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IT IS ALL ABOUT SEEING THINGS IN PERSPECTIVE — PART I : SOLUTIONS TO ALHAZEN'S PROBLEM AND BEYOND

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Abstract. This article, the first of a two-part series, characterizes general cubic curves within a projective framework based on the interaction between involutions, perspectivity, and conjugacy. We begin by establishing the properties of characteristic involutions associated with the Miquel-Steiner point of a complete quadrilateral. The primary application of this theory is the presentation of a novel, projectively-grounded solution to the classical Alhazen's reflection problem. The theoretical foundations developed here regarding perspectivity and involution pairs serve as the basis for the specialized study of Möbius isogonal cubics and generalizations of P-conjugacy presented in the companion paper, "Part II: Involutions, Perspectivity and Conjugacy on Cubics".

Throughout this article, we denote by (ABC) the circumcircle of a triangle $\triangle ABC$. More generally, (XYZ) will denote the circle through points X, Y, Z. We also denote by $P_{A,BC}$ the complete quadrilateral formed by lines AB, AC, BP and CP. We use the symbol $\measuredangle$ for directed angles namely the directed angle $\measuredangle BAC$ is the angle between the lines $\angle(BA, CA)$ measured counterclockwise.
The following theorem establishes how to find the Miquel-Steiner point of a complete quadrilateral by the composition of three involutions.

While this part focuses on the solution to Alhazen's problem and the fundamental theory of characteristic involutions, a detailed exploration of the Möbius isogonal cubic and further generalizations of P-conjugacy follows in Part II [1].

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1. The Miquel-Steiner Point and Characteristic Involutions

Theorem 1.1. *Let $\triangle ABC$ be a triangle and let P be an arbitrary point in the plane. Define*

$$E = BP \cap AC, \qquad F = CP \cap AB.$$

By the Miquel–Steiner theorem, the circles (AFC), (AEB), (EPC), and (FPC) concur at a point, denoted by X (the Miquel-Steiner point of $P_{A,BC}$). Let M be the midpoint of BC, and let P' be the reflection of P across M. Define

$$U = P'C \cap BX, \qquad V = P'B \cap CX.$$

Let P'' be the point on ray AP' such that

$$AP' \cdot AP'' = AB \cdot AC.$$

Then X is the reflection of P'' across the internal angle bisector of $\angle BAC$.

Proof.

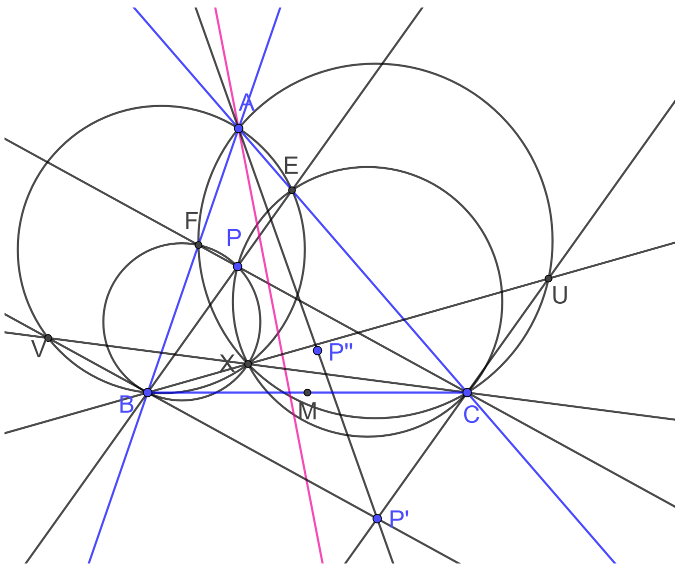

Figure 1. Geometric configuration for Theorem 1.1.

Consider the spiral similarity s_X centered at X associated with the complete quadrilateral formed by the lines AB, AC, BE, and CF. This similarity yields the direct correspondences

$$\triangle XCE \sim \triangle XFB, \qquad \triangle XCF \sim \triangle XEB.$$

Hence,

$$XB \cdot XC = XE \cdot XF.$$

Moreover, line XF is the reflection of line XE across the internal angle bisector of $\angle BXC$.

Let Ω be the circle centered at X with radius $\sqrt{XB \cdot XC}$. Define an involution τ_X as the composition of inversion with respect to Ω followed by reflection across the internal angle bisector of $\angle BXC$. Then

$$\tau_X(C) = B, \qquad \tau_X(E) = F.$$

Furthermore,

$$\tau_X(CE) = (FXB), \qquad \tau_X(BF) = (EXC).$$

It follows that

$$\tau_X(A) = P, \qquad \tau_X(CF) = (BXE), \qquad \tau_X(BE) = (CXF).$$

We refer to this involution as the *characteristic involution* of the spiral similarity at X. It exchanges the lines of the complete quadrilateral with the corresponding circles.

$PBP'C$ is a parallelogram. By Reims' theorem, $U \in (AFXC)$ and $V \in (ABXE)$.

It follows that A is the Miquel–Steiner point of the complete quadrilateral formed by the lines $P'C, P'B, BU$, and CV.

Hence, there exists a characteristic involution associated with the spiral similarity centered at A, which maps

$$B \mapsto C, \quad U \mapsto V, \quad P \mapsto P'.$$

Thus, the claim follows immediately.

Corollary 1.1. *Let $\triangle ABC$ be a triangle and P an arbitrary point. Let M_1 be midpoint of BC and P_1 the reflection of P across M_1. Line AP_1 intersects with the circle (BP_1C) again at D. Let X be the Miquel-Steiner point of $P_{A,BC}$. Then D is the isogonal conjugate of X.*

Proof. Assuming the same notation in the proof of Theorem 1.1. A is the

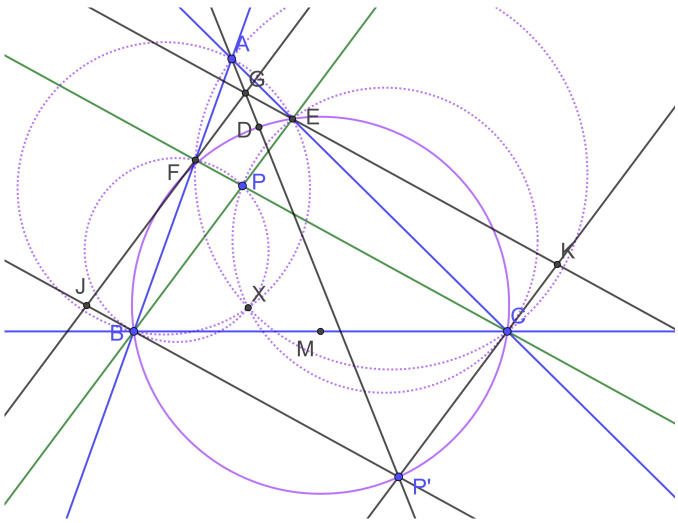

FIGURE 2. Geometric configuration for Corollary 1.1.

Miquel-Steiner point for the complete quadrilateral $X_{P',BC}$. Thus,

$$\angle XBA = \angle UBA = \angle UP'A = \angle CP'D = \angle CBD$$

We have also $\angle BAX = \angle DAC$. Then D and X are isogonal conjugates as required.

Remark 1.1. *$PBP'C$ is a parallelogram. If we construct from E and F parallels to CP and BP respectively. Let G be the intersection point. By Pappus' theorem A, G and P' are collinear. Let $J = P'B \cap GF$ and $K = P'C \cap GE$. Then*

$$\frac{BE}{CF} = \frac{GJ}{P'J} = \frac{\sin\angle GP'J}{\sin\angle P'JG} = \frac{\sin\angle AP'B}{\sin\angle CP'A} = \frac{DB}{DC}$$

Lemma 1.1. *Let $\triangle ABC$ be a triangle. For a point P, let E and F be the feet of the cevians BP and CP, respectively. The locus of points P satisfying*

$$\frac{BE}{CF} = \frac{AB}{AC}$$

is the union of two curves:

- *the rectangular hyperbola through A, B, C, centered at M, the midpoint of BC; and*
- *the circle (BHC), where H denotes the orthocenter of $\triangle ABC$.*

Proof. Let P be a point on the rectangular hyperbola. Since B and C are

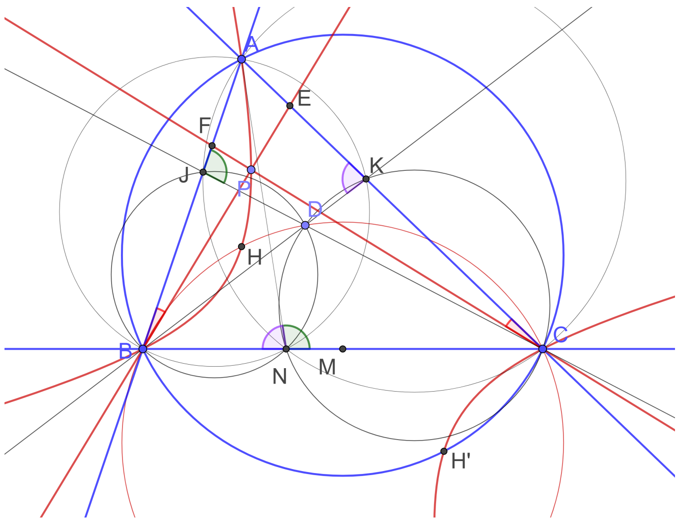

FIGURE 3. Geometric configuration for Lemma 1.1.

antipodes on this hyperbola, the equal-angle property yields

$$\angle EBA = \angle ACF.$$

Hence, the triangles $\triangle AEB$ and $\triangle AFC$ are similar, which shows that P lies on the locus.

Now let D be a point on the circle (BHC), and denote by K and J the feet of the cevians BD and CD, respectively. Observe that the circle (ABC) is the reflection of (BHC) across M, the midpoint of BC. Furthermore, applying the inversion with power $AB \cdot AC$ followed by reflection across the angle bisector of $\angle BAC$ maps (BHC) to line BC. By Theorem 1.1, the Miquel–Steiner point N of the complete quadrilateral $D_{A,BC}$ lies on BC.

Consequently, we have

$$\angle AKB = \angle ANB \quad \text{and} \quad \angle CJA = \angle CNA.$$

It follows that the opposite angles at sides AB and AC in the triangles $\triangle AKB$ and $\triangle AJC$ sum to π, and therefore they have equal sine values. Applying the sine law in these two triangles shows that D also lies on the locus.

2. Concurrency, Circumconics, and Polar Configurations

Lemma 2.1 (Focal Circle of a Tangent Point)**.** *Let $\mathcal{C}$ be a central conic with major axis α and minor axis β. Let D be a point on $\mathcal{C}$ and let ℓ be the tangent line to $\mathcal{C}$ at D. Let H be the intersection of ℓ with the minor axis β, and let D' be the reflection of D across β. Then the circle passing through the points H, D, and D' passes through the foci J and K of the conic $\mathcal{C}$.*

Proof. The construction rule is a direct corollary of the optical focal prop-

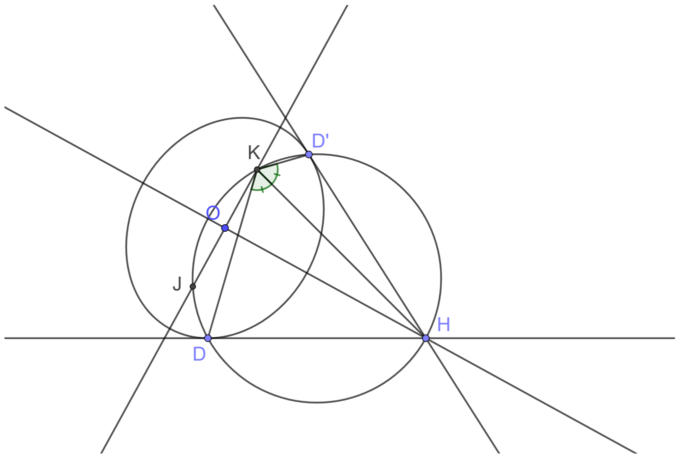

Figure 4. Geometric configuration for Lemma 2.1.

erty of conics, line KH is an angle bisector of $\angle D'KD$. Similarly, JH is an angle bisector of $\angle D'JD$. Hence, by symmetry KH and JH meet at H the midpoint of the arc DD' of the circumcircle of $\triangle D'KD$.

Lemma 2.2. *Let A and B be two antipodes on a rectangular hyperbola h. Let C be any point on h then the internal and external angle bisectors of ACB are parallel to the asymptotes of h.*

Proof. We use the rectangular hyperbola $xy = 1$. The asymptotes of this

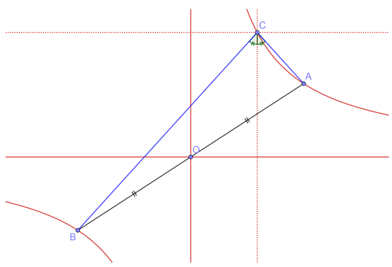

Figure 5. Configuration for Lemma 2.2.

rectangular hyperbola are the $xAxis$ and $yAxis$ respectively. Let $A = (a, \frac{1}{a})$ and $C = (c, \frac{1}{c})$. Then $B = (-a, -\frac{1}{a})$. The slope of AC

$$s_{AC} = \frac{\frac{1}{c} - \frac{1}{a}}{c - a} = -\frac{1}{ac}$$

The slope of BC

$$s_{BC} = \frac{\frac{1}{c} + \frac{1}{a}}{c + a} = \frac{1}{ac}$$

Thus, $s_{AC} = -s_{BC}$ which leads to the required result.

Theorem 2.1. *Let $\triangle ABC$ be a triangle. Define M_1, M_2 and M_3 as the midpoints of segments BC, CA and AB respectively. Let P be an arbitrary point. Denote by P_1, P_2 and P_3 the reflections across M_1, M_2 and M_3 respectively. Then*

(1) *Lines AP_1, AP_2 and AP_3 concurs at a point O which is the center of the circumconic of $AP_3BP_1CP_2$.*
(2) *Circles $(BP_1C), (CP_2A), (AP_3B)$ and $(P_1P_2P_3)$ concurs at a point D on the circumconic.*
(3) *E the midpoint of DP lies on the nine point circle $(M_1M_2M_3)$.*

Proof.[Proof of part 1] Let G the centroid of triangle ABC. Since AG :

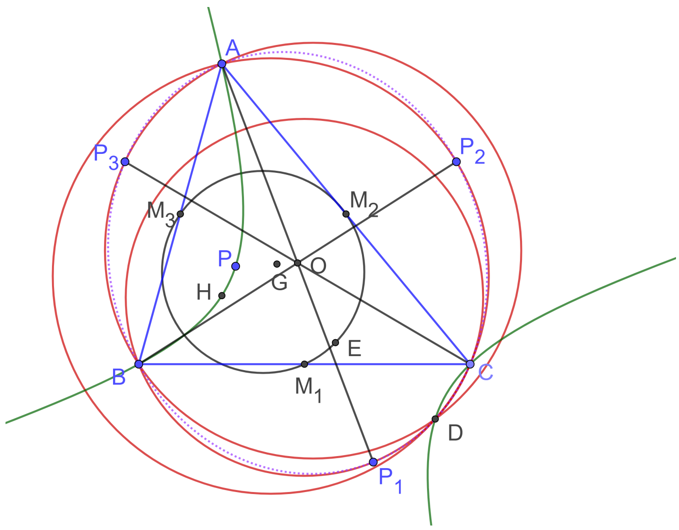

FIGURE 6. Geometric configuration for Theorem 2.1 part 1.

$GM_1 = 2 : 1$ and M_1 is the midpoint of PP_1, it follows that G is the centroid of triangle APP_1. Hence, O the midpoint of AP_1 is collinear with P and G such that $PG : GO = 2 : 1$. Similarly, O is also the midpoint of AP_2 and AP_3. Moreover, there exists a conic c centered at O that passes through A, B, C, P_1, P_2 and P_3.

Proof.[Proof of part 2 and 3] Define D_1, D_2 and D_3 as the reflections of P across BC, CA and AB respectively. Since D_1P_1CB, CD_2P_2A and AP_3D_3B are all isosceles trapezoids, then $D_1 \in (BP_1C), D_2 \in (CP_2A)$ and $D_3 \in (AP_3B)$. Define T_1, T_2 and T_3 as the second intersection of lines D_1P, D_2P and D_3P with the circle $(D_1D_2D_3)$.

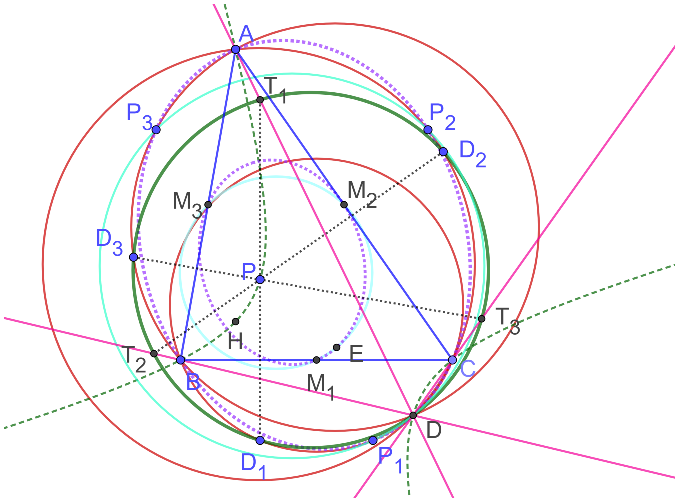

FIGURE 7. Geometric configuration for Theorem 2.1 part 2 and 3.

Consider the inversion centered at P such that the circle $(D_1D_2D_3)$ is invariant under this inversion. Then $D_1 \mapsto T_1,\ D_2 \mapsto T_2$ and $D_3 \mapsto T_3$ under this inversion. Let l_1, l_2, l_3, l_a, l_b and l_c be the polars of T_1, T_2, T_3, A, B and C respectively. Let f be a dilation centered at P with a scale factor of $\frac{1}{2}$. We have

$$f(l_1) = BC,\ f(l_2) = CA,\ f(l_3) = AB$$

This implies that

$$f(l_a) = T_2T_3,\ f(l_b) = T_3T_1,\ f(l_c) = T_1T_2$$

In addition,

$$\measuredangle CD_1B = \measuredangle BPC = \measuredangle(BP, PC) = \measuredangle(T_1T_3, T_1T_2) = \measuredangle T_3T_1T_2 = \measuredangle T_3D_1T_2$$

Also, we have

$$\frac{D_1B}{D_1C} = \frac{PB}{PC} = \frac{dist(P, T_1T_2)}{dist(P, T_1T_3)}$$

Since T_1, P and D_1 are collinear then

$$\frac{D_1B}{D_1C} = \frac{dist(D_1, T_1T_2)}{dist(D_1, T_1T_3)}$$

However, we have $\measuredangle D_1T_3T_1 = \measuredangle D_1T_2T_1$. Hence,

$$\frac{D_1B}{D_1C} = \frac{dist(D_1, T_1T_2)}{dist(D_1, T_1T_3)} = \frac{D_1T_2}{D_1T_3}$$

We conclude then that there is a spiral similarity centered at D_1 mapping B to C and T_2 to T_3, then there is also another spiral similarity centered at D_1 again and maps B to T_2 and T_3 to C. Hence lines CT_3 and BT_2 intersect at a point D which lies on the circles (BD_1C) and $(D_1D_2D_3)$. Similarly, lines AT_1, BT_2, CT_3 and circles $(BD_1C), (CD_2A), (AD_3B)$ and $(D_1D_2D_3)$

concur at D. In addition,

$$\begin{aligned}\measuredangle P_3DP1 &= \measuredangle P_3DB + \measuredangle BDP1 = \measuredangle P_3AB + \measuredangle BCP_1\\ &= \measuredangle ABD_3 + \measuredangle D_1BC = \measuredangle CBP + \measuredangle PBA\\ &= \measuredangle CBA = \measuredangle M_3M_2M_1 = \measuredangle P3P2P1\end{aligned}$$

We conclude that the points P_1, P_2, P_3 and D are concyclic. However, $f[(P_1P_2P_3)] = (M_1M_2M_3)$ then $f(D) = E$ the midpoint of PD lies on the nine point circle (M_1M_2M3).
Consider the rectangular hyperbola h passing through the five points A, B, C, P and H. Since D lies on circles $(BD_1C), (CD_2A), (AD_3B)$, we have

$$\measuredangle APB = \measuredangle BD_3A = \measuredangle BDA$$

Similarly, $\measuredangle BPC = \measuredangle CDB$ and $\measuredangle CPA = \measuredangle ADC$. Hence, by the equal angle property of rectangular hyperbola, D is the antipode with respect to h. Thus E is the center of h. By [2] the center of any conic passing through four points lies on the nine-points conic. Thus, E is also on the n the nine point-points conic of A, B, C and D. However, $f(c) = n$ which proves that D is also on c.

Corollary 2.1. *Let $\triangle ABC$ be a triangle. Define M_1, M_2 and M_3 as the midpoints of segments BC, CA and AB respectively. Let P be an arbitrary point. Denote by P_1, P_2 and P_3 the reflections across M_1, M_2 and M_3 respectively. Denote by X, Y and Z the Miquel-Steiner points of the complete quadrilaterals $P_{A,BC}, P_{B,CA},$ and$P_{C,AB}$. Then lines AX, BY and CZ are concurrent. Moreover, the circles $(AYZ), (BZX)$ and (CXY) are concurrent at the same point.*

Proof. By Theorem 2.1 lines $AP_1, AP_2,$ andAP_3 concurs at O the center of

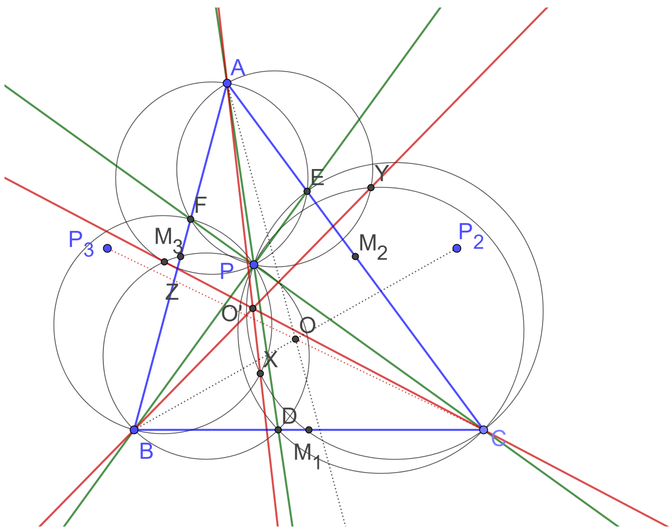

FIGURE 8. Geometric configuration for Corollary 2.1.

the circumconic of $AP_3BP_1CP_2$. Thus, by Theorem 1.1 lines AX, BY and CZ are concurrent at O' the isogonal conjugate of O with respect to the triangle $\triangle ABC$.

Consider τ_X the characteristic involution defined in the proof of Theorem 1.1. The circle $(CBFY)$ and $(AFPY)$ are mapped to the circles $(CBEZ)$ and $(AEPZ)$. Thus Y is mapped to Z. By spiral similarity centered at X that maps C to Z and Y to B we conclude that O' is the second intersection point of the circles (CXY) and (BZX). Similarly, we have the circles $(AYZ), (BZX)$ and (CXY) are concurrent at O'. Using Theorem 1.1, we observe that O and O' are isogonal conjugates with respect to the triangle $\triangle ABC$. Using barycentric coordinates. Let $P = (x, y, z)$ then $O = (y + z, z + x, x + y)$ and $O' = \big(a^2(x + y)(z + x) : b^2(y + z)(x + y) : c^2(y + z)(z + x)\big)$.

The following theorem generalizes the result in Theorem 2.1.

Theorem 2.2. *Let p be the power of an inversion centered at P (p can be also be negative). Let $\triangle ABC$ be a triangle. Let A', B' and C' be the poles of sides BC, CA and AB respectively. Then*

(1) *The triangles $\triangle A'B'C'$ and $\triangle ABC$ are perspective.*
(2) *The perspective point lies on a rectangular hyperbola that passes through A, B, C and P*

Proof.[Proof of part 1]

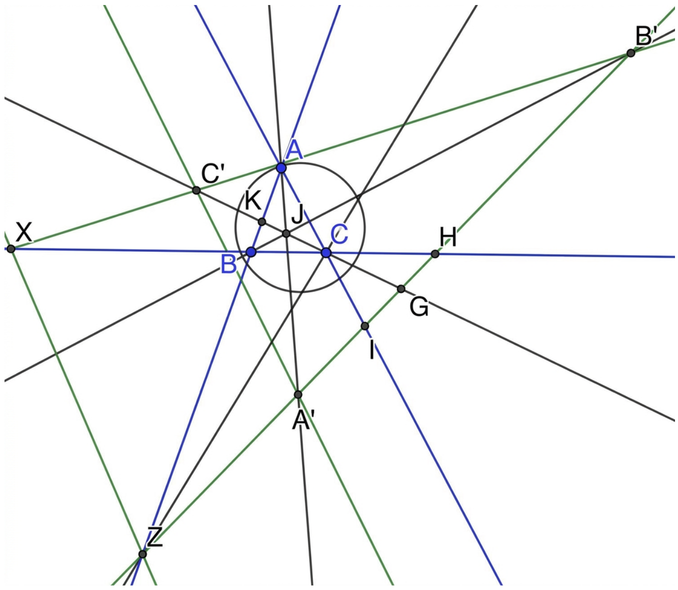

FIGURE 9. Geometric configuration for Theorem 2.2.

Define

$$K = CC' \cap AB, \qquad Z = AB \cap A'B', \qquad H = BC \cap A'B'$$

$$I = AC \cap A'B', \qquad G = CC' \cap A'B', \qquad J = B'B \cap AA', \qquad K' = GJ \cap AB$$

The vertices A, B and C are also the poles of $B'C'$,$C'A'$ and $A'B'$ respectively. In addition CC' is the polar of Z and ZC is the polar of G. We have:

$$(CB, CA; CC', CZ) = (H, I; G, Z) \overset{C}{=} (B, A; K, Z)$$

$$(CB,CA;CC',CZ)=(A',B';Z,G)\overset{J}{=}(A,B;Z,K')=(B,A;K',Z)$$

Therefore:

$$K=K'\Rightarrow J\in CC'$$

Proof.[Proof of part 2] We prove the second part analytically. Let the points be defined as:

$$A=(a,\tfrac{1}{a}),\quad B=(b,\tfrac{1}{b}),\quad C=(c,\tfrac{1}{c}),\quad D=(d,\tfrac{1}{d})$$

Let T be the translation that maps point D to the origin O. To find the equations of the polars, we apply the translation T, then its inverse T^{-1}. Thus, the polar equations of points A, B, and C are:

$$\begin{aligned}(a-d)(x-d)+\left(\tfrac{1}{a}-\tfrac{1}{d}\right)(y-\tfrac{1}{d})&=p\\(b-d)(x-d)+\left(\tfrac{1}{b}-\tfrac{1}{d}\right)(y-\tfrac{1}{d})&=p\\(c-d)(x-d)+\left(\tfrac{1}{c}-\tfrac{1}{d}\right)(y-\tfrac{1}{d})&=p\end{aligned}$$

where p is the power of inversion.

Solving this system by a computer algebra system yields the coordinates of the poles:

$$C'=\left(d-\frac{dp}{(a-d)(b-d)},\quad \frac{1}{d}-\frac{p}{\left(\frac{1}{a}-\frac{1}{d}\right)\left(\frac{1}{b}-\frac{1}{d}\right)d}\right)$$

$$B'=\left(d-\frac{dp}{(a-d)(c-d)},\quad \frac{1}{d}-\frac{p}{\left(\frac{1}{a}-\frac{1}{d}\right)\left(\frac{1}{c}-\frac{1}{d}\right)d}\right)$$

$$A'=\left(d-\frac{dp}{(b-d)(c-d)},\quad \frac{1}{d}-\frac{p}{\left(\frac{1}{b}-\frac{1}{d}\right)\left(\frac{1}{c}-\frac{1}{d}\right)d}\right)$$

The lines AA', BB', and CC' intersect at a point whose coordinates are computed by a computer algebra system as follows:

$$(k,\tfrac{1}{k})$$

where:

$$k=-d\cdot\frac{(a-d)(b-d)(c-d)+dp}{abcd^2p-(a-d)(b-d)(c-d)}$$

3. Alhazen's Problem and Barrow's Curve

We revisit the problem of Ibn al-Haytham (namely Alhazen's problem).

Given two points B,C outside a circle, find $R\in$ circle such that $\angle BRC$ is extremal.

Equivalently: determine the reflection point R where a ray from B reflects to pass through C.

We will use inversive geometry to give a simple solution. As shown in Figure 10. Let A be the center of the mirror circle ω, and let B,C be two points outside ω.

We denote by

$$\mathcal{I}_A^{AB\cdot AC}$$

the inversion centered at A with power $AB\cdot AC$. If ℓ_A is the internal angle bisector of $\angle BAC$, we define the composed involution

$$\mathcal{J}_A:=\rho_{\ell_A}\circ\mathcal{I}_A^{AB\cdot AC},$$

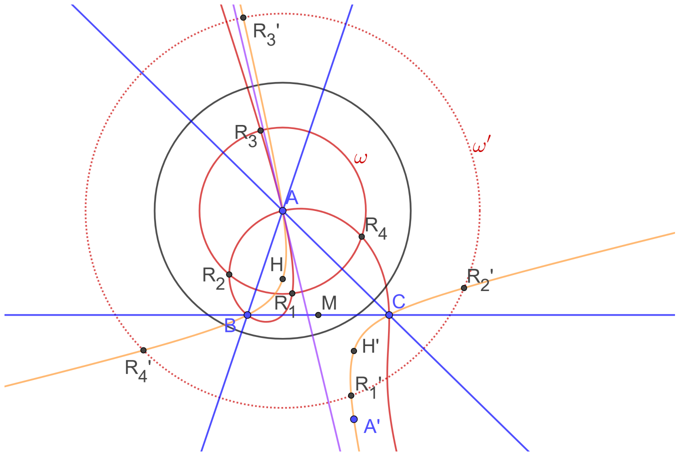

FIGURE 10. Geometric configuration for Alhazen's problem .

where ρ_{ℓ_A} denotes reflection across ℓ_A.

Define the circumconic rectangular hyperbola h through A, B, C, centered at M (the midpoint of BC). Let $\omega' = \mathcal{I}_A^{AB\cdot AC}(\omega)$. Then:

(1) Compute the intersections $h \cap \omega' = \{R_1', R_2', R_3', R_4'\}$ (up to four points).
(2) For each R_i', set

$$R_i = \mathcal{J}_A(R_i'), \quad i = 1, 2, 3, 4.$$

Each R_i lies on ω, and the ray BR_i reflects at R_i into CR_i, solving Alhazen's problem. Additionally, reflect the hyperbola h across ℓ_A to obtain h', then invert h' by $\mathcal{I}_A^{AB\cdot AC}$ to get the cubic curve

$$\mathcal{B} = \mathcal{I}_A^{AB\cdot AC}(h'),$$

known as *Barrow's curve* [3]. This cubic passes through the solution points R_1, R_2, R_3, R_4 of Alhazen's problem.

Explanation. Let R' be one of the intersection points of the hyperbola h with the circle ω'. Reflecting R' across M, we obtain a point P which also lies on h. Next, apply the involution $\mathcal{J}_A$ to R'. The resulting point R is precisely the Miquel–Steiner point of the complete quadrilateral $P_{A,BC}$ (by Theorem 1.1).

Furthermore, since P lies on h (by Lemma 1.1), the spiral similarity centered at R has the ratio

$$\frac{AB}{AC} = \frac{RE}{RC} = \frac{RB}{RF} = \frac{BE}{CF},$$

which implies that the corresponding angles are equal:

$$\angle CRE = \angle FRE \quad \text{and more importantly} \quad \angle CRA = \angle ARB.$$

Remark 3.1. *Barrow's curve is a focal circular cubic, possessing a singularity in the form of a knot. The curve is also known as the* strophoid [4]. *The focal point of Barrow's curve F_c is the point colloquially known as the A-Dumpty point, namely the midpoint of the chord cut by the A-symmedian*

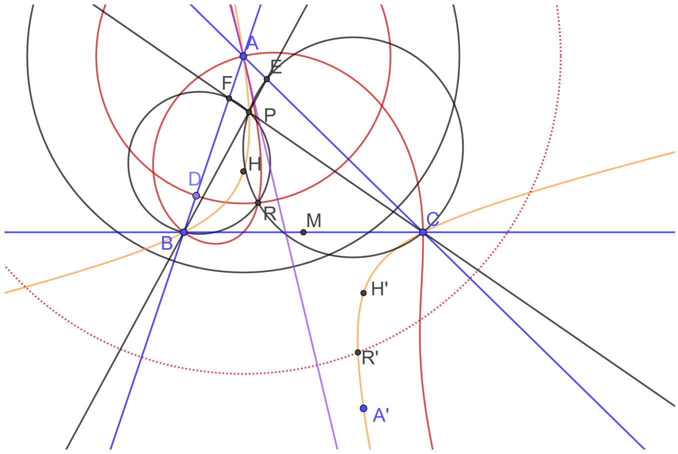

FIGURE 11. Geometric configuration illustrating the solution of Alhazen's problem by spiral similarity and inversive geometry

on the circumcircle of $\triangle ABC$. *The coincidence occurs when the intersection point* R' *equals* A', *the reflection of* A *across* M.

Definition 3.1. *We denote by*

$$\mathcal{I}_{F_c}^{F_cA^2}$$

the inversion centered at F_c *with power* $F_cA^2 = F_cB \cdot F_cC$. *If* ℓ_{F_c} *is the internal angle bisector of* $\angle BF_cC$ *(i.e. the line* F_cA*), we define the composed involution*

$$\mathcal{J}_{F_c} := \rho_{\ell_{F_c}} \circ \mathcal{I}_{F_c}^{F_cA^2}.$$

Moreover, Barrow's curve is invariant under $\mathcal{J}_{F_c}$, *i.e. the composition of inversion centered at* F_c *with power* $F_cA^2 = F_cB \cdot F_cC$ *followed by reflection across* ℓ_{F_c}.

To prove the invariance of Barrow's curve under $\mathcal{J}_{F_c}$, we observe that the action of $\mathcal{J}_{F_c}$ on the cubic corresponds to reflection across M on the rectangular hyperbola. Let ρ_M denote reflection across M, and define

$$\phi := \mathcal{J}_A \circ \mathcal{J}_{F_c} \circ \mathcal{J}_A \circ \rho_M.$$

ϕ is a Möbius transformation because it is a composition of inversions and reflections. Tracing the images of distinguished points under ϕ, we find:

$$B \xmapsto{\mathcal{J}_A} C \xmapsto{\mathcal{J}_{F_c}} B \xmapsto{\mathcal{J}_A} C \xmapsto{\rho_M} B,$$

so B is fixed by ϕ. Similarly, C is fixed. For A', we have

$$A' \xmapsto{\mathcal{J}_A} F_c \xmapsto{\mathcal{J}_{F_c}} \infty \xmapsto{\mathcal{J}_A} A \xmapsto{\rho_M} A',$$

so A' is also fixed. Similarly, A is fixed. Thus ϕ fixes four distinct points: A, A', B, C.

Now, any Möbius transformation has the form

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

Its fixed points satisfy $f(z) = z$, i.e. $cz^2 + (d - a)z - b = 0$. This is a quadratic equation in z, so a nontrivial Möbius transformation can have at most two distinct fixed points. Therefore, if a Möbius transformation fixes three distinct points, it must be the identity.

Applying this to ϕ, we conclude that ϕ is the identity. Consequently, the involution $\mathcal{J}_{F_c}$ on the cubic indeed corresponds to reflection across M on h, and Barrow's curve is invariant under $\mathcal{J}_{F_c}$.

Lemma 3.1. *Let $\triangle ABC$ be a triangle, and let P be an arbitrary point. Define*

$$E = BP \cap AC, \quad F = CP \cap AB.$$

Let X be the Miquel–Steiner point of the complete quadrilateral $P_{A,BC}$, and let M be the midpoint of BC. Denote by X' and P' the reflections of X and P across M. Define

$$H = BX' \cap AC, \quad G = CX' \cap AB.$$

Then P' is the Miquel–Steiner point of the complete quadrilateral $X'_{A,BC}$, and

$$\frac{BE \cdot BH}{CF \cdot CG} = \frac{BA}{CA}.$$

Proof.

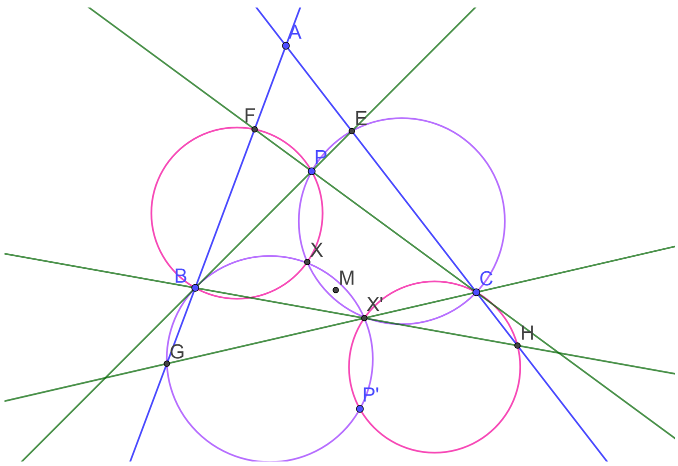

FIGURE 12. Geometric configuration for Lemma 3.1

The fact that P' is the Miquel–Steiner point of the complete quadrilateral $X'_{A,BC}$ follows immediately from Theorem 1.1. Consider the involution

$$\mathcal{J}_A := \rho_{\ell_A} \circ \mathcal{I}_A^{AB \cdot AC}.$$

We have

$$\mathcal{J}_A(B) = C, \quad \mathcal{J}_A(X) = P'.$$

It follows that the triangles $\triangle ABX$ and $\triangle AP'C$ are similar, and likewise $\triangle AXC$ and $\triangle AP'G$ are similar.

By Remark 1.1, we obtain

$$\frac{BE}{CF} = \frac{\sin \angle AP'B}{\sin \angle CP'A} = \frac{\sin \angle ACX}{\sin \angle XBA},$$

$$\frac{BH}{CH} = \frac{\sin \angle AXB}{\sin \angle CXA}.$$

Therefore,

$$\frac{BE \cdot BH}{CF \cdot CG} = \frac{\sin \angle ACX}{\sin \angle XBA} \cdot \frac{\sin \angle AXB}{\sin \angle CXA} = \frac{AX}{AC} \cdot \frac{AB}{AX} = \frac{BA}{CA}.$$

Corollary 3.1. *In a triangle $\triangle ABC$, the locus of points D such that*

$$BD \cap AC = H, \quad CD \cap AB = G, \quad \text{and} \quad BH = CG$$

is the reflection of Barrow's curve $\mathcal{B}$ *across M, the midpoint of BC.*

Proof. When P varies along the rectangular hyperbola, then by Lemma

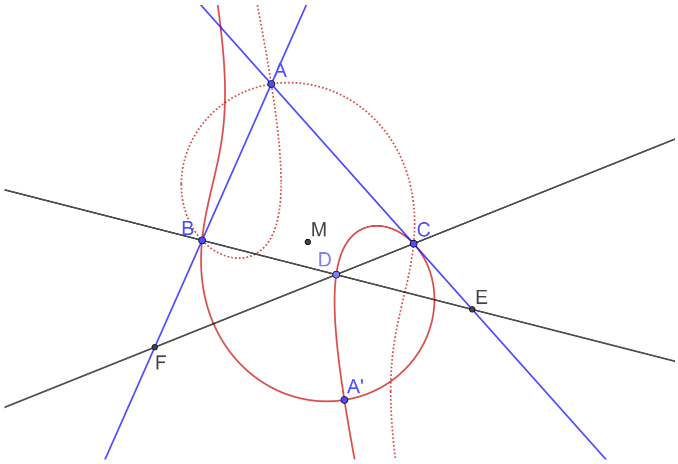

Figure 13. Geometric configuration for Corollary 3.1 equal cevian cubic

1.1 the ratio of the lengths of the cevians from B and C through P is fixed as

$$BE : CC = AB : AC.$$

In this case, the locus of X, the Miquel–Steiner point of $P_{A,BC}$, is precisely *Barrow's curve*. Furthermore, if we reflect X across M to obtain the point D, Lemma 3.1 implies that the cevians from B and C to D are equal.

4. Isoptic Cubics: Construction and Complex Analysis

To solve the generalized Ibn al-Haytham problem for conic mirrors, we will rely on the *isoptic cubic* (also called the focal circular cubic [5]).

Definition 4.1. *Let A, A', B, B' be four points in the plane, no three of which are collinear. The locus of points P that satisfy the optical property*

$$\measuredangle B'PA' = \measuredangle APB$$

is a cubic curve, called the isoptic cubic $\mathcal{P}$.

To construct the cubic, we proceed as follows. Let U be a point on the circle centered at A and passing through B. Let V be the rotation of B' around A' with an angle $\angle UAB$. Define G and H as the intersections of the circles (UBA) and $(VB'A')$. Then the loci of G and H form the cubic curve $\mathcal{P}$.

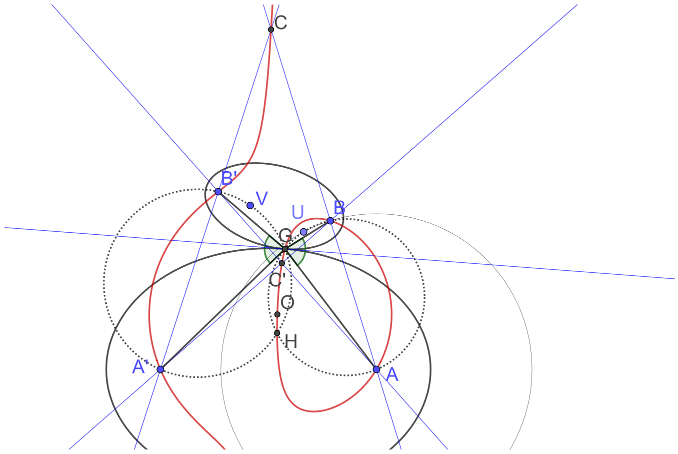

Figure 14. Geometric configuration for isoptic cubic

Furthermore, let $C = AB \cap A'B'$ and $C' = AB' \cap A'B$. Let O denote the Miquel–Steiner point of $C'_{C,AA'}$. The points C, C', and O also lie on $\mathcal{P}$, since they satisfy the optical property of the curve.

Observe that the isosceles triangles $\triangle BAU$ and $\triangle VA'B'$ are similar. Since G is the intersection of their circumcircles, we have

$$\measuredangle FGB = \measuredangle FVB = \measuredangle BUE = \measuredangle BGE.$$

Thus $G \in \mathcal{P}$. A similar argument shows that $H \in \mathcal{P}$ as well.

To establish the properties of this curve, we resort to complex numbers in the same manner as in [6]. By Theorem 1.1, there exists a characteristic involution $\mathcal{J}_O$ corresponding to inversion centered at O with power

$$k^2 = OA \cdot OA' = OB \cdot OB' = OC \cdot OC',$$

followed by reflection across ℓ_o, the common angle bisector of $\angle AOA'$, $\angle BOB'$, and $\angle COC'$.

Let a, b, c, a', b', c' and p denote the complex numbers of the points A, B, C, A', B', C' and P, respectively. Assume O is the origin and that ℓ_o coincides with the x-axis of our coordinate system. Scaling the diagram so that $k = 1$, the involution is given by

$$\mathcal{J}_O(z) = \frac{1}{z}, \quad \text{for any point } z \in \mathbb{C}.$$

Then, the condition that a point p satisfies the isoptic property of the curve is

$$\left(\frac{a-p}{p-b}\right) \Big/ \left(\frac{b^{-1}-p}{p-a^{-1}}\right) \in \mathbb{R}.$$

Expanding this expression and using the fact that $z \in \mathbb{R} \iff z = \bar{z}$, we obtain:

$$(p-a)(p-a^{-1})\overline{(p-b)(p-b^{-1})} - \overline{(p-a)(p-a^{-1})}(p-b)(p-b^{-1}) = 0.$$

$$(p^2+1-(a+a^{-1})p)\,\overline{(p^2+1-(b+b^{-1}))} - \overline{(p^2+1-(a+a^{-1})p)}\,(p^2+1-(b+b^{-1})) = 0.$$

$$p\overline{p}\Big[\big(p+p^{-1}-(a+a^{-1})\big)\,\overline{\big(p+p^{-1}-(b+b^{-1})\big)}-\overline{\big(p+p^{-1}-(a+a^{-1})\big)}\,\big(p+p^{-1}-(b+b^{-1})\big)\Big]=0.$$

Assume $p \neq 0$. Dividing by $p\overline{p}$, we observe by symmetry of the equation that p^{-1} is also a solution. Hence the point P', which corresponds to p^{-1}, satisfies the optical property of the curve.

Let

$$u = p + p^{-1}, \quad m = a + a^{-1}, \quad n = b + b^{-1}.$$

Then

$$\frac{u-m}{u-n} = \overline{\left(\frac{u-m}{u-n}\right)}.$$

That is, u, m, and n are collinear, which defines a line homothetic to ℓ_n, the Gauss–Newton line of the complete quadrilateral $A_{A',BB'}$ from the origin with ratio 2. Thus the solution curve $\mathcal{P}$ corresponds to the involution pairs (p, p^{-1}) where the midpoint of the pair lies on ℓ_n.

Hence, taking any two involution pairs (p, p^{-1}) and (q, q^{-1}) of the solution $\mathcal{P}$, the midpoints will share the Gauss–Newton line ℓ_n. Thus the complete quadrilateral share the same isoptic cubic curve solution with the two pairs (a, a^{-1}) and (b, b^{-1}). Let $q = b$, by the optical property from a we obtain

$$\measuredangle B'AP' = \measuredangle PAB.$$

Thus, AP and AP' are isogonal with respect to the lines AB and AB'. By a similar argument, we conclude that P and P' are isogonal conjugates with respect to the four triangles

$$\triangle AB'C, \quad \triangle ABC', \quad \triangle A'BC, \quad \triangle A'B'C'$$

of the complete quadrilateral. Therefore, P and P' are two foci of a conic tangent to the four lines comprising the complete quadrilateral.

Remark 4.1. *Let A be a point on the isoptic curve $\mathcal{P}$. For each pair of involution points (P, P') under the involution $\mathcal{J}_{\mathcal{O}}$, the pair of lines (AP, AP') defines an involution on the pencil of lines through A (which corresponds to reflection across a common angle bisector). Moreover, the lines $A'P$ and $A'P'$ intersect AP and AP' at points Q and Q', respectively, so that (Q, Q') forms another pair of involution points on $\mathcal{P}$ under $\mathcal{J}_{\mathcal{O}}$. Furthermore, if A' is the point paired with A by $\mathcal{J}_{\mathcal{O}}$, then the tangent to the cubic at A is paired with the line AA' with respect to the involution on the pencil from A.*

Using Remark 4.1, we can construct tangents to the cubic. By the isogonal conjugacy property, the involution on the pencil from A corresponds to reflection across a line ℓ_A through A, where ℓ_A is a common angle bisector of $\angle PAP'$ for each involution pair (P, P'). Consequently, the tangent at A is obtained as the reflection of AA' across ℓ_A. Observe that A lies on the isoptic curve determined by the system (P, P', A, A') (i.e,. a solution) which is the same cubic $\mathcal{P}$. Hence, A satisfies the optical property of the curve. This, in turn, explains the isogonality between the chord AA' and the tangent at A, taken with respect to the lines AP and AP'.

Remark 4.2. *If the complete quadrilateral $A_{A',BB'}$ is tangential, then its isoptic curve becomes a* Barrow's curve*, namely a strophoid, in which the incenter I of the quadrilateral serves as the double point of the cubic. This*

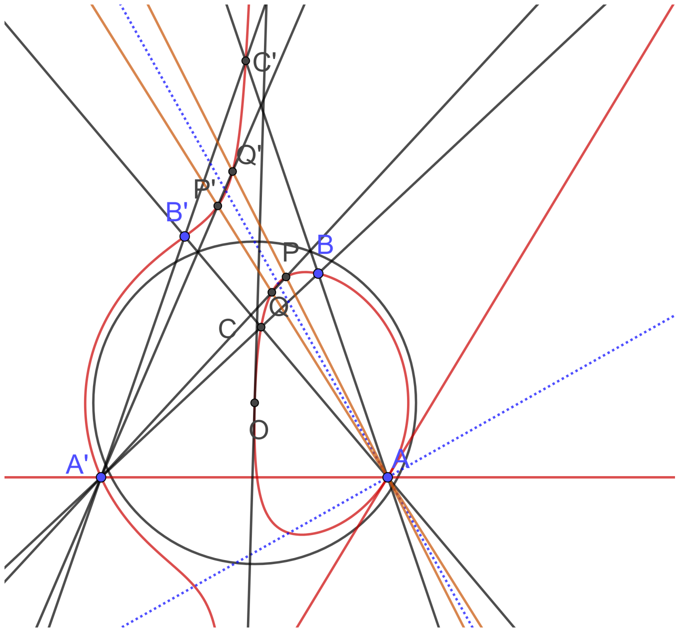

Figure 15. Geometric configuration of the involution on a pencil of lines from a point.

follows directly from the fact that the corresponding pairs of points act as the foci of the inconics associated with $A_{A',BB'}$ so I will be paired to itself by the involution.

Theorem 4.1. *Let (A, A') and (B, B') be two pairs of points. Define*

$$C = AB \cap A'B', \quad C' = AB' \cap A'B.$$

Let O denote the Miquel–Steiner point of the complete quadrilateral formed by the lines AB, AB', $A'B$, and $A'B'$. Let $\mathcal{J}_O$ be the characteristic involution, and let $\mathcal{P}$ denote the isoptic cubic curve associated with the two pairs (A, A') and (B, B').

Further, set

$$D = AA' \cap CC', \quad E = BB' \cap CC', \quad F = AA' \cap BB'.$$

Let D', E', F' be the feet of the altitudes of the diagonal triangle $\triangle DEF$, and let D'', E'', F'' be the images of D', E', F' under the involution $\mathcal{J}_O$. Then:

1. *The points D', E', F' lie on the curve $\mathcal{P}$.*
2. *The points D'', E'', F'' are collinear; equivalently, O lies on the nine-point circle of the triangle $\triangle DEF$.*

Proof. The claim of part 1 follows directly from the harmonic relation

$$(A, A'; D, F) = -1.$$

This establishes a harmonic bundle on the pencil of lines through D'. Moreover, since the lines DD' and $D'B$ are perpendicular, they serve as the internal and external angle bisectors of $\angle A'D'A$. Consequently, the point D' satisfies the optical property that characterizes the curve $\mathcal{P}$. By applying the same reasoning cyclically to the other vertices, we conclude that the

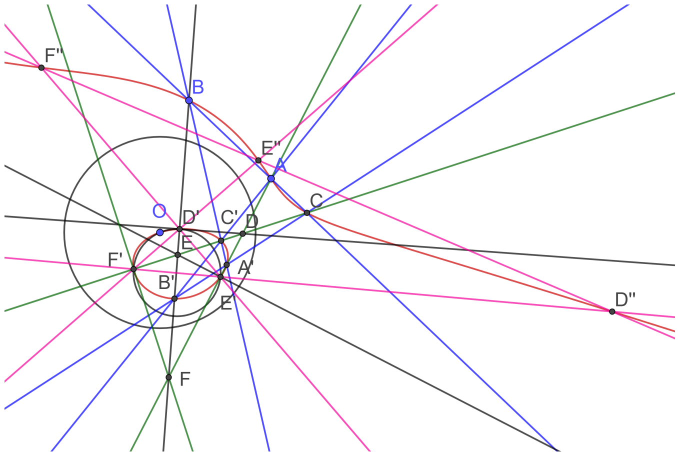

FIGURE 16. Geometric configuration for Theorem 4.1

points D', E', F' all lie on $\mathcal{P}$.
For part 2, consider the involution on the pencil of lines from F. This corresponds to the reflection across the line passing through F', E, C and C'. Since the lines $F'E'$ and $F'D'$ are paired under this involution, Remark 4.1 implies the collinearity of both triples

$$(F', E', D'') \quad \text{and} \quad (F', D', E'').$$

Hence, E', D', and F'' are collinear, as are D'', E'', and F'', thereby completing the four defining lines of the new complete quadrilateral.

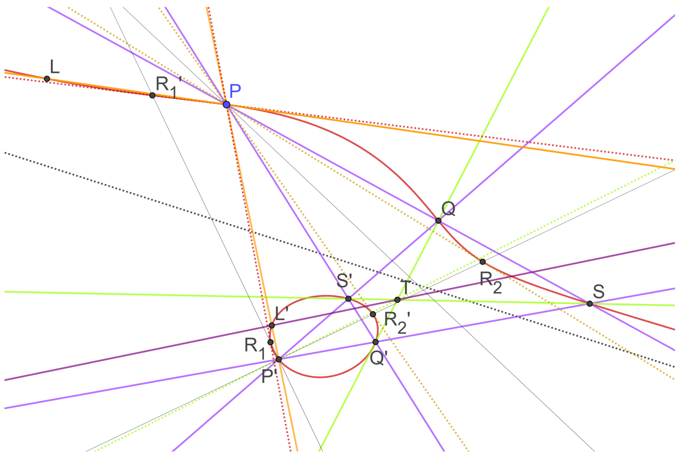

FIGURE 17. Geometric configuration of Remark 4.3.

Remark 4.3. *Let (P, P') be pairs of points on the isoptic cubic $\mathcal{P}$ under the involution $\mathcal{J}_{\mathcal{O}}$. The tangent to $\mathcal{P}$ at P meets $\mathcal{P}$ again at L. Thus, $L' = \mathcal{J}_{\mathcal{O}}(L)$, P and P' are collinear. Construct the tangents to $\mathcal{P}$ from P, we have four tangents they will touch the cubic on two pairs (R_1, R_1') and (R_2, R_2') under the involution $\mathcal{J}_{\mathcal{O}}$ such that $P' = R1R1' \cap R_2R_2'$ (some of tangent points and intersection can be complex points). Now consider and pair (Q, Q') on $\mathcal{P}$ under the involution $\mathcal{J}_{\mathcal{O}}$ lines then $S = PQ \cap P'Q'$ and*

$S' = PQ' \cap P'Q$ will be pair under the involution $\mathcal{J}_{\mathcal{O}}$. Let $T = QQ' \cap SS'$, by Theorem 4.1 TL' is an altitude thus $TL' \perp PP'$. Hence, the locus of T is the perpendicular line to PP' at L'.

5. Projective Generalization to Arbitrary Cubics

Surprisingly, our study for isoptic curves can be extended to an arbitrary cubic curves that can be constructed via Schroeter's ruler construction of cubics [7]. Consider three pairs of involution points (A, A'), (B, B'), and (C, C') in the plane, with the condition that no three of these points are collinear. For any two pairs (P, P') and (Q, Q') among them, we may define a new pair (S, S') by

$$S = PQ \cap P'Q', \qquad S' = PQ' \cap P'Q.$$

.
Theorem 5.2 can be viewed as a corollary of the propositions established in [7]. Nevertheless, we shall present its proof by employing the projective transformations introduced in Theorem 5.1. In this way, the validity of the claims follows from the correctness of their analogy with the isoptic cubic curve that we have already proved. It is worth noting that certain intersection points may not be visible, as they correspond to complex points.

Theorem 5.1. *Given three pairs of points (A, A'), (B, B') and (C, C') in general position in the plane such that no three of them are collinear. There exists a projectivity that maps the pairs to three pairs (X, X'), (Y, Y') and (Z, Z') such that*

$$\measuredangle XZY = \measuredangle Y'ZX' \quad \textit{and} \quad \measuredangle XZ'Y = \measuredangle Y'Z'X'$$

Proof. The two angle conditions

$$\measuredangle XZY = \measuredangle Y'ZX' \quad \text{and} \quad \measuredangle XZ'Y = \measuredangle Y'Z'X'$$

are equivalent to the statement that the points Z and Z' are *isogonal conjugates* with respect to the quadrilateral $X_{X',YY'}$. In Euclidean geometry, isogonality is defined by equality of angles, but in projective geometry it is characterized by cross-ratios involving the circular points at infinity.

For the original configuration $(A, A'), (B, B'), (C, C')$, there exist unique (possibly complex) points U and V such that C and C' are isogonal conjugates with respect to the quadrilateral $A_{A',BB'}$ and the reference directions U and V. Projectively, this means that the pencils of lines at C and C' satisfy the cross-ratio equalities

$$(CA, CB; CU, CV) = (CB', CA'; CU, CV),$$
$$(C'A, C'B; C'U, C'V) = (C'B', C'A'; C'U, C'V).$$

These relations define the lines CU, CV, $C'U$, and $C'V$ uniquely.

A projectivity of the plane is determined by the images of four points. Choose a projectivity $\mathbf{P}$ such that

$$\mathbf{P}(U) = \mathbf{I} = (1 : i : 0), \qquad \mathbf{P}(V) = \mathbf{J} = (1 : -i : 0),$$

the two circular points at infinity. These points lie on every Euclidean circle and encode Euclidean angle measure via Laguerre's formula.

Under the projectivity **P**, the cross-ratio conditions defining isogonality with respect to U and V become cross-ratios with respect to **I** and **J**. By Laguerre's formula, such cross-ratios correspond exactly to Euclidean angles between lines. Thus the projective isogonality of C and C' with respect to U and V becomes Euclidean isogonality of

$$Z = \mathbf{P}(C), \qquad Z' = \mathbf{P}(C')$$

with respect to the quadrilateral $X_{X',YY'}$.

Consequently,

$$\angle XZY = \angle Y'ZX', \qquad \angle XZ'Y = \angle Y'Z'X',$$

as required.

This completes the construction of the desired projectivity **P**.

Theorem 5.2. *Let (A, A'), (B, B'), and (C, C') be three pairs of points in the plane, no three of which are collinear. Then the locus of points P such that the pairs of lines (PA, PA'), (PB, PB'), and (PC, PC') correspond to pairs of an involution $\mathcal{J}_P$ on the pencil of lines through P is a cubic curve Γ.*

Moreover, the pairing (A, A'), (B, B'), and (C, C') induces an involution $\mathcal{J}_\Gamma$ on the cubic curve itself.

In addition:

1. *The line PP' and the tangent to Γ at P are paired under the involution $\mathcal{J}_P$, where P' denotes the image of P under $\mathcal{J}_\Gamma$.*
2. *For any pair (Q, Q') associated with the involution $\mathcal{J}_\Gamma$, the lines (PQ, PQ') form a pair under the involution $\mathcal{J}_P$. Moreover, the intersections $S = PQ \cap \Gamma$ and $S' = PQ' \cap \Gamma$ yield a pair (S, S') under $\mathcal{J}_\Gamma$, and the lines QS' and $Q'S$ meet at P'. Moreover, the locus of $T = QQ' \cap SS'$ is a line that passes through L' the intersection of $PP' \cap \Gamma$.*
3. *The pairs (R_1, R_1') and (R_2, R_2') of contact points of the tangents from P to Γ are paired under $\mathcal{J}_\Gamma$, and $P' = R1R1' \cap R_2R2'$.*

Noting that intersection points and contact points need not be real; they may occur as complex points.

Theorem 5.3. *Let $\mathcal{Q}$ be a complete quadrilateral $A_{A',BB'}$. Define the pair of points $D = AB \cap A'B'$ and $D' = AB' \cap A'B$. Given any two points C and C' in the plane, the four conics uniquely determined by the following sets of five points are concurrent:*

$$(C, C', A, B, D'), \quad (C, C', A', B', D'), \quad (C, C', A, D, B'), \quad (C, C', A', B', D)$$

Proof. By Theorem 5.2, there exists a cubic Γ defined as the locus of points M such that the pairs of lines

$$(MA, MA'), \quad (MB, MB'), \quad (MC, MC')$$

form an involution in the pencil of lines passing through M. By construction, the tangents to Γ at C and C' intersect at a point H, which necessarily lies on Γ. Let H' denote the involutive pair of H on Γ; equivalently, H' is the third point of intersection of the line CC' with the cubic Γ.

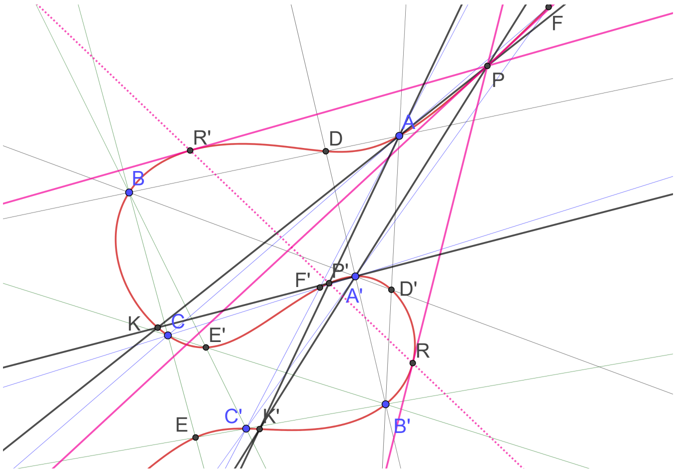

FIGURE 18. Geometric configuration illustrating Theorem 5.2.

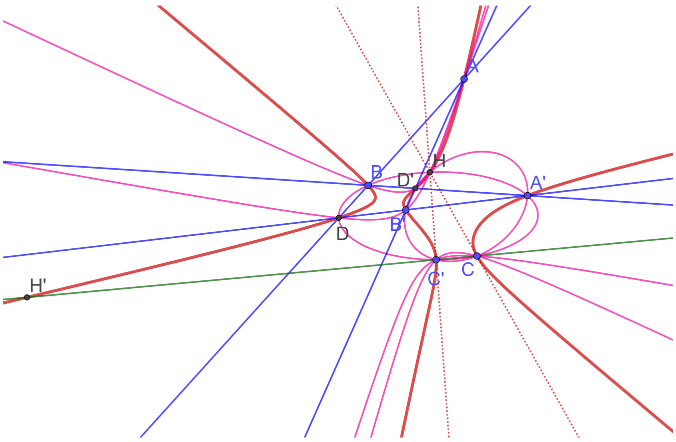

FIGURE 19. Geometric configuration illustrating Theorem 5.3.

Consider a projectivity Π that maps C and C' to the circular points at infinity, $\mathbf{I}$ and $\mathbf{J}$. Under Π, the conics in the theorem are transformed into circumcircles associated with the triangles of the projected complete quadrilateral $\Pi(\mathcal{Q})$. Furthermore, the cubic Γ is mapped to the isoptic cubic $\Pi(\Gamma)$ of this configuration.

According to the Miquel-Steiner Theorem, these circles concur at a point H^*. Let H^{**} be the point at infinity on $\Pi(\Gamma)$, which serves as the image of H^* under the characteristic involution associated with $\Pi(\mathcal{Q})$ on the isoptic cubic $\Pi(\Gamma)$. We observe that H^{**} lies on the line at infinity connecting $\mathbf{I}$ and $\mathbf{J}$, and the lines from H^* to $\mathbf{I}$ and $\mathbf{J}$ are tangent to $\Pi(\Gamma)$.

By applying the inverse projectivity Π^{-1}, we conclude that $H = \Pi^{-1}(H^*)$ is the common point of concurrence for the original four conics:

$$(C, C', A, B, D'), \quad (C, C', A', B', D'), \quad (C, C', A, D, B'), \quad (C, C', A', B', D)$$

Additionally, it follows that $H' = \Pi^{-1}(H^{**})$, completing the proof.

Remark 5.1. *In the context of Theorem 5.3, if the pairs* (A, A'), (B, B'), (D, D'), *and* (C, C') *form a Möbius involution on* Γ *(as defined in the following section), then the point of concurrence* H *lies on the circle passing through* C, C', *and* O. *Here,* O *denotes the Miquel-Steiner point of the complete quadrilateral* $A_{A',BB'}$, *and* O *serves as the center of the Möbius involution. This property arises because this circle is the image of the line passing through* C, C', *and* H' *under the Möbius involution.*

Theorem 5.4. *Let* $\triangle ABC$ *be a triangle, and let* D, E, *and* F *lie on the sides* BC, CA, *and* AB, *respectively. Let* U *and* V *be two points in the plane. Then the three conics uniquely determined by the following sets of five points are concurrent:*

$$(A, E, F, U, V), \qquad (B, D, F, U, V), \qquad (C, D, E, U, V).$$

Proof.

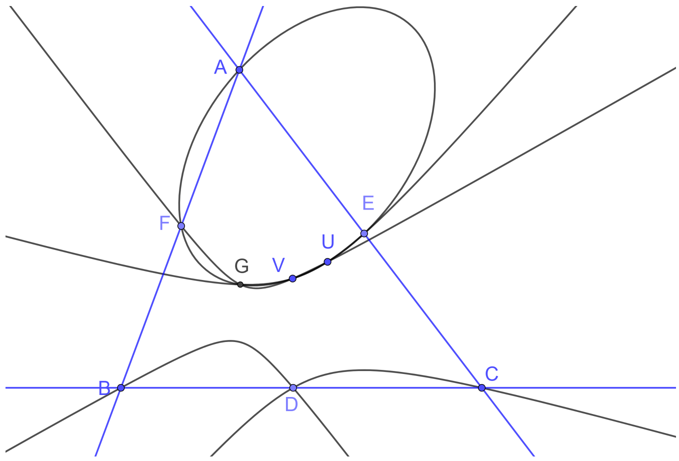

FIGURE 20. Geometric configuration illustrating Theorem 5.4.

The statement follows as a direct corollary of the Miquel triangle theorem. Consider a projectivity Π that sends the points U and V to the circular points at infinity, **I** and **J**. Under this projectivity, the three conics become circles. By the classical Miquel theorem, these circles concur at a single point. Applying the inverse projectivity Π^{-1} then shows that the original three conics are likewise concurrent.

Remark 5.2. *Using the same approach, many circle problems can be generalized to conics by fixing two points on each conic. Observe that a family of conics sharing three points behaves analogously to circles sharing a single point. For example, consider only the conics passing through the points* O, U, *and* V.

Figure 21 illustrates a conic version of Miquel's Theorem. Here, we have a triangle formed by conic sides connecting (A, B), (B, C), *and* (C, A). *Points* D, E, *and* F *are chosen on these sides. The theorem states that the conics passing through* (A, E, F), (B, D, F), *and* (C, E, D) *share a common point.*

The proof relies on the fact that this configuration is projectively equivalent to circles passing through a common point O'. *This is achieved by using a* *projectivity that sends points U and V to the two circular points at infinity,* $\mathbf{I}$ *and* $\mathbf{J}$. *We then apply an inversion centered at O', which transforms all circles passing through O' into straight lines. This effectively maps the problem to the classical Miquel configuration for a triangle.*

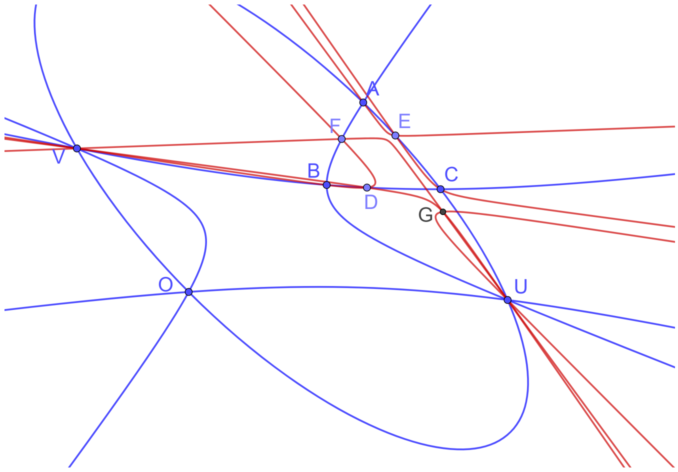

Figure 21. Geometric configuration illustrating the Conic Miquel Theorem in a triangle with sides as conics.

Similarly, Figure 22 depicts the Conic Miquel-Steiner Theorem, where the sides of a complete quadrilateral are conics sharing the three common points O, U, and V.

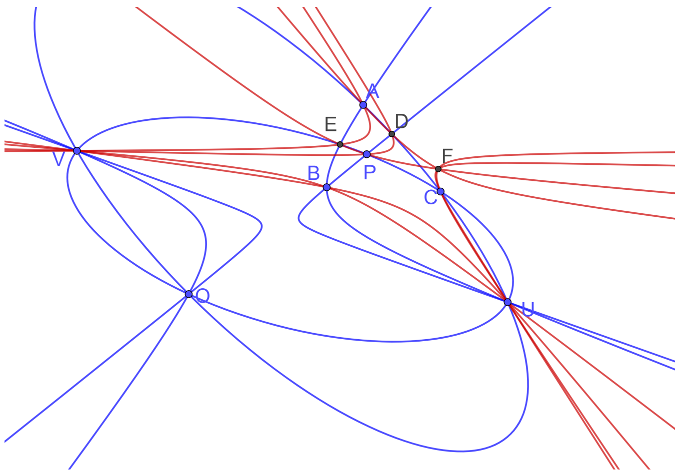

Figure 22. Geometric configuration illustrating the Conic Miquel-Steiner Theorem in a complete quadrilateral with sides as conics.

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