [5, 19] for the coordinatization procedure. We write $\mathrm{d}(P, Q)$ for the congruence class of $\overline{PQ}$, and $\mathrm{d}(P, Q) \leq \mathrm{d}(P', Q')$ when the geometry is over an ordered field in which the standard inner product induces a total order on segment lengths.*

Proposition 5.1 (Line preservation). *The image of a line under a reflection is a line.*

Proof.[Proof sketch] Let n be a line and m a fold line. By F1, each point of n has a unique image under $\mathcal{F}_m$. If $P, Q \in n$ are distinct, then $\mathcal{F}_m(P)$ and $\mathcal{F}_m(Q)$ are distinct (by F2 and the uniqueness in F1), and the set of all images $\{\mathcal{F}_m(R) : R \in n\}$ is a line by A1 and the incidence structure.

Proposition 5.2 (Isometry). *The reflection $\mathcal{F}_m$ preserves congruence: $\overline{PQ} \cong \overline{\mathcal{F}_m(P)\,\mathcal{F}_m(Q)}$.*

Proof.[Proof sketch] By Definition 5.3, $\overline{PQ} \cong \overline{P'Q'}$ whenever some composition of reflections carries $P \mapsto P'$ and $Q \mapsto Q'$. Taking the single reflection $\mathcal{F}_m$ itself as that composition gives $\overline{PQ} \cong \overline{\mathcal{F}_m(P)\,\mathcal{F}_m(Q)}$ immediately. Thus every reflection preserves congruence; the claim then extends to arbitrary compositions of reflections, which are exactly the isometries of the plane.

6. The closure principle

In origami mathematics, it is customary to assume that all points and lines marked on the sheet are available for subsequent fold operations, regardless of which layer they may lie on in a physically folded paper. This is described by saying that the paper is "transparent" [7, 18].

Closure Principle T (Transparency). *The set of constructible points and lines is closed under the fold operation and under intersection. That is: if P is constructible and m is a constructible line, then $\mathcal{F}_m(P)$ is constructible; if n is a constructible line, then $\mathcal{F}_m(n)$ is constructible; and if m and n are constructible lines that intersect, then their intersection point is constructible.*

Remark 6.1. *Closure Principle T is not a first-order axiom but a metatheoretic convention governing the constructive setting, where one considers the set of objects generated from an initial configuration by iterated application of the construction postulates (cf. Definition 1 of* [8]*). In a first-order formalization, closure of the sorts under definable operations is automatic; the principle becomes substantive only in the constructive context.*

7. Classification of elementary fold operations

We now come to a central point: the axioms of the ambient geometry and the fold axioms determine the complete catalogue of possible elementary fold operations. The enumeration is a theorem, proved in [13]; we summarize it here.

7.1. Incidence constraints.

An incidence constraint is a condition on the fold line χ involving a given object α (point or line) and the image $\mathcal{F}_\chi(\beta)$ of a given object β. Each constraint consumes a number of degrees of freedom of χ, called its *codimension*. Since a line in the plane has two degrees of freedom, the possible codimensions are 1 and 2. A detailed analysis based on the geometry of reflections, as established by the fold axioms, yields six incidence constraints, listed in Table 1; see [13] for the full derivation.

7.2. Elementary operations as combinations of constraints.

Definition 7.1 (Elementary fold operation)**.** *An* elementary single-fold operation *is a combination of incidence constraints from Table 1 whose total codimension equals* 2 *(the number of degrees of freedom of a fold line in the plane) and which, for data in general position, admits a finite nonzero number of solutions.*

Each codimension-2 constraint (I_1, I_2, I_6) defines an operation by itself. The codimension-1 constraints (I_3, I_4, I_5) must be combined in pairs, giving six possible combinations, of which one ($I_5 + I_5$) is invalid (it has zero or infinitely many solutions). This yields a total of eight elementary operations.

TABLE 1. Incidence constraints for 2D origami. P and Q are points; m and n are lines.

Label	Definition	Codimension
I_1	$\mathcal{F}_\chi(P) = Q$, with $P \neq Q$	2
I_2	$\mathcal{F}_\chi(m) = n$, with $m \neq n$	2
I_3	$\mathcal{F}_\chi(P) \in m$, with $P \notin m$	1
I_4	$\mathcal{F}_\chi(P) = P$	1
I_5	$\mathcal{F}_\chi(m) = m$, and $\exists P \in m$, $\mathcal{F}_\chi(P) \neq P$	1
I_6	$\mathcal{F}_\chi(m) = m$, and $\forall P \in m$, $\mathcal{F}_\chi(P) = P$	2

Theorem 7.1 (Classification of 2D elementary fold operations [13])**.** *Under Definition 7.1, there are exactly eight elementary single-fold operations, listed in Table 2.*

TABLE 2. The eight elementary single-fold operations.

Label	Incidences	Folding action
O1	I_1	Fold P onto Q
O2	I_2	Fold line m onto line n
O3	I_6	Fold along a given line m
O4	$I_4 + I_4$	Fold along the line through P and Q
O5	$I_4 + I_5$	Fold through P, perpendicular to m
O6	$I_3 + I_4$	Fold P onto m, through Q
O7	$I_3 + I_3$	Fold P onto m and Q onto n (Beloch's fold)
O8	$I_3 + I_5$	Fold P onto m, perpendicular to n

Remark 7.1. *The classification follows from an exhaustive analysis of the incidence constraints, their codimensions, and the requirement of finitely many solutions in general position* [13]*. It is a consequence of the fold axioms and the ambient geometry, not an additional assumption. The eight operations constitute the* menu *from which the construction postulates (next section) make a selection.*

8. CONSTRUCTION POSTULATES

The construction postulates assert the existence of fold lines satisfying specific incidence constraints. Different selections of postulates from the menu of Table 2 yield systems of different constructive power.

8.1. **Selection of postulates.** Not all eight operations of Table 2 need to be adopted as postulates. Operation O3 (fold along a given line) does not create a new line; in the presence of Closure Principle T, the reflected images it produces are already constructible by other operations. Therefore O3 is excluded from the set of postulates.[1]

[1]The inclusion of O3 is motivated by practical origami, where the paper is not transparent and one may need to fold along a line visible only on a lower layer [13]. In the

Furthermore, Ghourabi et al. [8] showed that Beloch's fold (O7), stated without side conditions as a *general fold principle* (G)—"given points P, Q and lines m, n, fold P onto m and Q onto n"—subsumes all other operations as special or degenerate cases (Theorem 1 of [8]). Thus a *minimal* set of postulates consists of O7 alone.

Nevertheless, as is customary in axiomatic systems (cf. Hilbert's axioms, which contain known redundancies for pedagogical clarity), we state the full set of seven postulates for completeness.

8.2. **The postulates.** Each postulate has the form: *given certain objects satisfying stated hypotheses, there exists a line χ such that certain incidence constraints hold.*

Postulate C1 (O1: Fold point onto point). *Given distinct points P and Q, there exists a line χ such that $\mathcal{F}_\chi(P) = Q$.*
Solutions: 1 (unique). *Existence:* always.

Remark 8.1. *Postulate C1 asserts the existence of the perpendicular bisector of any two distinct points. It is the first genuinely constructive assertion in the system: none of the fold axioms F1–F4 guarantee that a reflecting line exists for a given pair of points. Adding C1 to the axiom system ensures that midpoints, and hence a metric structure compatible with the reflection, are available.*

Postulate C2 (O2: Fold line onto line). *Given distinct lines m and n, there exists a line χ such that $\mathcal{F}_\chi(m) = n$.*
Solutions: 1 if $m \parallel n$; 2 if $m \nparallel n$. *Existence:* always.

Postulate C3 (O4: Fold through two points). *Given distinct points P and Q, there exists a line χ such that $P \in \chi$ and $Q \in \chi$.*
Solutions: 1 (unique). *Existence:* always.

Postulate C4 (O5: Fold perpendicular through a point). *Given a point P and a line m, there exists a line χ such that $P \in \chi$ and $\chi \perp m$.*
Solutions: 1 (unique). *Existence:* always.

Postulate C5 (O6: Fold point onto line through a point). *Given a point P not on line m, and a point Q with $\mathrm{d}(P,Q) \geq \mathrm{d}(Q,m)$, there exists a line χ with $Q \in \chi$ such that $\mathcal{F}_\chi(P) \in m$.*
Solutions: 0–2. *Existence:* iff $\mathrm{d}(P,Q) \geq \mathrm{d}(Q,m)$.

Postulate C6 (O7: Beloch's fold). *Given points P, Q and lines m, n, with $P \notin m$, $Q \notin n$, ($P \neq Q$ or $m \neq n$), and $\mathrm{d}(P,Q) \geq \mathrm{d}(m,n)$, there exists a line χ such that $\mathcal{F}_\chi(P) \in m$ and $\mathcal{F}_\chi(Q) \in n$.*
Solutions: 0–3. *Existence:* iff $\mathrm{d}(P,Q) \geq \mathrm{d}(m,n)$ [15].

Postulate C7 (O8: Fold point onto line, perpendicular to line). *Given a point P and lines m, n with $P \notin m$ and $m \nparallel n$, there exists a line χ with $\chi \perp n$ such that $\mathcal{F}_\chi(P) \in m$.*
Solutions: 0–1. *Existence:* iff $m \nparallel n$.

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present mathematical context, where Closure Principle T holds, this motivation does not apply.

9. Properties of the system

9.1. **Redundancy and independence.** Postulate C6 (Beloch's fold) subsumes all others. Specifically, Ghourabi et al. [8] showed that a general fold principle consisting of C6 without side conditions realizes each of the other operations under appropriate degenerate or incident configurations of its parameters. Thus the system could be reduced to a single postulate. The remaining postulates C1–C5 and C7 are retained for clarity and because they correspond to natural geometric constructions with simpler existence conditions.

Note also that Postulate C3 is equivalent to Axiom A1 (both assert the existence of a line through two given points).

Conversely, C6 is independent of the remaining postulates: C1–C5 and C7 can solve at most quadratic equations, whereas C6 solves cubics.

The set of constructible points generated from two initial points by iterated application of the postulates, together with intersection of constructible lines, forms a field. The algebraic characterization depends on which postulates are available [1]:

- Postulates C1–C5 and C7 (excluding Beloch's fold) yield the field of numbers constructible by straightedge and compass, i.e., the *Euclidean closure* of $\mathbb{Q}$—the smallest subfield of $\mathbb{R}$ closed under square roots of positive elements.
- The full set C1–C7 (including Beloch's fold) yields the *origami field*—the smallest subfield of $\mathbb{R}$ closed under real roots of cubic polynomials. This coincides with the field constructible by straightedge, compass and angle trisector.

The origami field properly contains the Euclidean field: it includes, for example, the numbers $\cos(2\pi/7)$ and $\cos(2\pi/9)$, which allow the construction of the regular heptagon and nonagon. It does not, however, include solutions to irreducible quintic equations; those require multi-fold operations [2, 16].

10. Models of the axiom system

We have presented the axioms A1–A4 and F1–F4 as properties of "the" Euclidean plane, but a natural foundational question arises: *what are all the models of these axioms?* That is, what structures $(\mathbf{Pt}, \mathbf{Ln}, \mathrm{Inc}, \mathrm{Sym})$ satisfy A1–A4 and F1–F4? This question is worth addressing because it reveals the precise geometric content of the axiom system and shows that the construction postulates play a role not merely in selecting fold operations but in determining the coordinate field and the metric structure.

10.1. **Coordinatization.** By a standard result of affine geometry (see, e.g., [18]), axioms A1–A4 characterize *Desarguesian affine planes*: for every model of A1–A4, there exists a field K and a bijection between the points of the model and K^2 under which lines correspond to cosets of one-dimensional subspaces (or, equivalently, to solution sets of equations $ax + by = c$ with $(a, b) \neq (0, 0)$), and incidence corresponds to membership. The field K is uniquely determined up to isomorphism by the incidence structure.

10.2. **The reflection structure.** The fold axioms F1–F4 equip each line with a reflection mapping $\mathcal{F}_m$. In coordinates, F1–F3 assert that $\mathcal{F}_m$ is a non-trivial involutory bijection whose fixed-point set is exactly m; F4 adds that $\mathcal{F}_m$ maps any non-fixed point to a point equidistant from m on the opposite side, with m perpendicular to the segment connecting the point and its image. These conditions force $\mathcal{F}_m$ to be an orthogonal reflection with respect to some nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on K^2. Note that K must have characteristic different from 2, since the midpoint $(P + Q)/2$ must be definable.

At this stage, the bilinear form need not be universal: it is possible that some pairs of distinct points do not admit a perpendicular bisector within the model. This is where the first construction postulate enters.

10.3. **The effect of Postulate C1.** Postulate C1 asserts that for every pair of distinct points P, Q, there exists a line m with $\mathcal{F}_m(P) = Q$. Geometrically, this is the existence of the perpendicular bisector of $\overline{PQ}$. In coordinates, the perpendicular bisector of (x_1, y_1) and (x_2, y_2) exists in K^2 if and only if the midpoint $\frac{1}{2}(x_1 + x_2, y_1 + y_2)$ lies in K^2 (guaranteed by $\mathrm{char}(K) \neq 2$) *and* the direction perpendicular to the segment $(x_2 - x_1, y_2 - y_1)$ is representable in the geometry. The latter condition is precisely the requirement that the bilinear form be *universal*: every nonzero vector in K^2 has a well-defined orthogonal complement.

Over an ordered field, a universal nondegenerate symmetric bilinear form is equivalent, after a linear change of coordinates, to the standard inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$. We thus obtain:

Theorem 10.1 (Models)**.** *The models of axioms A1–A4 and F1–F4 over an ordered field K are precisely the planes K^2 ($\mathrm{char}(K) \neq 2$) equipped with the standard incidence structure and orthogonal reflections with respect to a nondegenerate symmetric bilinear form on K^2.*

Adding Postulate C1 forces the bilinear form to be universal. Over an ordered field, this is equivalent (after a linear change of coordinates) to the standard inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$.

Remark 10.1. *The ordering assumption is natural for origami geometry, where the existence conditions for construction postulates involve distance comparisons (e.g., $\mathrm{d}(P, Q) \geq \mathrm{d}(Q, m)$ for Postulate C5). Without ordering, models over fields such as finite fields or $\mathbb{C}$ are possible but lack the geometric features that make the construction postulates meaningful. We restrict attention to ordered fields henceforth.*

Remark 10.2. *Theorem 10.1 may be compared with Bachmann's program [5], which founds plane geometry on the group generated by line reflections. Bachmann's "metric planes" are characterized group-theoretically; the present axioms A1–A4, F1–F4 give an incidence-theoretic counterpart. The class of models identified by Theorem 10.1—Desarguesian ordered planes with orthogonal reflections—corresponds to a subclass of Bachmann's metric planes. Making this correspondence fully precise is an interesting direction for future work.*

10.4. **The role of the construction postulates.** Theorem 10.1 shows that the axiom system A1–A4, F1–F4 determines a *class* of planes, parameterized by the ordered field K and the choice of bilinear form. The construction postulates then play a double role: they constrain the metric structure (C1 forces universality of the form) and they determine the *coordinate field* of the origami-constructible sub-plane.

To make the second role precise, fix a model K^2 with K an ordered field and the standard inner product (as guaranteed by Theorem 10.1 after C1 is assumed), and choose two initial points, say $(0, 0)$ and $(1, 0)$. The *constructible sub-plane* generated by iterated application of a given set of postulates (together with line intersection, Closure Principle T) consists of all points and lines so obtainable. The coordinates of the constructible points form a subfield $K_0 \subseteq K$, and the following hierarchy, due to Alperin [1], holds:

(1) Postulates C1–C3 (fold point onto point, line onto line, through two points) generate $K_0 = \mathbb{Q}$ within K. These are the *Thalian* constructions [1]: they perform only field operations (addition, subtraction, multiplication, division) on coordinates, solving no polynomial equations of degree ≥ 2.
(2) Adding postulate C4 (fold perpendicular through a point, i.e., angle bisection) enlarges K_0 to the *Pythagorean closure* of $\mathbb{Q}$ within K—the smallest subfield closed under the operation $a \mapsto \sqrt{1 + a^2}$. This field is strictly larger than $\mathbb{Q}$ but does not contain all square roots.
(3) Adding postulate C5 (fold point onto line through a point) further enlarges K_0 to the *Euclidean closure*—the smallest subfield closed under square roots of positive elements. This is the classical field of straightedge-and-compass constructible numbers.
(4) Adding postulate C6 (Beloch's fold) enlarges K_0 to the *origami field* $\mathcal{O}$—the smallest subfield closed under real square roots and real cube roots. This field properly contains the Euclidean closure.

In this way, the axiom-vs-postulate distinction becomes not merely terminological but *algebraically substantive*: the axioms determine the ambient geometry (an ordered plane with reflections), while the postulates select a coordinate subfield from the hierarchy $\mathbb{Q} \subset \mathcal{P} \subset \mathcal{E} \subset \mathcal{O} \subseteq K$.

Remark 10.3. *The full model K^2 may be much larger than the constructible sub-plane. For instance, if $K = \mathbb{R}$, the origami field $\mathcal{O}$ is a countable dense subfield of $\mathbb{R}$, and almost all points of $\mathbb{R}^2$ are not origami-constructible. Conversely, one may take $K = \mathcal{O}$ itself, obtaining the smallest ordered field in which the full system of postulates C1–C7 is "complete" in the sense that every constructible point already has coordinates in K.*

11. Conclusion

We have proposed a formal axiom system for origami geometry, structured in three layers: ambient axioms (A1–A4), fold axioms (F1–F4), and construction postulates (C1–C7). The fold axioms describe the *properties* of reflection but do not assert the existence of any specific fold line; that role belongs to the construction postulates. This separation ensures that the axiom-vs-postulate distinction is not merely terminological but algebraically substantive.

The characterization of models (Section 10) shows that axioms A1–A4, F1–F4 identify, over ordered fields, the class of Desarguesian planes equipped with a nondegenerate symmetric bilinear form, placing origami geometry within the tradition of Bachmann's reflection-based foundations [5]. The first construction postulate (C1, existence of perpendicular bisectors) forces the form to be universal—equivalent to the standard inner product over ordered fields—and subsequent postulates determine the coordinate field from the hierarchy $\mathbb{Q} \subset \mathcal{P} \subset \mathcal{E} \subset \mathcal{O}$, where $\mathcal{P}$ is the Pythagorean closure, $\mathcal{E}$ the Euclidean closure, and $\mathcal{O}$ the origami field [1].

Several directions for further development suggest themselves.

(1) *The Bachmann connection.* Theorem 10.1 identifies the models at the incidence-theoretic level; making the correspondence with Bachmann's group-theoretic metric planes [5] fully precise would yield a deeper structural understanding of origami geometry.
(2) *Logical foundations.* The present work complements the model-theoretic approach of Beklemishev, Dmitrieva and Makowsky [6]; a synthesis of both perspectives—connecting our reflection-based primitives to their orthogonality-based axioms—is an interesting open problem.
(3) *Extension to three dimensions.* The framework extends naturally to 3D, where a medium is folded along a plane with three degrees of freedom, producing ten incidence constraints and 36 elementary fold operations [17]. The 3D analogue of Beloch's fold (placing three points onto three planes) admits up to seven solutions [14, 4]. The subsumption question—whether this single operation subsumes all 36 3D axioms—the determination of existence conditions, and the characterization of the field of 3D-origami-constructible numbers will be treated in a companion article.

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