



**A UNIFIED INVERSE PROOF FOR ALL EIGHTEEN  
 MORLEY TRIANGLES**

SPIRIDON A. KURUKLIS

**Abstract.** In a previous work [12], the validity of Morley’s general theorem is established through a “holistic” approach, relying on adjacency to propagate the equilateral property from one to another. While effective, that method depends on the connectivity of Morley triangles agglomeration.

This paper presents a fundamentally different, independent methodology: a *uniform indirect proof* covering all 18 Morley triangles. Instead of deriving the equilateral nature of each Morley triangle, starting from the inner, we employ a *reverse construction*. This constructive approach offers a clean and structured argument, by using only fundamental properties of angle bisectors and similar triangles, for demonstrating that all Morley triangles are equilateral. As an implication of these results a simple proof of Morley’s general theorem is provided.

1. INTRODUCTION

One of the most unexpected discoveries in mathematics, unnoticed until the turn of the 20<sup>th</sup> century, is the celebrated **Morley’s theorem**: *In a triangle, the interior trisectors, proximal to sides respectively, meet at the vertices of an equilateral triangle.* (Fig.1.) Evidently, *the proximal to a side trisector* bisects the angle between the side and the other trisector.

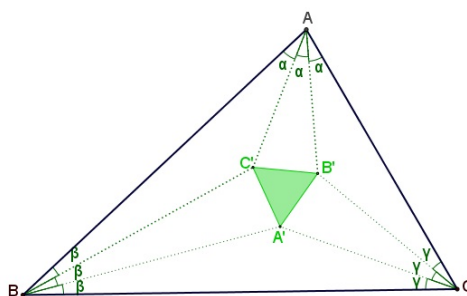


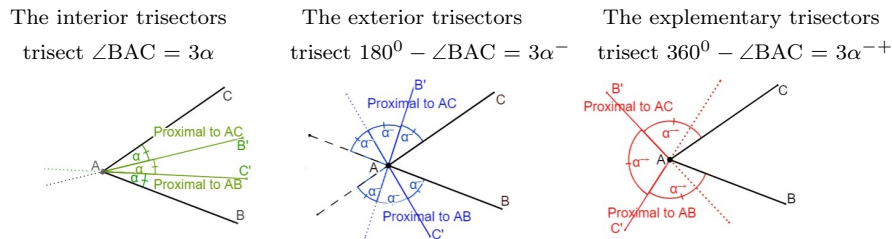
Fig.1. Morley’s theorem:  $\triangle A'B'C'$  is equilateral

The theorem has received remarkable attention with many publications using various techniques and different tools attempting to provide a proof as simple and elegant as its statement [1] [2] [3] [4] [5] [6] [9] [10][16]. Notably, it refers to the *interior* trisectors, while outside each angle of a triangle there are two more angles, referred to as *exterior* and *explementary* (Fig.2).

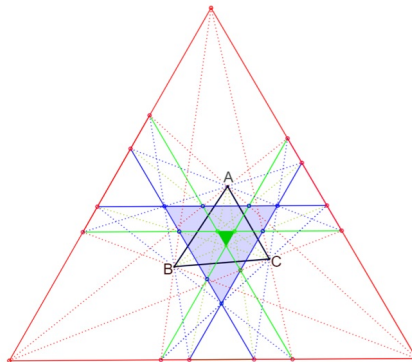
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In fact, Frank Morley, working on an independent geometrical problem, observed the theorem among complicated equations [13]. In his book *Inversive Geometry* [14], written with his son Frank Vigor Morley in 1933, the theorem was stated as an exercise and its proof was proposed by trigonometry. The following generalization of the theorem was discovered also by Morley [15] but first published by Taylor and Marr in 1914 [17]. *In a triangle the intersections of trisectors, proximal to a side, lie on nine lines, per three parallel, forming equilateral triangles.* (Fig.3.) This may be called **Morley's general theorem**. Its name is justified as this readily implies that 18 triangles with vertices intersections of trisectors proximal to sides respectively are equilateral. In contrast to Morley's theorem the general one has received rather limited attention.



**Fig.2.** The three types of trisectors of an angle with the corresponding proximal trisectors. with  $\alpha^- = 60^\circ - \alpha$ ,  $\alpha^+ = 60^\circ + \alpha$ ,  $\alpha^{-+} = 120^\circ - \alpha$



**Fig.3.** Morley's general theorem: In a triangle the intersections of trisectors, proximal to a side, lie on nine lines, per three parallel, forming equilateral triangles

The intersections of trisectors proximal to a side are depicted in Fig.3 and surely are on three triples of parallel lines. Looking carefully we count 27 equilaterals. In fact 18 have vertices intersections of trisectors from all three angles of the  $\triangle ABC$  and they are referred as *Morley triangles*, while 9 have vertices intersections of trisectors from just two angles of  $\triangle ABC$  and have been named *Guy Faux triangles* [9]. The Morley triangles may be distinguished as follows:

- 3 triangles with vertices intersections of same type trisectors for all angles (inner, central, peripheral Morley triangles, Fig.4 left).
- 6 triangles with vertices intersections of distinct type trisectors for each angle (complete Morley triangles, Fig.4 right).
- 3 triples of triangles each with vertices intersections of one type trisectors for one angle and a corresponding type for the other two angles (mix inner, mix central, mix peripheral, Fig.5).

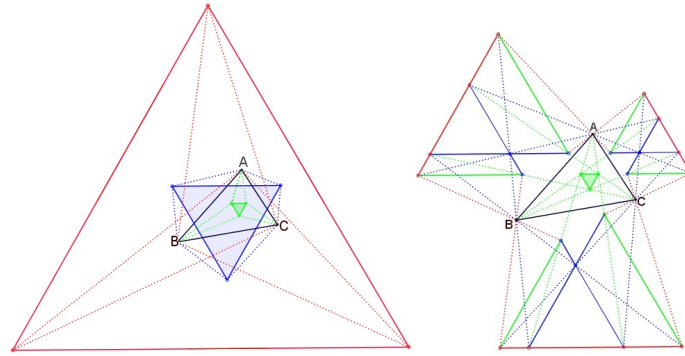


Fig.4. The inner, central, peripheral and 6 complete Morley triangles

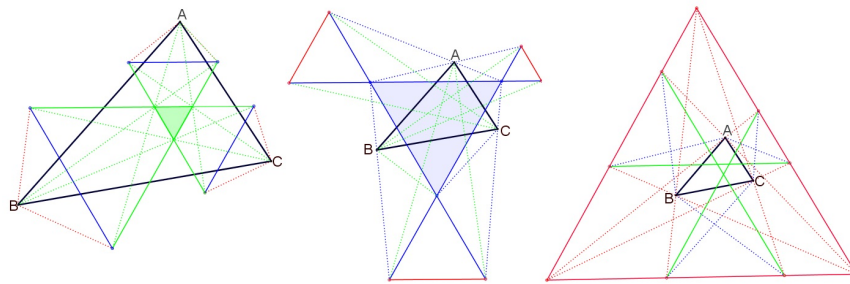


Fig.5. Triples of mix inner, mix central and mix peripheral Morley triangles

In a previous work [12], a holistic approach was presented covering the general theorem. However, that followed a method of adjacency, propagating the property from one triangle to the next. In contrast, the present work extends the **indirect (or backwards) approach**, historically used for the inner triangle [3]. Incidentally [7] and [8] have dealt with all Morley triangles using trigonometry.

For proving that in a given triangle each of the 18 Morley triangles is equilateral, a  $\triangle ABC$  is provided, similar to the given, in which the corresponding Morley triangle is in fact equilateral. Starting from an equilateral  $\triangle A'B'C'$ ,  $\triangle ABC$  is determined by the vertices of constructed isosceles triangles with bases the sides of  $\triangle A'B'C'$ .

For showing that sides of two isosceles are trisectors of  $\triangle ABC$  proximal to one of its sides, it is established that they are bisectors in the triangle formed by their other two sides and the side of  $\triangle ABC$ , using the following lemma from [11] illustrated in Fig.6.

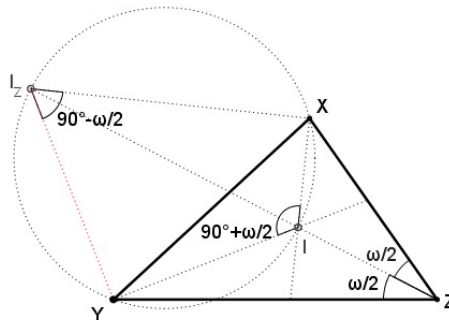


Fig.6. The incenter  $I$  and excenter  $I_z$  of  $\triangle XYZ$  are determined by a bisector and an angle

**Lemma 1.1.** *A point I inside  $\triangle XYZ$  on the bisector of  $\angle YZX$  is its incenter if and only if  $\angle XIY = 90^\circ + \frac{1}{2}\angle YZX = \angle^i YZX$ .*

*Similarly, a point  $I_Z$  outside  $\triangle XYZ$  on the bisector of  $\angle YZX$  is its excenter if and only if  $\angle XI_ZY = 90^\circ - \frac{1}{2}\angle YZX = \angle^e YZX$ .*

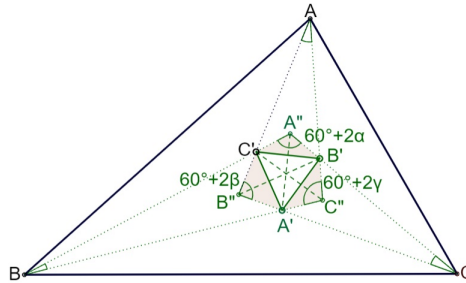
The proof presented here is inverse in nature: starting from an equilateral triangle, the given triangle is reconstructed so that the corresponding Morley triangle arises naturally as a consequence of the construction.

## 2. ALL MORLEY TRIANGLES ARE EQUILATERAL INDEED

Next,  $\triangle ABC$  is produced from an equilateral  $\triangle A'B'C'$  by constructing isosceles  $\triangle B'A''C'$ ,  $\triangle C'B''A'$  and  $\triangle A'C''B'$ , referred in the following as  $A''$ ,  $B''$  and  $C''$  isosceles. Then the vertices of  $\triangle ABC$  are determined as intersections of sides of these isosceles, i.e.  $A''$  is the intersection of  $BC'$  and  $CB'$  and so on symmetrically.

For finding out the sizes of base angles for each isosceles, initially it is assumed that in  $\triangle ABC$ , the intersections of its trisectors, proximal to sides respectively, meet at the vertices of an equilateral. This analysis step is omitted for the sake of space. Details of the analysis there are in [11].

**2.1. INNER MORLEY TRIANGLE.** To yield  $\triangle ABC$ , outside an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $60^\circ + 2\alpha$ ,  $60^\circ + 2\beta$ ,  $60^\circ + 2\gamma$  respectively. (Fig.7.)



**Fig.7.** Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with interior trisectors, proximal to sides respectively, meeting at  $\triangle A'B'C'$  vertices

According to the construction  $\angle CA''B$ ,  $\angle AB''C$  and  $\angle BC''A$  coincide with the corresponding vertex angles of isosceles  $\triangle C'A''B'$ ,  $\triangle A'B''C'$  and  $\triangle B'C''A'$ , in which  $A''A'$ ,  $B''B'$  and  $C''C'$  are bisectors respectively as  $\triangle A'B'C'$  is equilateral. The isosceles equal angles have sizes  $60^\circ - \alpha$ ,  $60^\circ - \beta$ ,  $60^\circ - \gamma$ . So the angles between the lines determining  $\triangle ABC$  and the equilateral sides across vertices  $A'$ ,  $B'$  and  $C'$  are  $\angle A'B'C = \angle A'C'B = 60^\circ + \alpha$ ,  $\angle B'C'A = \angle B'A'C = 60^\circ + \beta$  and  $\angle C'A'B = \angle C'B'A = 60^\circ + \gamma$ . Hence the angles between isosceles sides determining vertices  $A$ ,  $B$  and  $C$ , namely  $\angle B'AC'$ ,  $\angle C'BA'$  and  $\angle A'CB'$ , have sizes  $\alpha$ ,  $\beta$  and  $\gamma$  respectively.

Then, as Fig.4 indicates,  $\angle CA'B = 360^\circ - (60^\circ + \gamma) - 60^\circ - (60^\circ + \beta) = 120^\circ + \alpha = 90^\circ + \frac{1}{2}\angle CA''B$ . Thus  $A'$  is incenter of  $\triangle CA''B$  as the unique point of  $A''A'$ , bisector of  $\angle CA''B$ , from which side  $BC$  is seen under such an angle. Fig.4. Cyclically,  $B'$  and  $C'$  are incenters of  $\triangle AB''C$  and  $\triangle BC''A$  respectively.

As  $B'$  and  $C'$  are incenters of  $\triangle AB''C$  and  $\triangle BC''A$ ,  $AC'$  bisects  $\angle BAB'$  and  $AB'$  bisects  $\angle CAC'$ . So they are  $\angle CAB$  trisectors proximal to  $AB$  and  $AC$  respectively. Since  $\angle C'AB' = \alpha$ ,  $\angle CAB = 3\alpha$ . Cyclically,  $BA'$  and  $BC'$  are  $\angle ABC$  trisectors, proximal to  $AC$  and  $AB$ , while  $CB'$  and  $CA'$  are  $\angle BCA$  trisectors, proximal to  $CA$  and  $CB$  respectively. Hence  $\angle ABC = 3\beta$  and  $\angle BCA = 3\gamma$ . Also  $\triangle A'B'C'$  vertices are intersections of  $\triangle ABC$  trisectors proximal to sides.  $\square$

2.2. CENTRAL MORLEY TRIANGLE. Without loss of generality assume  $\alpha < 30^0$  and  $\beta < 30^0$ .

To yield  $\triangle ABC$ , on an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $60^0 - 2\alpha$ ,  $60^0 - 2\beta$ , and  $|60^0 - 2\gamma|$ , respectively, so that  $\triangle A'B'C'$  is inside both  $A''$  and  $B''$  isosceles while  $\triangle A'B'C'$  is inside or outside  $C''$  for  $\gamma < 30^0$  or  $\gamma > 30^0$ . (Fig.8.)

The base angles of  $A''$ ,  $B''$  and  $C''$  isosceles are  $60^0 + \alpha = \alpha^+$ ,  $60^0 + \beta = \beta^+$ , and  $60^0 + \gamma = \gamma^+$  for  $\gamma < 30^0$  or  $120^0 - \gamma = \gamma^{-+}$  for  $\gamma > 30^0$  respectively. Then  $\angle AB'C' = \angle BA'C' = \gamma^+ - 60^0$  or  $180^0 - \gamma^{-+} - 60^0$  and in both cases  $\angle AB'C' = \angle BA'C' = \gamma$ . Similarly  $\angle AC'B' = \angle CA'B' = \beta$  and  $\angle BC'A' = \angle CB'A' = \alpha$ .

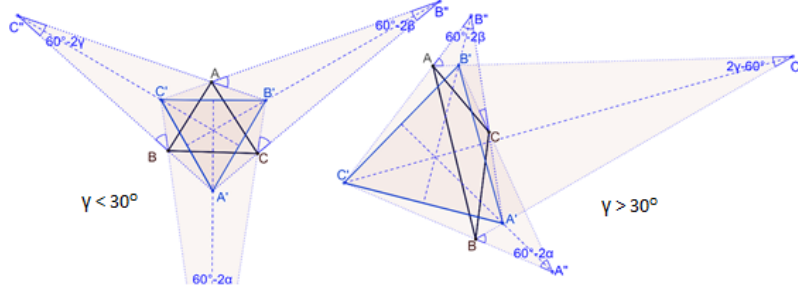


Fig.8. Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with exterior trisectors, proximal to sides respectively, meeting at  $\triangle A'B'C'$  vertices

According to the construction the isosceles vertex angles coincide with the corresponding angles of  $\triangle CA''B$ ,  $\triangle AB''C$  and  $\triangle BC''A$ . So  $\angle CA''B = 60^0 - 2\alpha$ ,  $\angle AB''C = 60^0 - 2\beta$ , and  $\angle BC''A = |60^0 - 2\gamma|$ , while  $A''A'$ ,  $B''B'$  and  $C''C'$  are their corresponding interior bisectors. Also,  $A'$  and  $B'$  are inside  $\triangle CA''B$  and  $\triangle AB''C$  respectively, while  $C'$  is inside  $\triangle BC''A$  for  $\gamma < 30^0$  and outside it for  $\gamma > 30^0$ . Regardless of  $C'$  placement, using the base angles of  $A''$  and  $B''$  isosceles,  $\angle BC'A = \angle BC'A' + \angle A'C'B' + \angle B'C'A = (\alpha^+ - 60^0) + 60^0 + (\beta^+ - 60^0) = 120^0 - \gamma$ . Then  $\angle BC'A = \angle^i BC''A$  or  $\angle BC'A = \angle^e BC''A$  for  $C'$  inside or outside  $\triangle BC''A$ , respectively. Hence  $C'$  is incenter or excenter of  $\triangle BC''A$ , as the unique points inside or outside the triangle from which side  $BA$  is seen with angle  $\angle^i BC''A$  or  $\angle^e BC''A$  respectively. Similarly  $A'$  and  $B'$  are corresponding incenters of  $\triangle CA''B$  and  $\triangle AB''C$ .

As  $C'$  is incenter or excenter of  $\triangle BC''A$ ,  $AC'$  bisects  $\angle BAC''$ . As  $B'$  is incenter of  $\triangle CA''B$ ,  $AB'$  bisects  $\angle CAB''$ . Thus  $AC'$  and  $AB'$  are exterior trisectors of  $\angle CAB$  proximal to  $AB$  and  $AC$  respectively. Since the angle between them is  $\angle BAB'' = \gamma + \beta = \alpha^-$ ,  $\angle CAB = 3\alpha$ . Similarly  $BA'$  and  $BC'$  are exterior trisectors of  $\angle ABC$ ,  $CB'$  and  $CA'$  are exterior trisectors of  $\angle BCA$ ,  $\angle ABC = 3\beta$ ,  $\angle BCA = 3\gamma$ , while the exterior trisectors of  $\triangle ABC$ , proximal to sides respectively, meet at  $\triangle A'B'C'$  vertices.  $\square$

2.3. PERIPHERAL MORLEY TRIANGLE. To yield  $\triangle ABC$ , construct inside an equilateral  $\triangle A'B'C'$  isosceles  $A''$ ,  $B''$  and  $C''$  having vertex angles  $180^\circ - 2\alpha$ ,  $180^\circ - 2\beta$  and  $180^\circ - 2\gamma$ . (Fig.9.)

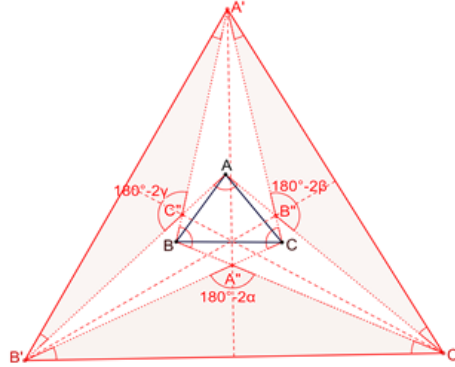


Fig.9. Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with explementary trisectors, proximal to sides respectively, meeting at  $\triangle A'B'C'$  vertices

So  $A''$ ,  $B''$  and  $C''$  isosceles base angles are  $\alpha$ ,  $\beta$  and  $\gamma$  respectively. Hence  $\angle AC'B = 60^\circ - \alpha - \beta = \gamma$ , and cyclically  $\angle BA'C = 60^\circ - \beta - \gamma = \alpha$  and  $\angle CB'A = 60^\circ - \gamma - \alpha = \beta$ . The construction makes  $\angle CA''B$  and  $\angle C'A''B'$  opposite and so  $\angle CA''B = \angle C'A''B' = 180^\circ - 2\alpha$ , while  $A''A'$  is bisector of  $\angle CA''B$ . Also  $A'$  is outside  $\triangle CA''B$  while  $\angle CA'B = \angle C'A'B' = \alpha$ , and so  $\angle CA'B = \angle^e CA''B$ . Thus  $A'$  is excenter of  $\triangle CA''B$  as the unique point outside it from which side  $CB$  is seen with angle  $\angle^e CA''B$ . Cyclically,  $B'$  and  $C'$  are excenters of  $\triangle AB''C$  and  $\triangle BC''A$  respectively.

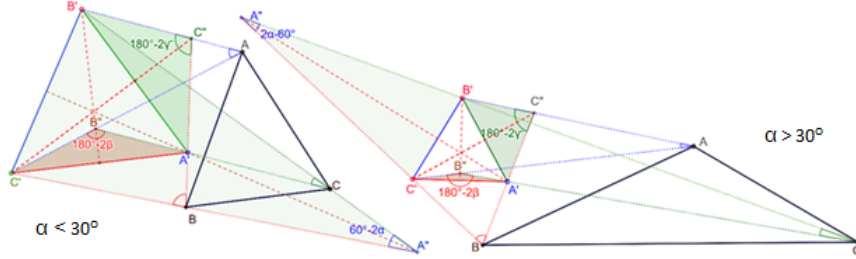
As  $B'$  and  $C'$  are excenters of  $\triangle AB''C$  and  $\triangle BC''A$ ,  $AC'$  and  $AB'$  are exterior bisectors of  $\angle CAB''$  and  $\angle BAC''$ , respectively. Hence  $AC'$  and  $AB'$  are exterior bisectors of  $\angle CAC'$  and  $\angle BAC'$ . Thus the extensions of  $AC'$  and  $AB'$  are explementary trisectors of  $\angle CAB$  proximal to  $AB$  and  $AC$  respectively. From  $\triangle C'AB'$ , the angle between them is  $\angle C'AB' = 180^\circ - (60^\circ - \gamma) - (60^\circ - \beta) = \alpha^-$ , and so  $\angle CAB = 3\alpha$ . Similarly the extension of  $BA'$  and  $BC'$  are explementary trisectors of  $\angle ABA$  proximal to  $BC$  and  $BA$ , as well as the extensions of  $CB'$  and  $CA'$  are explementary trisectors of  $\angle CAB$ , proximal to  $AC'$  and  $AB'$ , respectively. Hence  $\angle ABC = 3\beta$  and  $\angle CAB = 3\gamma$  while the explementary trisectors of  $\triangle ABC$ , proximal to sides respectively, meet at  $\triangle A'B'C'$  vertices.  $\square$

2.4. A COMPLETE MORLEY TRIANGLE. To yield  $\triangle ABC$ , on an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $|60^\circ - 2\alpha|$ ,  $180^\circ - \beta$  and  $180^\circ - 2\gamma^-$ , so that  $\triangle A'B'C'$  is inside  $A''$  isosceles for  $\alpha > 30^\circ$  and outside it for  $\alpha < 30^\circ$ , while  $B''$  and  $C''$  isosceles are inside and outside  $\triangle A'B'C'$  respectively.

The base angles of  $A''$  isosceles are  $60^\circ + \alpha = \alpha^+$  for  $\alpha < 30^\circ$  and  $120^\circ - \alpha = \alpha^-$  for  $\alpha > 30^\circ$ , while of  $B''$  and  $C''$  are  $\beta$  and  $\gamma^-$  respectively. (Fig.10.)

According to the construction, vertex angle of  $A''$  isosceles coincides with the corresponding angle of  $\triangle BA''C$  while vertex angles of  $B''$  and  $C''$  are complementary with the corresponding angles of  $\triangle CB''A$  and  $\triangle AC''B$ . So,  $A''A'$  is interior bisector of the first while  $B''B'$  and  $C''C'$  are exterior bisectors of the other two.

$A'$  is outside or inside  $\triangle CA''B$  for  $\alpha < 30^0$  or  $\alpha > 30^0$ , with  $\angle CA''B = 60^0 - 2\alpha$  or  $\angle CA''B = 2\alpha - 60^0$ , respectively. Then from  $B''$  and  $C''$  isosceles,  $\angle CA'B = \angle C''A'B'' = (60^0 - \beta) + \gamma^- = 60^0 + \alpha$ . So  $\angle CA'B = \angle^e CA''B$  or  $\angle CA'B = \angle^i CA''B$ . Thus  $A'$  is excenter or incenter of  $\triangle CA''B$ .



**Fig.10.** Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with types of trisectors different for each angle, proximal to sides respectively, meeting at  $\triangle A'B'C'$  vertices

$B'$  is outside  $\triangle AB''C$  with  $\angle AB''C = 180^0 - \beta$ . Also  $\angle AB'C = \gamma^- - (\alpha^{-+} - 60^0) = \beta$  or  $\angle AB'C = \gamma^- - (180^0 - \alpha^{-+} - 60^0) = \beta$  for  $\alpha < 30^0$  or  $\alpha > 30^0$ . As  $\angle AB'C = \beta$ ,  $\angle AB'C = \angle^e AB''C$ . So  $B'$  is excenter of  $\triangle AB''C$ .

$C'$  is outside  $\triangle BC''A$  with  $\angle BC''A = 180^0 - 2\gamma^-$ . Also,  $\angle BC'A = \gamma^-$  since it is  $\alpha^{-+} - 60^0 + \beta$  for  $\alpha < 30^0$  or  $(180^0 - \alpha^{-+} - 60^0) + \beta$  for  $\alpha > 30^0$ . So in both cases  $\angle BC'A = \angle^e BC''A$ . Hence  $C'$  is excenter of  $\triangle BC''A$ .

As  $C'$  is excenter of  $\triangle BC''A$ ,  $AC'$  bisects  $\angle BAC''$  and so does  $\angle BAB'$ . As  $B'$  is excenter of  $\triangle AB''C$ ,  $AB'$  bisects  $\angle CAB''$  and so does  $\angle CAC'$ . Thus  $AC'$  and  $AB'$  are  $\angle CAB$  exterior trisectors proximal to  $AB$  and  $AC$  respectively. From  $\triangle C'AB'$ ,  $\angle C'AB' = 180^0 - (60^0 + \gamma^-) - (60^0 - \beta) = \alpha^-$  and so  $\angle CAB = 3\alpha$ .

As  $C'$  is excenter of  $\triangle BC''A$ ,  $BC'$  is exterior bisector of  $\angle ABC''$  and so of  $\angle A'BC$ . As  $A'$  is excenter or incenter of  $\triangle CA''B$ ,  $BA'$  is exterior or interior bisector of  $\angle ABC''$  and so the extension of  $BA'$  is exterior bisector of  $\angle CBC'$ . Hence  $BC'$  and the extension of  $BA'$  are  $\angle ABC$  complementary trisectors proximal to  $BA$  and  $BC$ , respectively.

From  $\triangle A'BC'$ ,  $\angle A'BC' = \beta^+$  as it is  $(60^0 + \gamma^-) - (\alpha^+ - 60^0)$  for  $\alpha < 30^0$  or  $(\gamma^- + 60^0) - (180^0 - \alpha^{-+} - 60^0)$  for  $\alpha > 30^0$ . Thus the angle between  $BC'$  and the extension of  $BA'$  is  $180^0 - \angle A'BC' = 180^0 - \beta^+ = \beta^{-+}$ . So  $\angle ABC = 3\beta$ .

As  $A'$  is incenter or excenter of  $\triangle CA''B$ ,  $CA'$  bisects  $\angle BCA''$  and so does  $\angle BCB'$ . As  $B'$  is excenter of  $\triangle AB''C$ ,  $CB'$  bisects  $\angle ACB''$  and so does  $\angle ACA'$ . Hence  $CA'$  and  $CB'$  are  $\angle BCA$  interior trisectors proximal to  $CB$  and  $CA$  respectively.

From  $\triangle B'CA'$ ,  $\angle B'CA' = \gamma$  as it is  $(60^0 - \beta) - (\alpha^+ - 60^0)$  for  $\alpha < 30^0$  or  $(60^0 - \beta) - (180^0 - \alpha^{-+} - 60^0)$  for  $\alpha > 30^0$ . Thus  $\angle BCA = 3\gamma$ .

Therefore the  $\angle BCA$  interior,  $\angle CAB$  exterior and  $\angle ABC$  complementary trisectors, proximal to its sides respectively, meet at  $\triangle A'B'C'$  vertices.  $\square$

2.5. A MIX INNER MORLEY TRIANGLE. Assume  $\beta < 30^0$ . To yield  $\triangle ABC$ , on an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $60^0 + 2\alpha$ ,  $60^0 - 2\beta$  and  $|60^0 - 2\gamma|$  respectively so that  $A''$  isosceles is inside  $\triangle A'B'C'$ ,  $B''$  isosceles is outside it, while  $C''$  isosceles is outside it for  $\gamma < 30^0$  and inside it for  $\gamma > 30^0$ . (Fig.11.)

Hence  $A''$ ,  $B''$  and  $C''$  isosceles base angles are  $60^0 - \alpha = \alpha^-$ ,  $60^0 + \beta = \beta^+$ , and  $60^0 + \gamma = \gamma^+$  for  $\gamma < 30^0$  or  $120^0 - \gamma = \gamma^{-+}$  for  $\gamma > 30^0$ .

According to the construction  $A''$  vertex angle is opposite to the corresponding angle of  $\triangle CA''B$ , while  $B''$  and  $C''$  vertex angles coincide with the corresponding angles of  $\triangle AB''C$  and  $\triangle BC''A$ . Thus,  $A''A$ ,  $B''B'$  and  $C''C'$  are their corresponding interior bisectors.

$A'$  is inside  $\triangle CA''B$ , with  $\angle CA''B = 60^0 + 2\alpha$ , while  $\angle CA'B = 120^0 + \alpha$  as it is  $360^0 - \beta^+ - 60^0 - \gamma^+$  for  $\gamma < 30^0$  or  $180^0 - [\beta^+ - (\gamma^{-+} - 60^0)]$  for  $\gamma > 30^0$ . Thus  $\angle CA'B = \angle^i CA''B$  and so  $A'$  is incenter of  $\triangle CA''B$ .

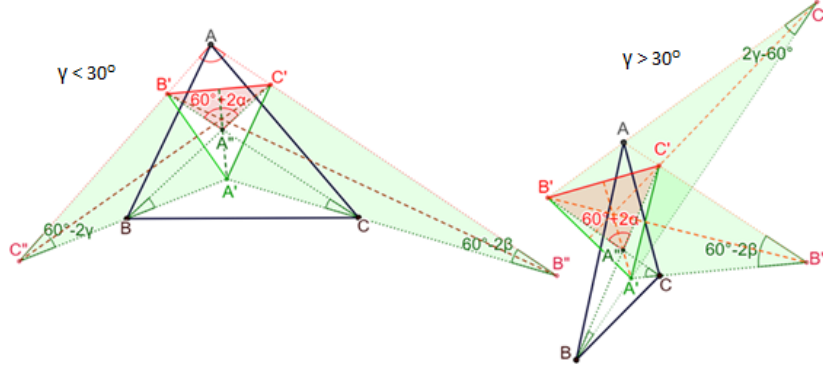


Fig.11. Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with complementary trisectors of an angle and interior for the other two, proximal to sides, meeting at  $\triangle A'B'C'$  vertices

$B'$  is outside  $\triangle AB''C$ , with  $\angle AB''C = 60^0 - 2\beta$ , while  $\angle AB'C = 60^0 + \beta$  as it is  $180^0 - \gamma^+ - (60^0 - \alpha^-)$  for  $\gamma < 30^0$  or  $\gamma^{-+} - (60^0 - \alpha^-)$  for  $\gamma > 30^0$ . Then  $\angle AB'C = \angle^e AB''C$  and so  $B'$  is excenter of  $\triangle AB''C$ .

$C'$  is outside or inside  $\triangle AB''C$  for  $\gamma > 30^0$  or  $\gamma < 30^0$  with  $\angle BC''A = 60^0 - 2\gamma$  or  $\angle BC''A = 2\gamma - 60^0$ , while  $\angle BC'A = 180^0 - \beta^+ - (60^0 - \alpha^-) = 60^0 + \gamma$ . Thus  $\angle BC'A = \angle^e BC''A$  or  $\angle BC'A = \angle^i BC''A$  and so  $C'$  is excenter or incenter of  $\triangle BC''A$  for  $\gamma < 30^0$  or  $\gamma > 30^0$ , respectively.

As  $B'$  is excenter  $\triangle AB''C$ ,  $AB'$  is exterior bisector of  $\angle BAB''$  and so is of  $\angle CAC'$ . As  $C'$  is excenter of  $\triangle CA''B$ ,  $AC'$  is exterior bisector of  $\angle BAC''$  and so is of  $\angle BAB'$ . Hence the extensions of  $AB'$  and  $AC'$  are complementary trisectors of  $\angle CAB$  proximal to  $AC$  and  $AB$  respectively. From  $\triangle A'BC'$ ,  $\angle B'AC' = \alpha^{-+}$  as it is  $180^0 - [180^0 - (\gamma^+ + 60^0)] - [180^0 - (\beta^+ + 60^0)] = \alpha^{-+}$  for  $\gamma < 30^0$  or  $180^0 - [180^0 - (\beta^+ + 60^0)] - (\gamma^{-+} - 60^0) = \alpha^{-+}$  for  $\gamma > 30^0$ . So  $\angle CAB = 3\alpha$ .

As  $A'$  is incenter of  $\triangle CA''B$ ,  $BA'$  bisects  $\angle CBA''$  and so does  $\angle CBC'$ . As  $C'$  is excenter or incenter  $\triangle AC''B$ . In first case,  $BC'$  is exterior bisector of  $\angle AB''C''$  and so it bisects  $\angle CBA'$  since  $BA'$  extends  $C''B$ . In second case,  $BC'$  bisects  $\angle ABC''$  and so does  $\angle ABA'$ . So  $BA'$  and  $BC'$  bisect  $\angle CBC'$  and  $\angle ABA'$ . Thus  $BA'$  and  $BC'$  are interior trisectors of  $\angle ABC$  proximal to  $BC$  and  $BA$  respectively. From  $\triangle A'BC'$ ,  $\angle A'BC' = \beta$  as it is  $180^0 - (\gamma^+ + 60^0) - (60^0 - \alpha^-)$  for  $\gamma < 30^0$  or  $180^0 - [(120^0 - \gamma) - 60^0] - (60^0 - \alpha^-)$  for  $\gamma > 30^0$ . So  $\angle ABC = 3\beta$ .

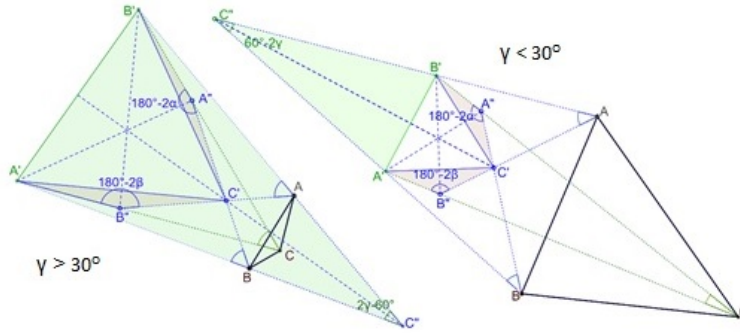
As  $A'$  is incenter of  $\triangle CA''B$ ,  $CA'$  bisects  $\angle CBA''$  and so does  $\angle CBC'$ . As  $B'$  is excenter of  $\triangle AB''C$ ,  $CB'$  is exterior bisector of  $\angle ACB''$  and so  $CB$

bisects  $\angle ACA'$  as  $CA'$  is extension of  $CB''$ . Thus  $CA'$  and  $CB'$  are interior trisectors  $\angle BCA$  proximal to  $CB$  and  $CA$  respectively. From  $\triangle B'AC'$ ,  $\angle B'CA' = 180^\circ - (\beta^+ + 60^\circ) - (60^\circ - \alpha^-) = \gamma$ . So  $\angle BCA = 3\gamma$ .

Therefore the explementary of  $\angle CAB$ , and the interior trisectors of  $\angle ABC$  and  $\angle BCA$ , proximal to its sides respectively, meet at  $\triangle A'B'C'$  vertices.  $\square$

2.6. A MIX CENTRAL MORLEY TRIANGLE. To yield  $\triangle ABC$ , on an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $180^\circ - 2\alpha$ ,  $180^\circ - 2\beta$  and  $|60^\circ - 2\gamma|$  respectively so that  $A''$  and  $B''$  isosceles are outside  $\triangle A'B'C'$  while  $C''$  isosceles is outside it for  $\gamma < 30^\circ$  and inside it for  $\gamma > 30^\circ$ . (Fig.12.)

The base angles of  $A''$ ,  $B''$  and  $C''$  isosceles are  $\alpha$ ,  $\beta$ , and  $60^\circ + \gamma = \gamma^+$  or  $120^\circ - \gamma = \gamma^{-+}$  for  $\gamma < 30^\circ$  or  $\gamma > 30^\circ$ , respectively. The vertex angles of  $A''$  and  $B''$  isosceles are complementary to the corresponding angles of  $\triangle CA''B$  and  $\triangle AB''C$  while of  $C''$  coincides with the corresponding angle of  $\triangle BC''A$ . So,  $A''A$  and  $B''B'$  are exterior bisectors in the first two and  $C''C'$  is interior for the third.



**Fig.12.** Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with interior trisectors of an angle and exterior for the other two, proximal to sides, meeting at  $\triangle A'B'C'$  vertices

$A'$  is outside  $\triangle CA''B$  with  $\angle CA''B = 180^\circ - 2\alpha$ , while  $\angle CA'B = 180^\circ - \beta - 60^\circ - \gamma^+ = \alpha$  for  $\gamma < 30^\circ$  or  $\angle CA'B = \gamma^{-+} - 60^\circ - \beta = \alpha$  for  $\gamma > 30^\circ$ . Hence  $\angle CA'B = \angle^e CA''B$  and so  $A'$  is excenter of  $\triangle BA''C$ .

$B'$  is outside  $\triangle AB''C$  and as before it is an excenter of  $\triangle AB''C$ .

$C'$  is outside or inside  $\triangle BC''A$  for  $\gamma < 30^\circ$  or  $\gamma > 30^\circ$  with  $\angle AC''B = 60^\circ - 2\gamma$  or  $\angle BC''A = 2\gamma - 60^\circ$ , while  $\angle BC'A = \angle A''C'B'' = \alpha + 60^\circ + \beta = 120^\circ - \gamma$  in both cases, and so  $\angle BC'A = \angle^e AC''B$  or  $\angle BC'A = \angle^i AC''B$ , and thus  $C'$  is excenter or incenter of  $\triangle BC''A$  respectively.

As  $C'$  is excenter or incenter of  $\triangle BC''A$ ,  $AC'$  bisects  $\angle BAC''$ . As  $B'$  is excenter of  $\triangle AB''C$ ,  $AB'$  bisects the angle between  $AB''$  and the extension of  $AC$ . Thus,  $AC'$  and  $AB'$  are exterior trisectors of  $\angle CAB$  proximal to  $AC'$  and  $AB$ , while the extension of  $B'A$  is proximal to  $AC$ . From  $\triangle B'AC'$ ,  $\angle C'AB' = \alpha^-$  as it is  $(60^\circ + \beta) - (180^\circ - \gamma^+ - 60^\circ)$  for  $\gamma < 30^\circ$  or  $(60^\circ + \beta) - (\gamma^{-+} - 60^\circ)$  for  $\gamma > 30^\circ$ . So  $\angle CAB = 3\alpha$ .

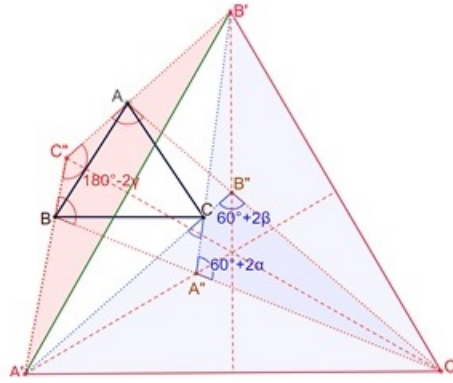
Similarly,  $BC'$  and  $BA'$  are exterior trisectors of  $\angle ABC$  proximal to  $BA$  and  $BC$  respectively and  $\angle ABC = 3\beta$ .

As  $A'$  and  $B'$  are excenters of  $\triangle CA''B$  and  $\triangle CB''A$  respectively, each of  $CA'$  and  $CB'$  bisects the angle between the other and its corresponding side of  $\angle BCA$ . Thus,  $A'C$  and  $B'C$  are interior trisectors of  $\angle BCA$  proximal to  $AB$  and  $AC$  respectively. From  $\triangle B'CA'$ ,  $\angle B'CA' = 180^\circ - (60^\circ + \alpha) - (60^\circ + \beta) = \gamma$  and so  $\angle BCA = 3\gamma$ .

Therefore the interior trisectors of  $\angle BCA$ , and the exterior of  $\angle CAB$  and  $\angle ABC$ , proximal to sides respectively, meet at  $\triangle A'B'C'$  vertices.  $\square$

2.7. A MIX PERIPHERAL MORLEY TRIANGLE. To yield  $\triangle ABC$ , on an equilateral  $\triangle A'B'C'$ , construct  $A''$ ,  $B''$  and  $C''$  isosceles with vertex angles  $60^\circ + 2\alpha$ ,  $60^\circ + 2\beta$  and  $180^\circ - 2\gamma$  respectively, so that  $A''$  and  $B''$  are inside and  $C''$  outside it.

The base angles of  $A''$ ,  $B''$  and  $C''$  are  $60^\circ - \alpha = \alpha^-$ ,  $60^\circ - \beta = \beta^-$  and  $\gamma$ , respectively. The vertex angles of  $A''$  and  $B''$  are complementary to the corresponding angles of  $\triangle CA''B$  and  $\triangle AB''C$  while the vertex angle of  $C''$  coincides with the corresponding angle of  $\triangle BC''A$  and so  $A''A'$  and  $B''B$  are exterior bisectors of  $\angle CA''B$  and  $\angle AB''C$  while  $C''C'$  is interior bisector, of  $\angle BC''A$ . (Fig.13.)



**Fig.13.** Construction of  $\triangle ABC$  from an equilateral  $\triangle A'B'C'$  with exterior trisectors of an angle and complementary for the other two, proximal to sides, meeting at  $\triangle A'B'C'$  vertices

$A'$  is outside  $\triangle CA''B$ , with  $\angle CA''B = 60^\circ + 2\alpha$ ,  $\angle CA'B = 60^\circ - \beta^- = 60^\circ - \alpha$ ,  $\angle CA'B = \angle^e CA''B$  and so  $A'$  is excenter of  $\triangle CA''B$ .

$B'$  is outside  $\triangle AB''C$  and as before  $B'$  is excenter of  $\triangle AB''C$ .

$C'$  is outside  $\triangle BC''A$ , with  $\angle AC''B = 180^\circ - 2\gamma$  and  $\angle BC'A = \alpha^- - (60^\circ - \beta^-) = \gamma$ . Hence  $\angle BC'A = \angle^e BC''A$  and so  $C'$  is excenter of  $\triangle BC''A$ .

As  $B'$  is excenter of  $\triangle CB''A$ ,  $AB'$  is exterior bisector  $\angle CAB''$ . So  $AC''$  is exterior bisector of  $\angle CAC'$ . As  $C'$  is excenter of  $\triangle BC''A$ ,  $AC'$  is exterior bisector of  $\angle BAC''$ . Hence  $AB'$ , which is extension of  $AC''$ , and the extension of  $AC'$  are complementary trisectors of  $\angle CAB$  respectively. But from  $\triangle C'AB'$ ,  $\angle C''AC' = (\gamma + 60^\circ) + (60^\circ - \beta^-) = \alpha^-$  and so  $\angle CAB = 3\alpha$ .

Similarly  $BA'$  and the extension of  $BC'$  are complementary trisectors of  $\angle ABC$ , proximal to sides  $BC$  and  $BA$  respectively, with  $\angle ABC = 3\beta$ .

As  $C'$  is excenter of  $\triangle BC''A$ ,  $CA'$  bisects  $\angle BCA''$  and so  $CA'$  bisects the angle between  $CB$  and the extension of  $CB''$ . Similarly  $CB'$  bisects the angle between  $CA$  and the extension of  $CA''$ . Hence  $CA'$  and  $CB'$  are exterior bisectors of  $\angle BCA$ . But from  $\triangle B'CA'$ ,  $\angle A''CA' = (60^\circ - \alpha^-) + (60^\circ - \beta^-) = \gamma^-$ , and so  $\angle BCA = 3\gamma$ .

Thus  $\triangle A'B'C'$  vertices are intersections of the exterior trisectors of  $\angle BCA$ , and the complementary of  $\angle CAB$  and  $\angle ABC$ , proximal to sides respectively.  $\square$

## 3. IMPLICATIONS

In prime notation a side of  $\triangle ABC$  and a side of a Morley triangle  $\triangle A'B'C'$  are *corresponding* if they are denoted by the same letters, i.e.  $BC$  and  $B'C'$ .

**Corollary 3.1.** *The intersections of proximal and distal trisectors to a side, determine the perpendicular bisector to its corresponding side of a Morley triangle.*

**Proof.** The common vertex is intersection of trisectors proximal to same triangle side. Thus the equilaterals sides opposite to common vertex have same perpendicular bisector. So their sides with the common vertex must be collinear.  $\square$

**Corollary 3.2.** *Two Morley triangles with a common vertex have two pairs of sides collinear and their third parallel.*

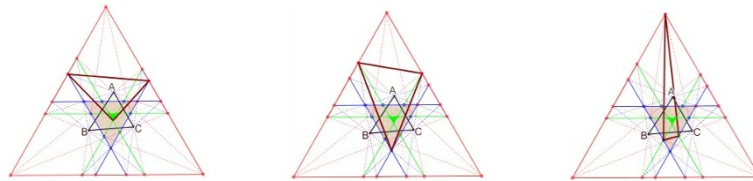
**Corollary 3.3.** (Morley's general theorem.) *The intersections of angle trisectors, proximal to a side, are located on 3 triples of parallel lines forming equilateral triangles.*

**Proof.** Each intersection of trisectors proximal to a side is vertex of two Morley triangles, according to their definitions. Using Corollary 2, it can be checked that a Morley triangle side is collinear to a side of either inner, central or peripheral triangle. In fact along the line of a side there are 6 vertices of Morley triangles.

Therefore the intersections of trisectors, proximal to a side, are located on lines determined by the sides of these triangles. As they are equilaterals these 9 lines intersect with  $60^\circ$  angles and so are per 3 parallel.  $\square$

**Corollary 3.4.** (Extension of Morley's theorem.) *In a triangle, the trisectors of same type for all angles, a distinct type for each, or a type for one and its corresponding type for the other two, proximal to sides respectively, meet at the vertices of an equilateral.*

Interior of an angle and      Exterior of an angle      Explementary of an angle  
explementary for the other two    interior for the other two    exterior for the other two



**Fig.14.** *Non equilateral triangles with vertices intersections of trisectors proximal to sides*

The intersections of trisectors, proximal to triangle sides respectively, determine 27 triangles from which 18 are the previously studied Morley triangles. However, as Fig.14. indicates, the rest 9 cannot be equilateral.

**Remark.** It is worth noting that the inverse construction presented here highlights the limits of Morley's general theorem, which is inseparably connected to angle trisectors. In particular, equal subdivisions of the angles into  $n$  parts lead to an equilateral triangle only in the classical case  $n = 3$ . A  $p$ - $q$  version of Morley's theorem is known and can be established via compositions of rotations or complex-analytic methods; however, such approaches lie beyond the scope of elementary Euclidean constructions and do not admit an inverse isosceles construction of the type developed in this work.

## 4. CONCLUSION

Morley's theorem and all its valid variants are of the same basic mathematical nature. Each of them may be proved with a standardized argument using only fundamental properties of angle bisectors and similar triangles.

The above results could have been discovered and proved in antiquity. The ancient Greeks studied the triangle methodically but ignored its angle trisectors. Across the ages their behavior did not attract attention until Morley observed the formation of equilaterals and in the labyrinth of trisectors in a triangle noticed that the ones proximal to a side meet on nine lines. Although many mathematicians have dealt with the subject this elementary approach seems overseen. The formation of isosceles around Morley's triangles, served as Ariadne's thread for showing they are in fact equilaterals while their arrangement justifies Morley's amazing observation.

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