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AROUND MENELAUS' AND CEVA'S THEOREMS

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Abstract. In this article we consider the successive parallel projections of a point along three fixed directions on three lines in general position or passing through a point, the initial point lying on one of these lines. We discuss some problems related to the closing property of the created paths, proved to lead to an application of Ceva's theorem. This application reveals also a related line and a connection to Menelaus' theorem, different from the standard one defined by the tripole-tripolar relation. The connection between the configuration related to Ceva's theorem and the configuration related to Menelaus' theorem leads to a quadratic transformation and its study. A problem of variation of a vertex of the triangle leads also to a property characteristic for any hyperbola.

1. Introduction

Figure 1 shows a key configuration we deal with in this article. We have three lines in general position, forming the sides of the triangle ABC. We

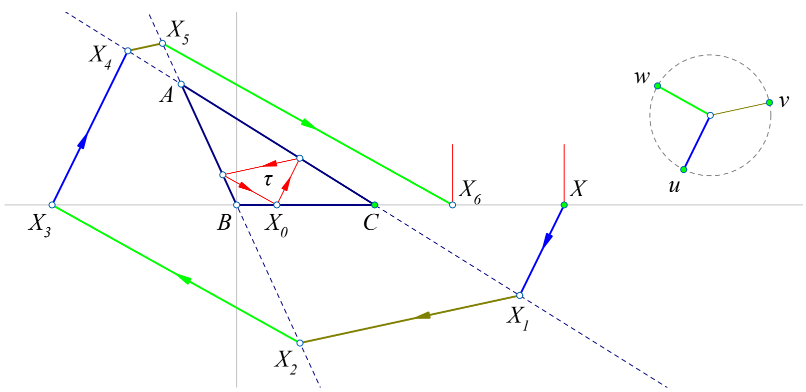

Figure 1. Path of six successive parallel projections

have also three directions determined by the vectors $\{u, v, w\}$ also in general position.

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The figure illustrates the following procedure creating the path shown. Starting with an arbitrary point $X \in BC$ we project parallel to u on CA to the point X_1. We denote such an operation by the symbol $X \xrightarrow{||u} CA \ni X_1$. Applying this successively, we obtain the sequence of points

$$(1) \qquad BC \ni X \xrightarrow{||u} CA \ni X_1 \xrightarrow{||v} AB \ni X_2 \ldots X_5 \xrightarrow{||w} BC \ni X_6\ .$$

A question we'll deal with for a while is: when the created path closes, i.e. when point X_6 coincides with X, for more than one initial points $X \in BC$. If this happens, then we say that the triangle ABC has the *"closing path property"* w.r.t $\{u, v, w\}$ and the side BC.

It is easily seen, that under general conditions, for any triple of directions $\{u, v, w\}$, there is always a point $X_0 \in BC$ such that the resulting path creates a, twice traversed, inscribed in ABC triangle τ (see Figure 1). Thus, there is always a point X_0 satisfying the closing condition, and the question reduces to: whether there is a second one $X \neq X_0$.

This triangle τ, whose sides are parallel to the projecting directions, plays a key role in our subject. We'll see below (theorems 2.1 , 2.2), that the closing path property is equivalent to the triangle τ being *"Cevian"*, i.e. being perspective to the triangle ABC, and the projection directions being determined by the sides of τ. Figure 2 displays a typical case of a Cevian triangle $A'B'C'$. Point P is the *"perspector"* realizing the perspectivity of the inscribed triangle $\tau = A'B'C'$ and ABC. By Ceva's theorem ([1,

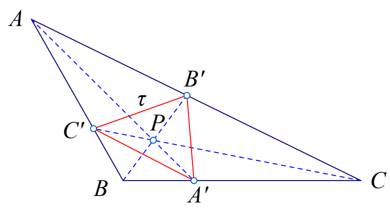

FIGURE 2. Triangle $\tau = A'B'C'$ perspective to ABC

I,p.501]), this happens precisely when Ceva's condition for the signed ratios holds:

$$(2) \qquad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1\ .$$

Before to proceed further to the closing path property, let us see some facts about this triangle τ . Inscribing a triangle with prescribed directions of sides into a given triangle ABC can be done in six different ways (see Figure 3). Thus, it is essential from which side of the triangle we start and which direction we use in each step for the definition of the next point of the path. For a certain ordering we may have the coincidence as desired, but the corresponding path for the same triples but another ordering may fail to close. In other words, one of these six triangles can be perspective to ABC but some other may fail to have this property.

Returning to figure 1, we'll see first, that if there is a second $X \neq X_0$ for which the path closes, then it closes for every $X \in BC$. Below we'll study

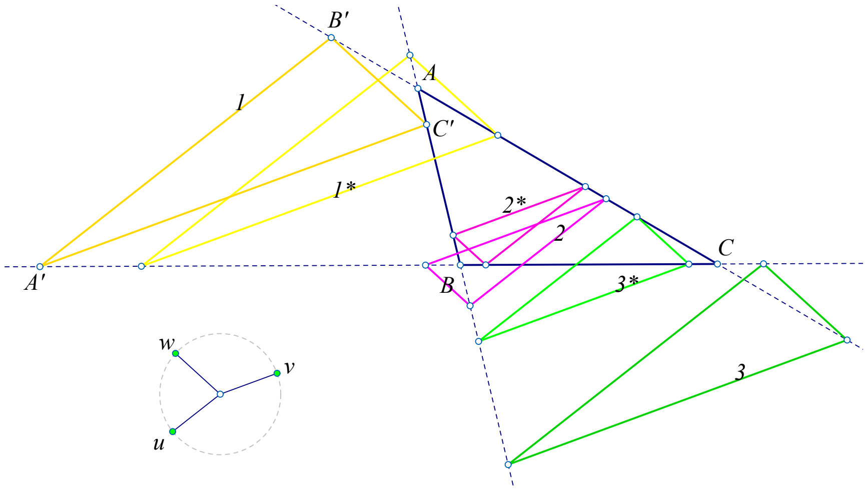

FIGURE 3. Six triangles with sides parallel to $\{u, v, w\}$

a quantitative condition for this phenomenon involving the directions of the triple (u, v, w).

In the sequel we work with the Cartesian coordinates suggested by figure 1. In this, point $B(0,0)$ coincides with the origin, point $C = (c, 0)$ and $A = (a_1, a_2)$.

We start with the representation of X in the form $X = (1-\lambda)C + \lambda B$ and seek the relation of λ to μ for the expression of the next point of the path $X_1 = (1-\mu)C + \mu A$. A short calculation shows that the two parameters are related through a quotient of determinants of two dimensional vectors $\{X(x_1, x_2), Y(y_1, y_2)\}$, which we denote by $CBAu$:

$$\mu = f_1(\lambda) = \lambda \cdot CBAu \;=\; \lambda \frac{det((B-C), u)}{det((A-C), u)} \; . \tag{3}$$

Continuing that way we obtain analogously the relations

$$\begin{aligned}
X_2 &= (1-\nu)A + \nu B && \Rightarrow & \nu = f_2(\mu) &= (1-\mu)ACBv \; , \\
X_3 &= (1-\lambda')B + \lambda' C && \Rightarrow & \lambda' = f_3(\nu) &= (1-\nu)BACw \; , \\
X_4 &= (1-\mu')C + \mu' A && \Rightarrow & \mu' = f_4(\lambda') &= (1-\lambda')CBAu \; , \\
X_5 &= (1-\nu')A + \nu' B && \Rightarrow & \nu' = f_5(\mu') &= (1-\mu')ACBv \; , \\
X_6 &= (1-\lambda'')B + \lambda'' C && \Rightarrow & \lambda'' = f_6(\nu') &= (1-\nu')BACw \; .
\end{aligned}$$

Interchanging the role of the points $\{B, C\}$ in the last representation of X_6 and writing $X_6 = (1-\lambda^*)C + \lambda^* B$, we see, that the relation must be $\lambda^* = f_7(\lambda'') = 1 - \lambda''$. Should $\{X, X_6\}$ coincide, then it would also hold $\lambda^* = \lambda$. Thus, starting from equation (3) and making successively the substitutions $\mu = f_1(\lambda), \nu = f_2(\mu), ..., \lambda^* = f_7(\lambda'')$ we would come to an equation involving the composition of these functions, in the form

$$(f_7 \circ f_6 \circ \cdots \circ f_1)(\lambda) \;=\; \lambda \; .$$

Introducing in this equation the expressions for the corresponding functions and doing some calculation and simplifications, which I omit, we come to the result expressed through the product:

$$[CBAu \cdot ACBv \cdot BACw - 1]\times$$
$$(4) \quad [(CBAu \cdot ACBv \cdot BACw + 1) \cdot \lambda - ((ACBv - 1) \cdot BACw + 1)] = 0 .$$

The first degree in λ equation (4) has the standard solution $\lambda = \lambda_0$ defining point X_0 and the triangle τ of figure 1. If we have also a second solution $\lambda \neq \lambda_0$, then the coefficients of the first degree function must vanish identically and we have the theorem:

Theorem 1.1. *With the notation and definitions of this section, and referring to figure 1, if, besides the standard solution X_0, there is a second one $X \neq X_0$, then the path $XX_1X_2...X_6$ closes $(X_6 = X)$ for every point $X \in BC$.*

2. The closedness condition

In this section we take a closer look at the closedness condition (4). The symbols like $ABCu$ denote oriented ratios on the line BC. In fact, $det(B - A, u)$ is twice the oriented area of the triangle ABu, which in figure 4 is positive oriented and $det(B - A, u) > 0$. In the same figure the triangle ACu is negative oriented and $det(C - A, u) < 0$. Obviously the

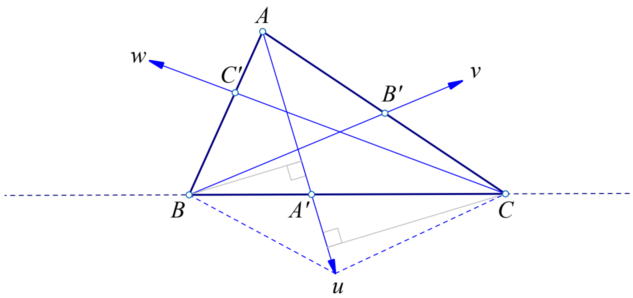

Figure 4. Signed ratio $A'B/A'C = ABCu$

quotient $ABCu$ of these triangle areas, drawing the altitudes from C and B, is seen to be equal to the signed ratio $A'B/A'C$. From this we deduce the following lemma.

Lemma 2.1. *The condition of closedness (4) is equivalent to Menelaus' condition for the collinearity of three points $\{A'', B'', C''\}$ on respective sides of the triangle ABC (see Figure 5). These points result by projecting the vertices $\{A, B, C\}$ parallel to the directions $\{u, v, w\}$ onto the opposite sides.*

Proof. We examine the factors of equation (4), assuming the non-vanishing of the first factor $CBAu \cdot ACBv \cdot BACw - 1 \neq 0$. This implies the vanishing of both coefficients appearing in the second factor:

$$CBAu \cdot ACBv \cdot BACw + 1 \;=\; (ACBv - 1) \cdot BACw + 1 \;=\; 0 .$$

Replacing the symbols with the corresponding expressions, we see that these conditions imply dependencies between the initial data $\{A, B, C, u, v, w\}$, which contradict the assumption of their being in *"general position"*. In fact, the second vanishing condition is equivalent with equation

$$0 = (ACBv - 1) \cdot BACw + 1 \quad \Leftrightarrow \quad \frac{a_2}{w_2} \cdot \frac{det(v,w)}{det(A,v)} = 0 \ ,$$

which implies the dependency of $\{v, w\}$ and is not acceptable. Thus, the only possibility for the closedness condition to hold, is the vanishing of the first factor in equation (4), reminiscent of the Menelaus collinearity criterion.

$$CBAu \cdot ACBv \cdot BACw - 1 \ = 0 \ . \tag{5}$$

In fact, using the preceding ratio expression of these symbols, it is precisely the Menelaus' condition of collinearity for the points shown in figure 5. In this $\{A'', B'', C''\}$ are the intersections of lines parallel to $\{u, v, w\}$ correspondingly from $\{A, B, C\}$ with their opposite sides:

$$CBAu = \frac{C''B}{C''A} \ , \quad ACBv = \frac{A''C}{A''B} \ , \quad BACw = \frac{B''A}{B''C} \ .$$

Thus, equation (5) is equivalent to:

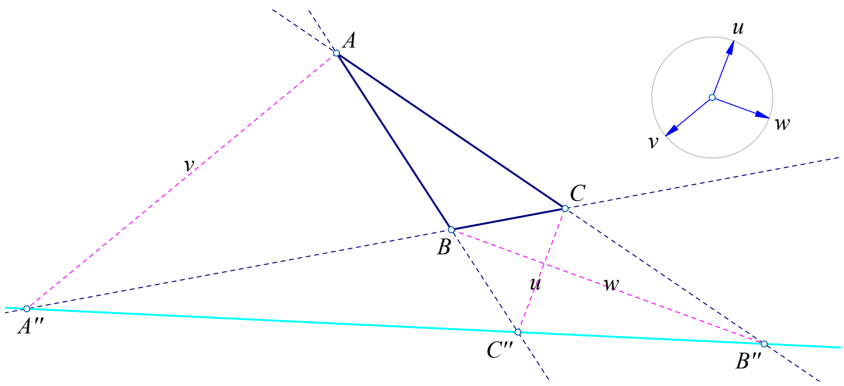

FIGURE 5. The closedness condition: $\{A'', B'', C''\}$ collinear

$$\frac{A''C}{A''B} \cdot \frac{B''A}{B''C} \cdot \frac{C''B}{C''A} = 1 \ , \tag{6}$$

which is Menelaus' criterion for the collinearity of the points $\{A'', B'', C''\}$ on respective sides of the triangle ABC.

Under the hypothesis of closedness, the hexagon $h = XX_1..X_5$ is closed for every starting point $X \in BC$. Figure 6 illustrates such a case. In this we have extended the sides of the hexagon and have created the parallelogram $p = XX_{12}X_3X_{45}$. Using the formulas for $\{X, X_3\}$ of **§** 1 we see that the middle A' of XX_3 (for $B = (0,0), C = (c,0)$):

$$A' \ = \ \left(c \cdot \frac{2a_1v_2w_2 - a_2(v_1w_2 + v_2w_1)}{2w_2(a_1v_2 - a_2v_1)} \ , \ 0\right) \tag{7}$$

is independent of λ and consequently, of the position of $X = (1-\lambda)C + \lambda B$. Thus, the segment $X_{12}X_{45}$ is a diagonal of the parallelogram p and passes through A', the middle of the other diagonal XX_3. The two triangles $\{X_1X_{12}X_2 \ , \ X_4X_{45}X_5\}$ are homothetic, w.r.t. A, hence A is also on the line $X_{12}X_{45}$. Analogous arguments show that the middles $\{B', C'\}$ of X_1X_4

are independent of the position of X and form the triangle $A'B'C'$ inscribed in ABC and with sides parallel to the directions of $\{u, v, w\}$, which is perspective w.r.t. ABC, the perspector P being the intersection of the diagonals of the parallelogram p. We have proved the following theorem.

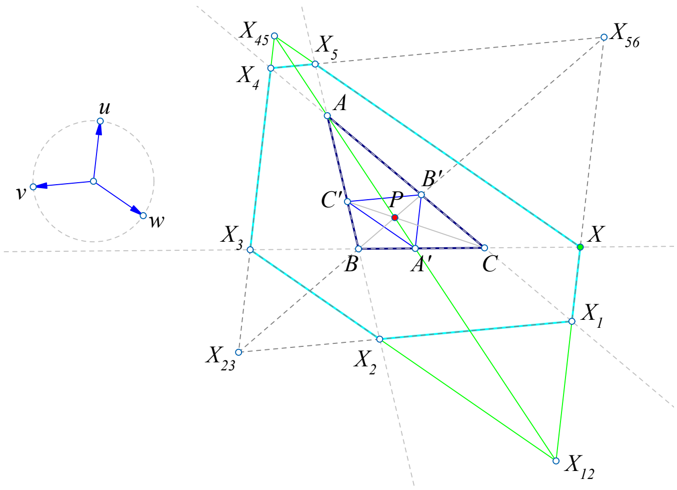

Figure 6. Middles $\{A', B', C'\}$ of $\{XX_3, X_1X_4, X_2X_5\}$

Theorem 2.1. *With the notation and conventions adopted so far, the closedness condition for the polygons $\{p = XX_1...X_5\}$ implies that the directions $\{u, v, w\}$ define a Cevian triangle $A'B'C'$ inscribed in ABC with sides $\{A'B', B'C', C'A'\}$ respectively parallel to $\{u, v, w\}$.*

The converse of the theorem is also true. To see this, assume the triangle $A'B'C'$ inscribed in ABC is Cevian with perspector P. Consider then a point $X \in BC$ and project it successively parallel to the sides of this triangle $\{u = A'B', v = B'C', w = C'A'\}$ onto the sides $\{CA, AB, BC\}$ to the points $\{X_1, X_2, X_3\}$ (see Figure 7).

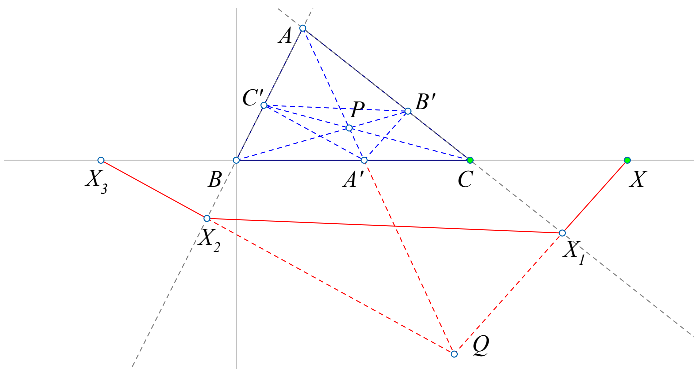

Figure 7. Projecting X parallel to the sides of $A'B'C'$

We show that A' is the middle of XX_3. For this extend the sides $\{XX_1, X_2X_3\}$ to their intersection Q. The triangles $\{X_1QX_2, B'A'C'\}$ have parallel sides and are homothetic w.r.t. A. Thus points $\{A, P, A', Q\}$ are collinear and we have the following relations.

$$\frac{A'X_3}{A'B} = \frac{C'X_2}{C'B} \Rightarrow A'X_3 = A'B\frac{C'X_2}{C'B} \quad \text{and}$$

$$\frac{A'X}{A'C} = \frac{B'X_1}{B'C} \Rightarrow A'X = A'C\frac{B'X_1}{B'C}, \quad \text{dividing} \quad \Rightarrow$$

$$\frac{A'X_3}{A'X} = \frac{A'B}{A'C} \cdot \frac{C'X_2}{C'B} \cdot \frac{B'C}{B'X_1} \text{ and from Ceva's: } \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1 \Rightarrow$$

$$\frac{A'X_3}{A'X} = \left(-\frac{B'A}{B'C} \cdot \frac{C'B}{C'A}\right) \cdot \frac{C'X_2}{C'B}\frac{B'C}{B'X_1} = -\frac{B'A}{B'X_1} \cdot \frac{C'X_2}{C'A} = -1 .$$

This shows that A' is the middle of XX_3. Analogously, projecting X_3 successively parallel to $\{u, v, w\}$ onto points $\{X_4 \in CA, X_5 \in AB, X_6 \in BC\}$ (not shown) we prove that A' is the middle of X_3X_6, hence $X_6 = X$ and we have the theorem.

Theorem 2.2. *If the triangle $A'B'C'$ inscribed in ABC is Cevian, then the triangle ABC has the closing path property w.r.t. to the directions $\{u = A'B', v = B'C', w = C'A'\}$ and points $X \in BC$.*

Remark 2.1. *Theorems 2.1 and 2.2 show that given the triangle ABC, the set of triples of directions $\{u, v, w\}$ producing the closing path property for points X on a selected side BC, can be identified with the complement of the union of sidelines of the triangle. This, because every such triple defines a Cevian triangle of ABC, which can be identified with its perspector P.*

Remark 2.2. *From our discussion so far follows that the closedness condition for the directions $\{u, v, w\}$ is equivalent to Ceva's criterion for the inscribed triangle $A'B'C'$ with sides $\{A'B', B'C', C'A'\}$ correspondingly parallel to $\{u, v, w\}$. It is also equivalent to Menelaus' criterion for the points $\{A'', B'', C''\}$ defined in lemma 2.1.*

Thus, shifting from configurations of points on the sides of the triangle of reference ABC to directions of parallel projections, we have a kind of duality similar to the ordinary duality of Menelaus' and Ceva's criteria expressed through the tripole-tripolar relation: if the lines $\{AA', BB', CC'\}$ pass through the point P then, the harmonic conjugate of these points $\{A'(BC), B'(CA), C'(AB)\}$ are on the tripolar trP of P and vice versa.

In our configuration, if the directions $\{u, v, w\}$ define a Cevian triangle with sides $\{A'B', B'C', C'A'\}$ correspondingly parallel to these vectors, then the points $\{A'', B'', C''\}$ of lemma 2.1 are on a line λ and vice versa.

In the next section we'll study a relation between these two kinds of duality.

3. A quadratic transformation

According to remark 2.1, every point P not on the sidelines of the triangle ABC defines a triangle $A'B'C'$ inscribed in ABC with perspector P, and its side-directions $\{u, v, w\}$ can be used to define a closing path and a line λ, as in lemma 2.1 (see Figure 8).

We notice that the trilinear pole (tripole) $tr(\lambda)$ of λ is in general different from the perspector P. This creates a transformation $F : P \mapsto tr(\lambda)$ of points on the complement of the sidelines of ABC, and rises some questions

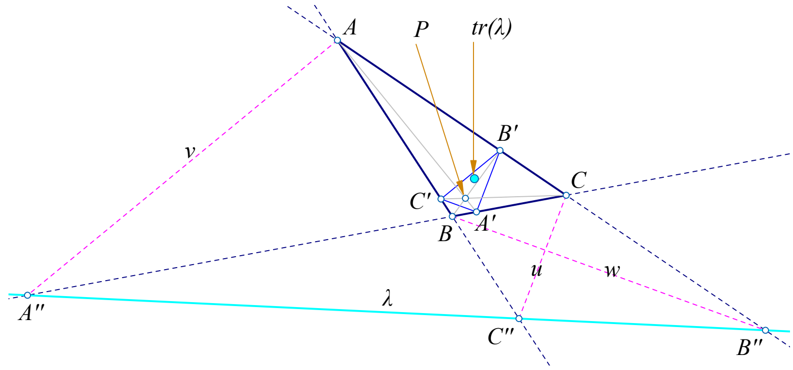

FIGURE 8. Perspector of $A'B'C'$: $P \neq tr(\lambda)$

about its properties. Here we take a closer look at this transformation F, proving that it is a *"quadratic"* ([2, p.46], [3, p.23]) one fixing the centroid G and the middles $\{A_0, B_0, C_0\}$ of the sides of the triangle ABC.

In fact, it is not difficult to express this transformation in terms of the barycentric coordinates (barycentrics) [4] of the triangle of reference ABC. Starting from the point $P(x:y:z)$, its traces on the sides and the directions of the sides of $A'B'C'$, represented by points of the line at infinity $\ell_\infty \ : \ x+y+z=0$, are correspondingly:

$$\text{traces}: \quad A'(0:y:z)\ ,\ B'(x:0:z)\ ,\ C'(x:y:0)\ ,$$
$$u = (x(y+z) : -y(z+x) : z(y-x))\ ,$$
$$v = (x(z-y)) : y(x+z) : -z(y+x))\ ,$$
$$w = (-x(y+z) : y(x-z) : z(x+y))\ .$$

From these we determine the points $\{A'', B'', C''\}$, the line λ of figure 8 and its tripole $tr(\lambda)$:

$$A'' = (0 : -y(z+x) : z(x+y))\ , \quad B'' = (x(y+z) : 0 : -z(y+x))\ ,$$
$$C'' = (-x(y+z) : y(z+x) : 0) \quad \Rightarrow$$
$$\text{line (coefficients) } \lambda \ : \ \left(\frac{1}{x(y+z)} \ : \ \frac{1}{y(z+x)} \ : \ \frac{1}{z(x+y)}\right) \quad \Rightarrow$$
$$(8) \qquad F(x:y:z) \ = \ tr(\lambda) \ = \ (x(y+z) : y(z+x) : z(x+y))\ .$$

Theorem 3.1. *The quadratic transformation F has the following properties:*

(1) *It fixes the centroid $G(1:1:1)$ of the triangle ABC and the middles $A_0(0:1:1), B_0(1:0:1)$ and $C_0(1:1:0)$ of the sides of ABC.*
(2) *It maps every point of the line $\{BC, CA, AB\}$ correspondingly to the middle $\{A_0, B_0, C_0\}$.*
(3) *It maps every line of the plane to a conic through the vertices of the medial triangle $A_0B_0C_0$.*
(4) *In particular, it maps the line at infinity ℓ_∞ to the inner Steiner ellipse of ABC.*
(5) *It maps the Steiner (outer) ellipse of the triangle ABC to the line at infinity.*

(6) *It maps the tripolar* $\mu: \ a^2x+b^2y+c^2z=0$ *of the 3rd Brocard point of the triangle* ABC $X(76)=(a^{-2}:b^{-2}:c^{-2})$ *to the Euler circle of the triangle* ABC *.*

(7) *Lines running outside the Steiner ellipse* κ *of* ABC *map via* F *to ellipses. Lines tangent to* κ *map to parabolas, and lines intersecting* κ *map to hyperbolas.*

Due to the simplicity of the transformation F, all these properties result easily by standard computations with barycentrics, therefore I leave their proofs as exercises.

4. The circumscribing conic

By the results of the preceding section, the closing path property of the triangle ABC w.r.t. the directions $\{u,v,w\}$ and the side BC is equivalent with these directions being parallel to the sides of an inscribed in ABC triangle $A'B'C'$ which is perspective to ABC (see Figure 9). The created in this case hexagon p_t through a starting point $X_t \in BC$ has parallel opposite sides, which intersect at the line at infinity. By the inverse of Pascal's *"hexagrammum mysticum theorem"* [5, p.65], there is a conic κ_t circumscribing this hexagon. For these conics $\{\kappa_t\}$ holds the following theorem.

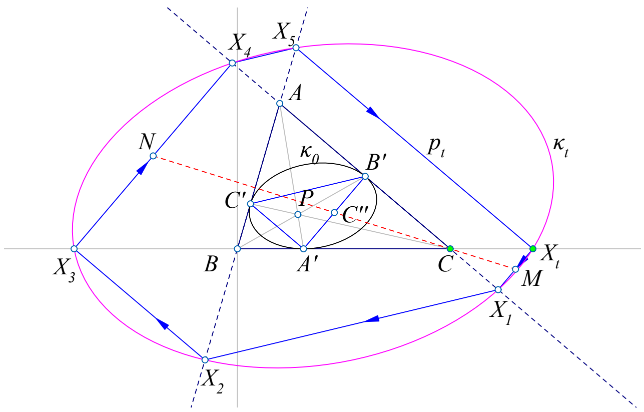

Figure 9. The conic κ_t circumscribing the hexagon p_t

Theorem 4.1. *The conics* $\{\kappa_t\}$ *are homothetic to the inscribed conic* κ_0 *of* ABC *with perspector* P*, the homothety center being the center of* κ_0*.*

Proof. Consider the line MN through the middles of two opposite sides, $\{X_tX_1, X_3X_4\}$ say, of the hexagon p_t (see Figure 9). By the parallelism of corresponding sides of the triangles $\{X_tCX_1, A'CB', X_3CX_4\}$, line MN passes through C and the middle C'' of $A'B'$. Thus, it is always the same, and independent of the particular position of $X_t \in BC$. The same happens with the lines joining the middles of the other couples of opposite sides of p_t. Since these lines are diameters of the conic, they pass through their centers, which therefore coincide with a certain point, K say. In case p_t coincides with the inscribed triangle $A'B'C'$, the corresponding conic κ_t becomes the inconic κ_0 of ABC with perspector P, i.e. the conic tangent to the sides of ABC at the vertices of $A'B'C'$. Thus, the center of all the conics $\{\kappa_t\}$ coincides with the center K of κ_0. Besides, the directions of MN and $A'B'$ are conjugate w.r.t. all the conics κ_t, and the same happens with the other two lines joining the middles of the other couples of parallel sides. Thus, all the conics $\{\kappa_t\}$ have three common pairs of conjugate diameter directions, hence they are homothetic.

5. A related hyperbola

We return now to equation (4), fixing the side BC and the projecting directions $\{u, v, w\}$ and considering variable the point A. In § 2 we saw that the possibility to have more than one solutions for the closing path problem reduces to a Menelaus collinearity criterion. This, making the appropriate substitutions for the symbols, takes the following form, in which we have replaced the coordinates (a_1, a_2) of the variable A with (x, y).

$$CBAu \cdot ACBv \cdot BACw - 1 = 0 \quad \Leftrightarrow$$

$$2(u_2v_2w_2)x^2 + v_1(u_1w_2 + u_2w_1)y^2 - ((u_1v_2 + u_2v_1)w_2 + (v_1w_2 + v_2w_1)u_2)xy$$
$$-2c(u_2v_2w_2)x + cu_2(v_1w_2 + v_2w_1)y = 0 \quad \Leftrightarrow$$
$$(v_1y - v_2x)((u_1w_2 + u_2w_1)y - 2u_2w_2x) -$$
$$2c(u_2v_2w_2)x + cu_2(v_1w_2 + v_2w_1)y = 0. \tag{9}$$

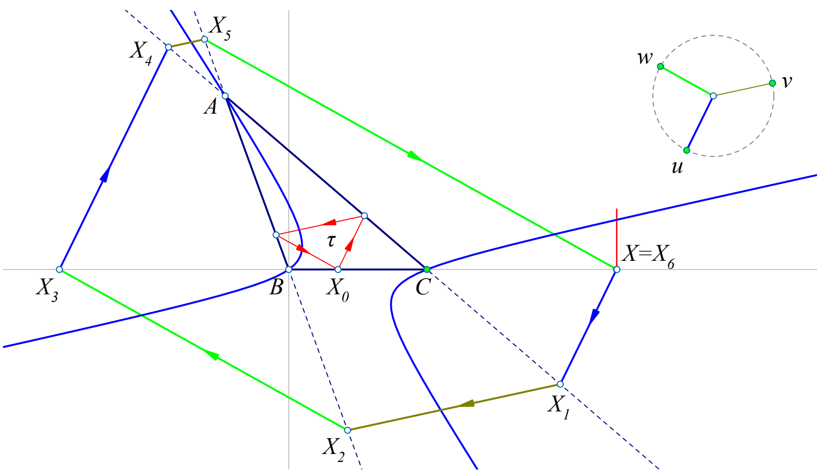

FIGURE 10. $A(x, y)$ for which the polygon $XX_1...X_6$ closes

Theorem 5.1. *With the definitions and the notation adopted so far, for fixed points $\{B, C\}$ and directions $\{u, v, w\}$, holds the following property. The point $A(x, y)$ defines a triangle ABC which has the closing path property w.r.t. $\{u, v, w\}$ and BC, if and only if A lies on the hyperbola of equation (9), passing through the points $\{B, C\}$ (see Figure 10).*

Proof. Obviously, the equation is satisfied for $\{B(0, 0), C(c, 0)\}$. A standard computation of the second invariant ([6]) of the equation in Cartesians, representing a conic:

$$\begin{aligned} J_2 &= (v_1(u_1w_2+u_2w_1))(2u_2v_2w_2)- \\ &\quad \frac{1}{4}((u_1v_2+u_2v_1)w_2+(v_1w_2+v_2w_1)u_2))^2 = \\ &\quad -\frac{1}{4}((u_1v_2-u_2v_1)w_2-(v_1w_2-v_2w_1)u_2)^2 , \end{aligned}$$

shows that J_2 is negative, hence the conic is a hyperbola.

Corollary 5.1. *With the notation of this section, the first couple of lines described by the equations (see Figure 11):*

$$v_2x - v_1y = 0 , \tag{10}$$

$$2u_2w_2x - (u_1w_2+u_2w_1)y = 0 , \tag{11}$$

$$(2v_2w_2)x - (v_1w_2+v_2w_1)y = 0 , \tag{12}$$

defines two lines through the origin $B(0,0)$ parallel to the asymptotes, and the third equation describes the tangent to the hyperbola at B.

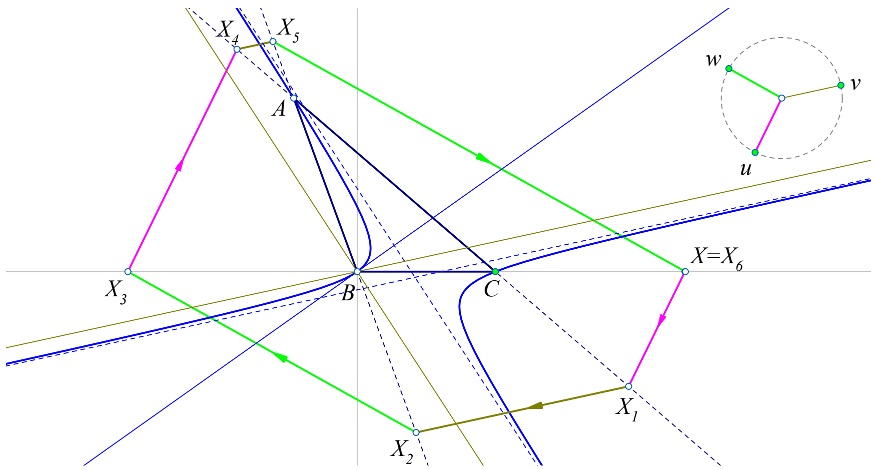

FIGURE 11. Parallels to the asymptotes and tangent at B

Proof. This follows from simple, easy to prove, general theorems for conics (and more general for algebraic curves), which pass through the origin of coordinates ([7, p.23, p.39]). According to these theorems, the quadratic part of the equation splits, as seen also in equation (9), in a product of lines representing parallels to the asymptotes, and the linear part represents the tangent of the hyperbola at the origin B.

Remark 5.1. *We notice that if we consider multiples $\{\kappa \cdot u \ , \ \lambda \cdot v \ , \ \mu \cdot w\}$, equation (9) of the hyperbola is multiplied with the factor $\kappa\lambda\mu$, i.e. the hyperbola depends only on the directions and not the specific vectors representing them.*

These projecting directions are, up to a constant multiple, completely determined from the three equations of corollary 5.1, i.e. the two asymptotes and the tangent at the origin B of the hyperbola. In fact $v=(v_1,v_2)$ is directly determined from the coefficients of the asymptote (10). From this *and equation* (12) *of the tangent at* B *are determined the coefficients of* $w = (w_1, w_2)$. *Finally, from equation* (11) *of the other asymptote are determined the coefficients of* $u = (u_1, u_2)$.

From this remark, comparing the coefficients of two corresponding equations (9) for two triples of vectors $\{(u, v, w), (u', v', w')\}$, we get the following corollary.

Corollary 5.2. *Suppose that two triples* $\{(u, v, w), (u', v', w')\}$ *of projecting direction vectors, operate on the sides of the triangle* ABC *under the same ordering and define the same hyperbola, i.e. the coefficients of the corresponding equations* (9) *are proportional. Then, the vectors point correspondingly to the same directions, i.e. it holds* $\{u' = \lambda u, v' = \mu v, w' = \nu w\}$, *for non-zero constants* $\{\lambda, \mu, \nu\}$.

Next theorem is a kind of inverse to the preceding remark, showing that any hyperbola can be determined through two of its points and three particular lines passing through one of these points: two lines parallel to the asymptotes, and the tangent at this point. This is equivalent with the determination of a hyperbola by prescribing five of its elements: two of its points, and its points at infinity, i.e. the directions of its asymptotes, and the tangent at a point. A geometric proof of existence and uniqueness of the hyperbola, using general principles of the theory of conics, can be found in §3.4 of the *"Gallery of conics by five elements"* [8]. Here we give an elementary heuristic proof producing the particular form of the equation of the hyperbola, needed for our discussion.

Theorem 5.2. *Every hyperbola can be completely determined by giving a point* B *on it, three lines* $\{\alpha, \beta, \gamma\}$ *through* B, *representing correspondingly two parallels to its asymptotes and the tangent at* B , *and giving also a second arbitrary point* C *of it.*

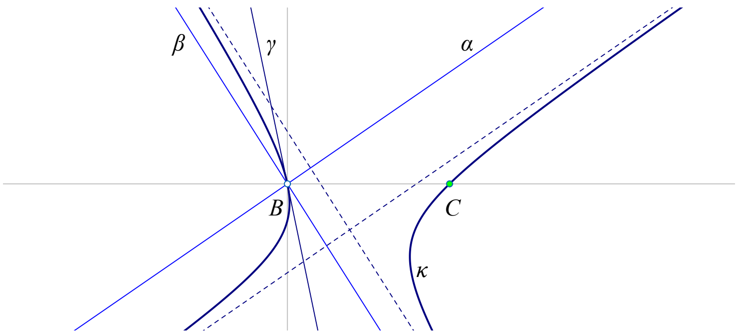

FIGURE 12. Hyperbola from the data $\{B, C, \alpha, \beta, \gamma\}$

Proof. Without loss of generality we may assume that the origin of the Cartesians is at $B(0, 0)$ and $C(c, 0)$ is on the x-axis (see Figure 12). In these coordinates the equation of the conic, if existing, will have the form:

$$l'x^2 + m'y^2 + n'xy + p'x + q'y \ = \ 0 \ . \tag{13}$$

As we noticed above, the tangent γ at B is described by a multiple of the linear part of this equation $p'x + q'y = (rp)x + (rq)y = 0$ for given $\{p, q\}$.

As we noticed also, the asymptotic directions are determined by splitting the quadratic part in two factors. For this we set $t = x/y$ and find the roots $\{t_1, t_2\}$ of $l't^2 + n't + m' = 0$, which exist and are real, since we assume that the conic is a hyperbola. The asymptotic directions are determined through the factors of the equation:

$$(14) \qquad l'\left(\frac{x}{y} - t_1\right)\left(\frac{x}{y} - t_2\right) = 0 \quad \Leftrightarrow \quad l'(x - t_1 y)(x - t_2 y) = 0 \ .$$

The given lines $\{\alpha : \ a_1x + a_2y = 0 \ , \ \ \beta : \ b_1x + b_2y = 0\}$ produce the quotients coinciding precisely with these roots: $\{t_1 = -a_2/a_1, t_2 = -b_2/b_1\}$, and the coefficients satisfy the equations

$$l't_1^2 + n't_1 + m' = 0 \quad \text{and} \quad l't_2^2 + n't_2 + m' = 0 \ .$$

Thus, (l', n', m') is a multiple, by a factor $s \neq 0$, of the exterior product $(l, n, m) = (t_1^2, t_1, 1) \times (t_2^2, t_2, 1)$. We have so far the description of the conic in the form:

$$(sl)x^2 + (sm)y^2 + (sn)xy + (rp)x + (rq)y = 0 \ .$$

The link between the two factors $\{s, r\}$ results from the hypothesis that $C(c, 0)$ satisfies this equation: $slc^2 + rpc = 0$, implying $s = -rp/(lc)$. Making the substitution and simplification into equation (13), we find the equation of the hyperbola,

$$(15) \qquad (pl)x^2 + (pm)y^2 + (pn)xy - c(pl)x - c(ql)y \ = \ 0 \ ,$$

easily seen to satisfy all the requirements.

Remark 5.2. *The coefficients* $(l, n, m) = (t_1^2, t_1, 1) \times (t_2^2, t_2, 1)$ *can be expressed through the given lines* $\{\alpha, \beta\}$ *and the fact we noticed above, that* $t_1 = -a_2/a_1, t_2 = -b_2/b_1$ *:*

$$\begin{gathered}(t_1^2, t_1, 1) \times (t_2^2, t_2, 1) = (t_1 - t_2, t_2^2 - t_1^2, t_1t_2(t_1 - t_2)) = \\ (t_1 - t_2)(1, -(t_1 + t_2), t_1t_2) = \\ (t_1 - t_2)\left(1 \ , \ \frac{a_2}{a_1} + \frac{b_2}{b_1} \ , \ \frac{a_2b_2}{a_1b_1}\right) = \lambda(a_1b_1 \ , \ a_1b_2 + a_2b_1 \ , \ a_2b_2) \ .\end{gathered}$$

Thus, disregarding the factor λ*, we can take*

$$(l, n, m) = (a_1b_1 \ , \ a_1b_2 + a_2b_1 \ , \ a_2b_2) \ ,$$

and the final expression of the hyperbola in terms of the given data will have the form:

$$B(0, 0) \ , \ C(c, 0) \ , \ \alpha : \ a_1x + a_2y = 0 \ , \ \ \beta : b_1x + b_2y = 0 \ , \ \ \gamma : px + qy = 0 \ ,$$

$$\begin{aligned}&(pa_1b_1)x^2 + (pa_2b_2)y^2 + (p(a_1b_2 + a_2b_1))xy - c(pa_1b_1)x - c(qa_1b_1)y \ = \ 0 \\ (16) \qquad &\Leftrightarrow \quad p(a_1x + a_2y)(b_1x + b_2y) - c(p(a_1b_1)x + q(a_1b_1)y) \ = \ 0 \ .\end{aligned}$$

6. The closing path property

With the notation and definitions of the preceding sections, we pass here into the investigation, whether the configuration of figure 10 can be realized for an arbitrary genuine hyperbola κ and, possibly, for two arbitrary points $\{B, C\}$ of it. The question is, whether from these data we can find directions, determined up to multiplicative factors by vectors $\{u, v, w\}$, producing that figure.

From the analysis made in the preceding section, in particular from equation (16), we know that the directions of the asymptotes and the tangent t at B, together with the point C, suffice to describe the hyperbola. Also, from the analysis made at the beginning of that section, we know that if such a closed path is possible, then the conic carrying the points $\{A, B, C\}$ satisfies equation (9), whose coefficients depend on the directions of the three aforementioned lines and the points $\{B, C\}$.

The idea is, to start from the last equation (16) and see how its coefficients link with the coefficients of (9). In principle, since the two equations express the same conic, they must be proportional, and the directions $\{u, v, w\}$, as we noticed in remark 5.1, must be expressible through $\{B, C\}$ and the three lines $\{\alpha, \beta, \gamma\}$ through B.

In order to compare the coefficients of the two equations we first pass from the equation of line $\alpha :\ a_1x + a_2y = 0$ to its direction $(a'_1, a'_2) = (-a_2, a_1)$, and substitute in equation (16) $\{a_1 = a'_2, a_2 = -a'_1\}$. Doing this also for the other two lines and dropping the primes, we come to the equivalent to (16) equation:

$$(qa_2b_2)x^2 + (qa_1b_1)y^2 - q(a_1b_2 + a_2b_1)xy \\ -c(qa_2b_2)x + c(pa_2b_2)y \;=\; 0\ . \tag{17}$$

Now both equations (9) and (17) involve in their coefficients, vectors defined up to multiplicative constants, and representing directions. Expressing the proportionality of coefficients we get the relations, λ denoting a proportionality factor:

$$\left.\begin{array}{rcl} \lambda\cdot qa_2b_2 &=& 2u_2v_2w_2\ , \\ \lambda\cdot qa_1b_1 &=& v_1(u_1w_2+u_2w_1)\ , \\ \lambda\cdot q(a_1b_2+a_2b_1) &=& (u_1v_2+u_2v_1)w_2+(v_1w_2+v_2w_1)u_2\ , \\ \lambda\cdot cqa_2b_2 &=& 2cu_2v_2w_2\ , \\ \lambda\cdot cpa_2b_2 &=& cu_2(v_1w_2+v_2w_1)\ . \end{array}\right\} \quad \Leftrightarrow$$

$$\left.\begin{array}{rcl} qb_2 &=& 2u_2w_2\ , \\ qb_1 &=& (u_1w_2+u_2w_1)\ , \\ q(a_1b_2+a_2b_1) &=& 2a_1u_2w_2+a_2(u_1w_2+u_2w_1)\ , \\ qb_2 &=& 2u_2w_2\ , \\ pa_2b_2 &=& u_2(a_1w_2+a_2w_1)\ . \end{array}\right\} \tag{18}$$

The second system is a simplification of the first, in which we have set $\lambda = 1$ and we took into account the equation $(a_1, a_2) = (v_1, v_2)$. The fourth equation is identical to the first, and the first triple of these equations is essentially the comparison of coefficients of the equations of the three lines of corollary 5.1, expressed correspondingly in terms of the vectors $\{a, b, t\}$ and $\{u, v, w\}$, where $t = (p, q)$ denotes the direction of the tangent at the origin B.

Solving the first couple of linear equations w.r.t to $\{w_1, w_2\}$ we find, up to factor:

$$w_1 \;=\; 2b_1u_2 - b_2u_1\ , \quad w_2 \;=\; b_2u_2\ . \tag{19}$$

Replacing the $\{w_1, w_2\}$ in the third equation, we come to the equation

$$(-a_1b_2qc)u_1 + ((a_2b_2 + 2a_1b_1)qc - 2a_2b_2p)u_2 \;=\; 0\;.$$

This determines u and subsequently w from (19), up to a multiplicative factors :

$$(20)\qquad u \;=\; \begin{pmatrix}(a_2b_2 + 2a_1b_1)qc - 2a_2b_2p\\ a_1b_2qc\end{pmatrix}\;,\quad w \;=\; \begin{pmatrix}a_2(2p - qc)\\ a_1qc\end{pmatrix}\;.$$

Replacing these in the fifth from equations (18), we find the compatibility condition:

$$(21)\;\; p(2a_2(a_1c - a_2)) + qc(a_2^2 - a_1^2) = 0 \qquad \Leftrightarrow \qquad c = \frac{2a_2^2p}{2a_1a_2p + (a_2^2 - a_1^2)q}\;.$$

Finally, replacing this value in equations (20) we find their expression, up to factor:

$$(22)\qquad u \;=\; \begin{pmatrix}a_1b_2q + 2a_2(b_1q - b_2p)\\ a_2b_2q\end{pmatrix}\;,\quad w \;=\; \begin{pmatrix}2a_2p - a_1q\\ a_2q\end{pmatrix}\;.$$

From all this we conclude the following theorem.

Theorem 6.1. *With the notation and definitions adopted so far, we have the following property. If the hyperbola κ has two points $\{B, C\}$ at distance c, satisfying the condition of equation (21), where $t = (p, q)$ is a direction vector of the tangent at B and $a = (a_1, a_2)$ is a direction vector of an asymptote, then there exist three direction vectors $\{u, v = a, w\}$, with the property: For every point A of κ, the triangle ABC has the closing path property w.r.t. these three directions and the side BC (see Figure 10).*

7. A characteristic of the hyperbola

In this section we show that every hyperbola has two points $\{B, C\}$ and three projecting directions $\{u, v, w\}$, as in theorem 6.1, characterizing it as the geometric locus of vertices $\{A\}$, such that the triangles ABC have the closing path property w.r.t. to these three directions.

Without loss of generality we may assume that the hyperbola κ is represented in the shifted *"canonical form"*, through equation

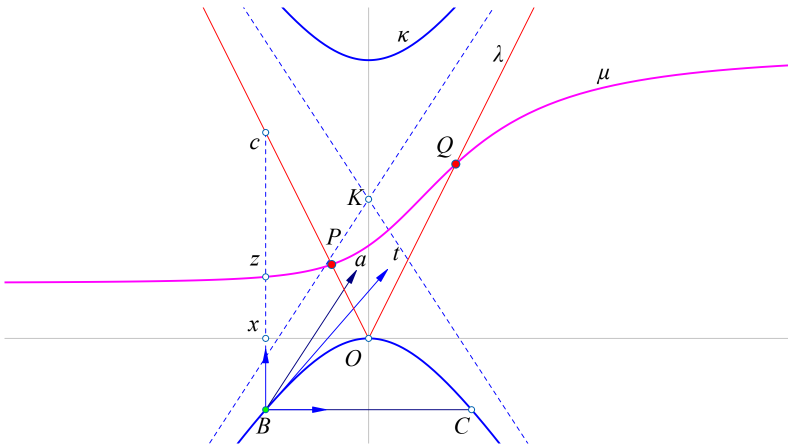

FIGURE 13. Graphs $\{\lambda, \mu\}$ of functions $c(x)$ and $z(x)$

(23) $$\frac{(y-a)^2}{a^2}-\frac{x^2}{b^2}=1 \quad \Leftrightarrow \quad a^2x^2-b^2y^2+2ab^2y=0\ ,$$

and point B lies in the lower branch, described by the function

(24) $$y=\frac{a}{b}(b-\sqrt{b^2+x^2})\ , x\in\mathbb{R}\ ,$$

with $\{a,b\}$ positive. For $B(x,y)\in\kappa$ we take $C(-2x,y)$, the reflection of B in the y-axis (see Figure 13). The distance $c(x)=|BC|=2|x|$. A direction vector for the asymptote is $(a_1,a_2)=(b,a)$ and a direction vector for the tangent at B is $(p,q)=(b^2(y-a),a^2x)$. Using these expressions for appropriate substitutions into the function on the right side of equation (21) we get:

(25) $$z(x)=\frac{2a_2^2p}{2a_1a_2p+(a_2^2-a_1^2)q}=\frac{2ab\sqrt{b^2+x^2}}{2b^2\sqrt{b^2+x^2}+(b^2-a^2)x}\ .$$

Figure 13 shows the graphs $\{\lambda,\mu\}$ of the functions $\{c(x),z(x)\}$. The compatibility condition expressed with formula (21) is equivalent with the existence of intersections of the two graphs, as shown in this figure. The intersection points $\{P,Q\}$ determine the corresponding values of x for which we have $c(x)=z(x)$. It turns out that we have always at least one intersection, which obviously produces two solutions, since reflecting everything on the y-axis we get also a second one.

Figure 13 shows a typical case for which $a>b$. In such a case it is obvious from formula (25), that $z(x)>0$ for all $x<0$, and this guaranties the existence of at least an intersection point $P\in\lambda\cap\mu$ for an $x_0<0$. Figure 14 shows the case for which still $a>b$, but a is sufficiently large and the graph μ of the function $z(x)$ splits in two branches, one of them (not shown) running below the x-axis and the other branch intersecting the graph λ of $c(x)=2|x|$ at one point only.

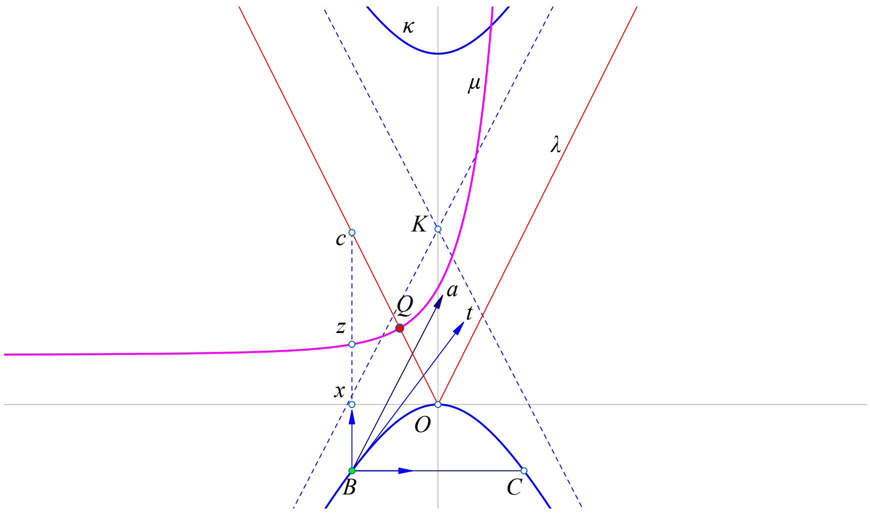

FIGURE 14. One only intersection $\lambda\cap\mu$ for $a>>b$

Finally figure 15 shows the case $b>a$ in which $z(x)>0$ for all $x\in\mathbb{R}$ guaranties the existence of two points in the intersection $\lambda\cap\mu$. In fact, in this case it is easily seen that $z(x) > 0$ for all $x \in \mathbb{R}$ and the graph μ of $z(x)$ runs on the upper half plane, as shown in the image. We notice, that in case $a = b$ the function $z(x) = a/b = 1$, and its graph produces again two intersections with λ. We formulate these results as a theorem.

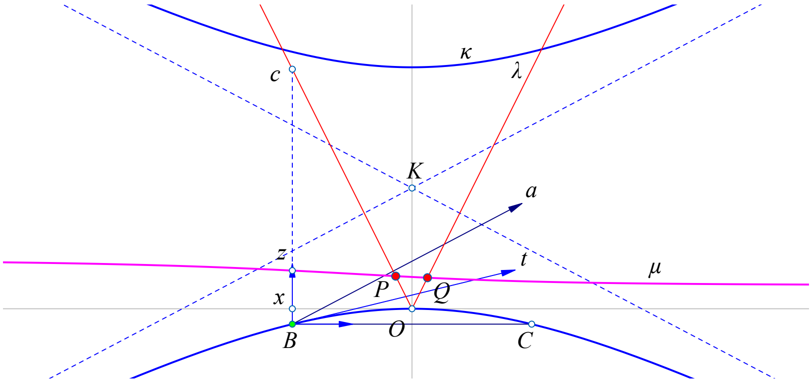

FIGURE 15. The intersections $\lambda \cap \mu$ in the case $b > a$

Theorem 7.1. *With the notation and definitions adopted so far, every hyperbola κ can be generated as a geometric locus in the following sense. There are two points $\{B, C \in \kappa\}$ and three direction vectors $\{u, v, w\}$ with the property : the triangle ABC has the closed path property w.r.t. $\{u, v, w\}$ and the side BC, if and only if $A \in \kappa$.*

8. Lines through a point

Figure illustrates the procedure introduced in § 1, now for lines passing through the same point. Again the question is when the path closes. In the figure the lines intersect at the origin of coordinates and their points are parameterized by multiples of corresponding vectors $\{a, b, c\}$. We use the same symbols to denote these lines.

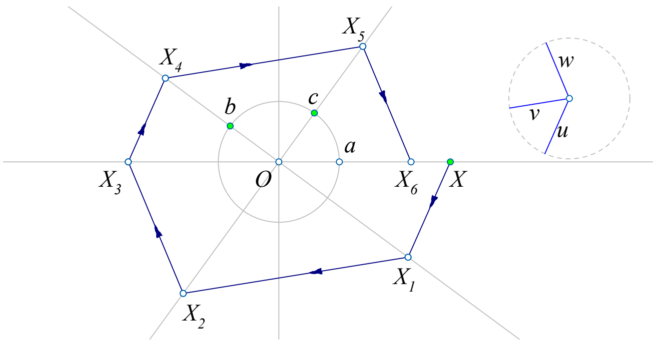

FIGURE 16. Projections on lines $\{a, b, c\}$ along $\{u, v, w\}$

We start again from a point $X \in a$ and project parallel to u onto a point $X_1 \in b$. Using the symbols $a \ni X \xrightarrow{||u} b \ni X_1$ we build the chain of successive projections

$$a \ni X \xrightarrow{||u} b \ni X_1 \xrightarrow{||v} c \ni X_2 \xrightarrow{||w} a \ni X_3 \ldots X_5 \xrightarrow{||w} a \ni X_6 . \quad (26)$$

A similar, but simpler, argument as the one applied for equation (4) shows the following theorem, in which we use the symbols $abu = det(a,u)/det(b,u)$:

Theorem 8.1. *The path* $p = XX_1...X_6$ *closes* ($X_6 = X$)*, precisely when it holds*

$$abu \cdot bcv \cdot caw = -1 \ .$$

It is also easily checked that this condition is equivalent with O being the middle of XX_3, which holding analogously for $\{X_1X_4, X_2X_6\}$, shows that if the path closes, then the resulting hexagon is symmetric in O. Combining this with theorem 2.1 we have the following theorem.

Theorem 8.2. *An hexagon with parallel opposite sides is symmetric, if and only if its main diagonals (those joining opposite vertices) intersect at a point.*

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