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A SPECIAL LINE THROUGH TWO CIRCLES' EXTERNAL CENTER OF SIMILARITY

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Abstract. In this article we show the construction of a special line through the external center of similarity of two given non congruent circles, neither of which is internal to the other. The special line generates two different circles each tangent to the given circles and also tangent to each other. We will use a special case of Apollonius' problem to aid us.

1. INTRODUCTION

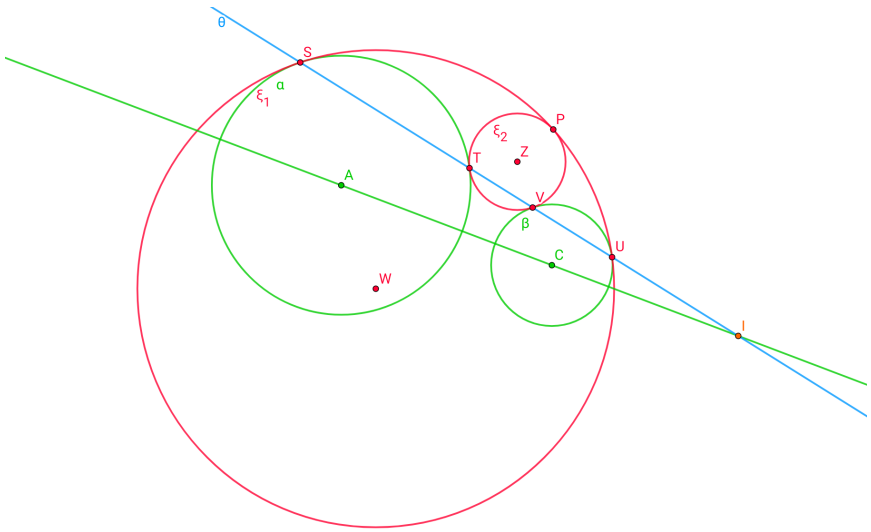

FIGURE 1. Special line θ through external center of similarity I of given circles $\{\alpha, \beta,\}$ defining circles $\{\xi_1, \xi_2\}$. These circles touch the given circles at the pairs of antihomologous points $\{S, U\}$ and $\{T, V\}$ respectively. They also touch each other at P.

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Given two non congruent circles, neither of which is internal to the other, we may through their external center of similarity draw a line intersecting the given circles in four points which can be identified as two pairs of antihomologous points. It is well known that for any such pair of antihomologous points, a unique circle touches the given two circles at these antihomologous points [6, p. 355]. Two different circles tangent to the two given circles at each pair of antihomologous points are thus associated with any line as described above. In general, the two circles tangent at the antihomologous points will not be tangent to each other.

However, the special line we construct, will intersect the given circles in two pairs of antihomologous such that the two different circles will also be tangent to each other.

Figure 1 shows the special line θ through the external center of similarity I intersecting the given circles $\{\alpha, \beta\}$ in the two pairs of antihomologous points $\{S, U\}$ and $\{T, V\}$. Touching the given circles at these points are the two circles $\{\xi_1, \xi_2\}$. We call the line θ special as the circles $\{\xi_1, \xi_2\}$ also touch each other (at P). A second special line exists (not shown) which is simply the reflection of θ in the line joining the centers of the given circles, AC.

We exclude the configurations of the given two circles when one is internal to the other. It may readily be verified that in these cases no special line such as θ will exist.

2. A SPECIAL CASE OF APOLLONIUS' PROBLEM

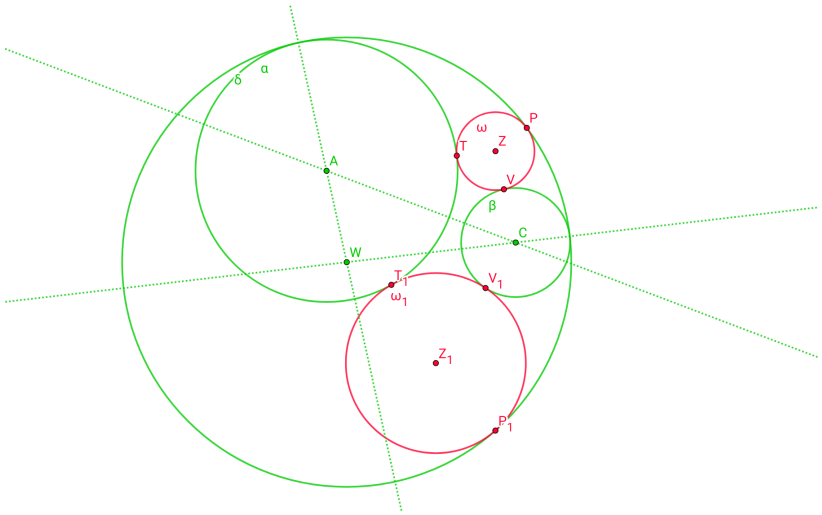

FIGURE 2. A special case of Apollonius' problem, where two $\{\alpha,\beta\}$ of the three given circles are internally tangent to the third δ. The two possible solution circles are $\{\omega,\omega_1\}$.

Apollonius' problem "consists in finding the circles which are simultaneously tangent to three given circles " [6, p. 436]. We next consider a special

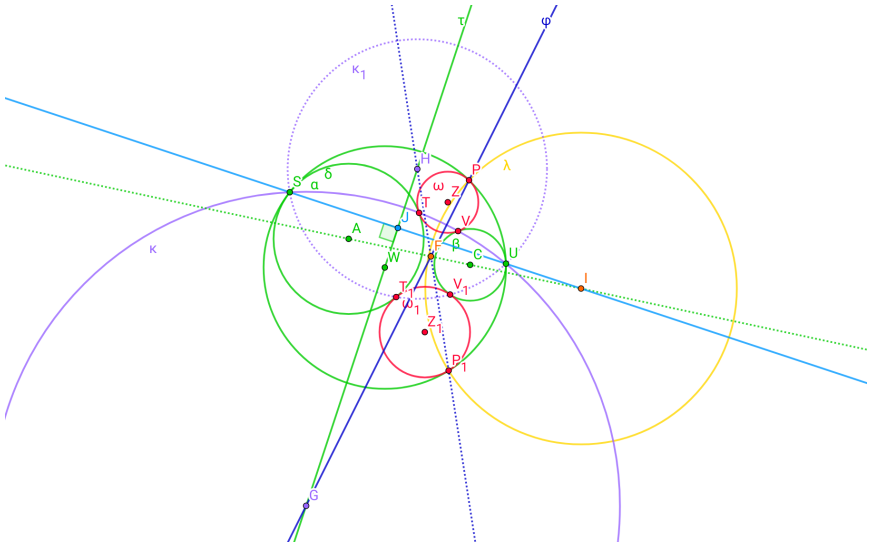

FIGURE 3. Construction of a special case of Apollonius' problem

case[1] of this problem where two of the given circles $\{\alpha,\beta\}$ are internally tangent to the third circle δ. There are two possible solutions $\{\omega,\omega_1\}$ (See Figure 2). In Figure 3 we show the full construction which we now proceed to explain.

(A) Find the internal and external centers of similarity $\{F, I\}$ of the two circles $\{\alpha,\beta\}$.
(B) Make tangents $\{IP, IP_1\}$ to the circle δ. These are radi of the circle of antisimilitude λ.
(C) Make the perpendicular bisector of the chord SU.
(D) Join the points $\{P, F\}$ (or $\{P_1, F\}$), and extend to intersect the perpendicular bisector in G (H).
(E) Make a circle κ (κ_1) centered at G (H) through points $\{S, U\}$ to intersect the circles $\{\alpha,\beta\}$ in $\{T, V\}$ ($\{T_1, V_1\}$) respectively. The three points $\{P, T, V\}$ ($\{P_1, T_1, V_1\}$) define the solution circle ω (ω_1) of Apollonius' problem. The center Z (Z_1) may easily be found.

A few lemmas will aid us in justifying this construction, and in the next section we finalise its justification by using some properties of a cyclic quadrilateral.

Lemma 2.1. *Any circle ω_i[2] centered on the line WP, internally tangent to the circle $\delta(W)$ at P and intersecting the circles $\{\alpha,\beta\}$, will intersect them in two pairs (one pair if tangency occurs) of antihomologous points relative to the external center of similarity I.*

This lemma holds equally true for any circle ω_i' analogous to ω_i centered on the line WP_1 and internally tangent to the circle $\delta(W)$ at P_1.

[1]When using the term "special case" in connection with Apollonius' problem, we always refer to this specific case.

[2]We use the i-index notation to indicate either a set of objects or an object that is not fixed in position.

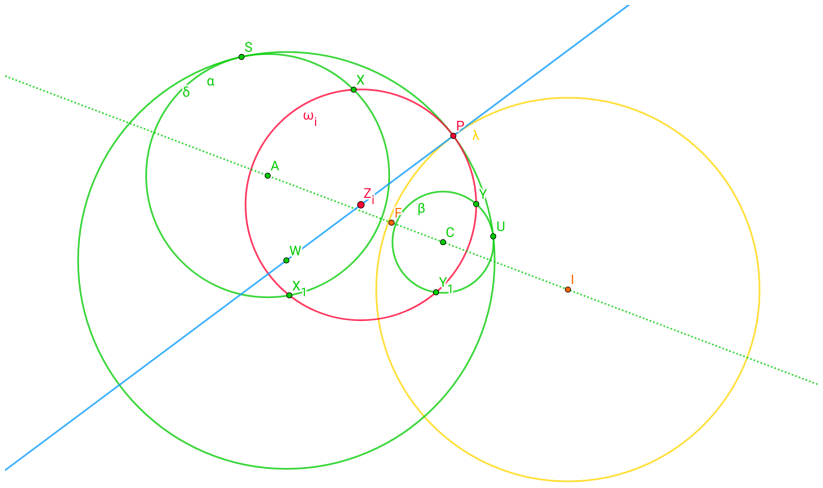

FIGURE 4. Figure for Lemma 2.1

Proof. (See Figure 4) All circles ω_i are orthogonal to the circle of antisimilitude $\lambda(I)$ as WP is orthogonal to IP. Thus, an inversion w.r.t. λ of the circles $\{\alpha,\beta,\omega_i\}$ will leave ω_i invariant, and interchange the circles $\{\alpha,\beta\}$. Their intersection points $\{X,Y\}$ and $\{X_1,Y_1\}$ with the circle ω_i will also interchange $(X \longleftrightarrow Y, X_1 \longleftrightarrow Y_1)$,thus rendering them collinear with the inversion center I and defining them as inverse and also antihomologous points.

Corollary 2.1. *A unique circle ω (ω_1) of the circles ω_i (ω_i') is tangent to the circles $\{\alpha,\beta\}$ at one pair of antihomologous points (i.e. only one pair exists in this case). It is a solution circle of Apollonius' problem*[3] *and P (P_1) is its tangent point with the circle δ.*

Proof. No circles other than one from the set of circles $\{\omega_i\}$ ($\{\omega_i'\}$), internally tangent to the circle δ at P (P_1), will intersect the circles $\{\alpha,\beta\}$ in one or two pairs of antihomologous points w.r.t. I, as an inversion w.r.t. the circle of antisimilitude λ would not leave those circles invariant.

As an alternative to the proof above, we may consider Casey's method of constructing solutions to the problem of Apollonius [2, pp. 121–123, prop. 10], [4]. By this method, it is readily found that I is the relevant point from which to make tangents to δ. The tangent points $\{P,P_1\}$ are points where the solution circles $\{\omega,\omega_1\}$ respectively, are tangential to the given circle δ.

Lemma 2.2. *Any circle $\kappa_i(G_i)$ centered on the perpendicular bisector of the chord SU and containing the points $\{S,U\}$, intersects the circles $\{\alpha,\beta\}$ in antihomogolous points $\{T_i,V_i\}$ w.r.t. to their external center of similarity I.*

Proof. (See Figure 5) The circles $\kappa_i(G_i)$ form an elliptic pencil with basic points $\{S,U\}$ of which also the circle δ is a member. The circle of antisimiltude λ is orthogonal to δ and every circle $\kappa_i(G_i)$, as its center I is on the

[3]The special case we are dealing with.

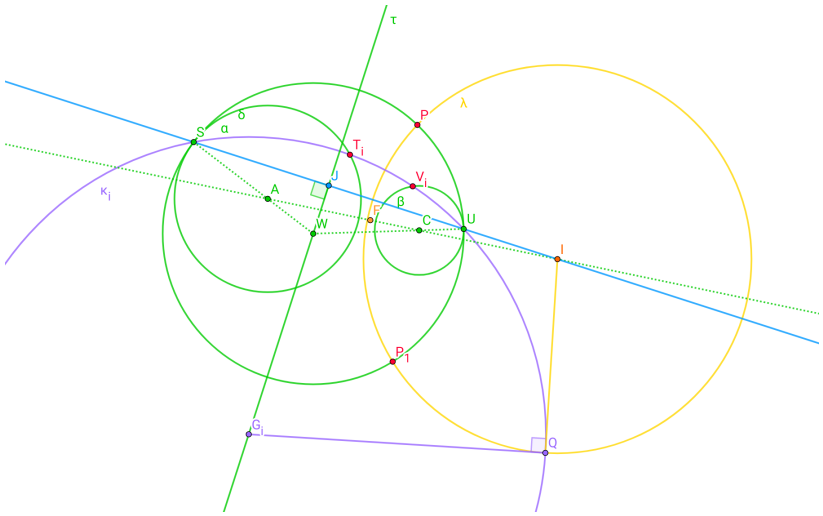

FIGURE 5. Figure for Lemma 2.2

radical axis of the elliptic pencil. Thus tangents to all member circles of the elliptic pencil from point I, will have the same distance, namely the radius of the circle λ. An inversion w.r.t. λ will leave every circle $\kappa_i(G_i)$ invariant and interchange the circles $\{\alpha,\beta\}$ and the points $\{T_i, V_i\}$. So T_i and V_i will be inverse points w.r.t. λ and collinear with its center I, the external center of similarity of the circles $\{\alpha,\beta\}$. These points will also be antihomologous w.r.t. I.

3. SOME PROPERTIES OF THE CYCLIC QUADRILATERAL ST_iV_iU

The cyclic quadrilateral ST_iV_iU has certain properties that we will use in justifying the construction in section 2.

PROPERTIES (See Figure 6)

(I) A unique circle $\gamma_i(B_i)$ exists tangent to the circles $\{\alpha, \beta\}$ at the antihomologous points $\{T_i, V_i\}$ [6, p. 355].

(II) The meet of opposite sides $ST_i \cap V_iU = E_i$ is the radical center of the circles $\{\alpha, \beta, \kappa_i\}$ as may easily be deduced by the known fact that E_i is a point on the radical axis of the circles $\{\alpha, \beta\}$ [5, p. 41], and that the lines SE_i and UE_i are radical axes of respectively $\{\alpha, \kappa_i\}$ and $\{\beta, \kappa_i\}$.

(III) E_i is also the internal similarity center of the circles $\{\delta, \gamma_i\}$. Consider the circle triplets $\{\alpha, \delta, \gamma_i\}$ and $\{\beta, \delta, \gamma_i\}$. With regard to the first triplet, the line ST_iE_i is an axis of similitude with points $\{S, T_i\}$ being external and internal similarity centers relative to the circles $\{\delta, \alpha\}$ and $\{\alpha, \gamma_i\}$ respectively. Thus the third center of similarity on this axis must necessarily be the internal similarity center relative to $\{\delta, \gamma_i\}$ [5, p. 151]. Similarly, the line UV_iE_i is an axis of similitude relative to the second triplet of circles $\{\beta, \delta, \gamma_i\}$ where the points $\{U, V_i\}$ are external and internal similarity centers relative to

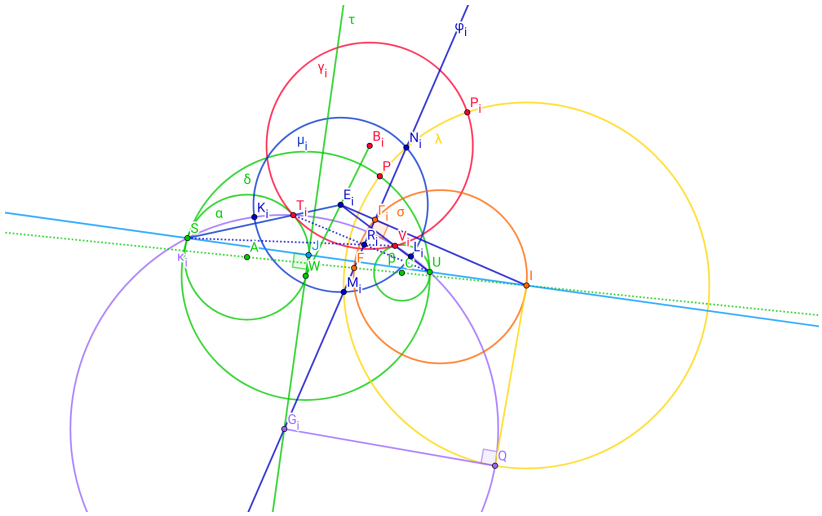

FIGURE 6. Configuration associated with some properties of the cyclic quadrilateral ST_iV_iU.

the circles $\{\delta, \beta\}$ and $\{\beta, \gamma_i\}$ respectively. As above, the third similarity center on the axis of similitude UV_iE_i must necessarily be the internal similarity center relative to $\{\delta, \gamma_i\}$. As E_i is the meet of the axes ST_i and UV_i, we may conclude it is identical with the internal similarity center relative to $\{\delta, \gamma_i\}$ and also, more importantly, that the circle centers $\{W, B_i\}$ of these circles are collinear with E_i.

(IV) It is known that the so-called diagonal triangle (here: E_iIR_i) of a cyclic quadrilateral is self-polar [3, p. 75]. Also, it is known that the center of the cyclic quadrilateral's circumcircle (here: G_i) is the orthocenter of the diagonal triangle and there will always be one vertex of this triangle inside the circumcircle (here: R_i) [1, pp. 181–182]. Hence, the line ϕ_i, joining the center G_i of the circumcircle κ_i, and the point $SV_i \cap UT_i = R_i$, is perpendicular to the diagonal line E_iI at the foot Γ_i.

(V) Make tangents to $\kappa_i(G_i)$ from E_i. The tangent points $\{K_i, L_i\}$ define the polar IR_i of E_i w.r.t. $\kappa_i(G_i)$. Make circle $\mu_i(E_i)$ with radius equal to $E_iK_i = E_iL_i$. The polar IR_i is also the radical axis of the circles $\{\mu_i, \kappa_i\}$. As the center I of the circle λ is on this axis and λ is orthogonal to κ_i, we must have λ also orthogonal to μ_i, and G_i is on their radical axis [6, p. 345]. The line $G_i\Gamma_i$ is perpendicular to the line joining the centers $\{E_i, I\}$ of the circles $\{\mu_i, \lambda\}$ respectively, and must necessarily be the radical axis of these circles. We see that E_i and Γ_i are inverse points w.r.t. λ [6, pp. 360–361].

(VI) As E_i is a point on the radical axis of the circles $\{\alpha, \beta\}$ [5, p. 41], and inversion of this radical axis w.r.t to λ (= circle of antisimilitude $\{\alpha, \beta\}$) is the circle of similitude σ [6, p. 393], Γ_i must be a point on the circle σ. Moreover, since the points $\{I, F\}$ define a diagonal of σ, making $\angle I\Gamma_iF$ a right angle, the line $G_i\Gamma_i = \phi_i \perp E_iI$ must necessarily be collinear with F.

Summing up:
For all circles κ_i, the associated cyclic quadrilateral ST_iV_iU generates a line ϕ_i, which is collinear with the internal similarity center F of the circles $\{\alpha, \beta\}$, the point $R_i = SV_i \cap UT_i$ and the tangent points $\{M_i, N_i\}$ on λ (tangents to λ from the point E_i).
ϕ_i is the polar of the point E_i w.r.t. the circle λ, and is also the radical axis of the circles $\{\mu_i, \lambda\}$.

Proposition 3.1. *The line joining the tangent point P and the internal center of similarity F of the circles $\{\alpha,\beta\}$, intersects the perpendicular bisector of the chord SU at the point G. G is the center of a circle κ which intersects $\{\alpha, \beta\}$ at the points $\{T, V\}$ respectively. These points are the tangent points for the solution circle ω of a special case of Apollonius' problem.*

Proof. (See Figure 3) The line GFP is one particular line ϕ of the set ϕ_i, and the quadrilateral $STVU$ is one particular quadrilateral of the set ST_iV_iU. Thus, we may use the properties (I)−(VI) from above, renaming points and circles appropriately. We see the following transformations as the line $\phi_i \to \phi$:

(1)
$$G_i \to G, \quad \kappa_i \to \kappa, \quad T_i \to T, \quad V_i \to V, \quad B_i \to Z, \quad \gamma_i \to \omega, \quad P_i \to P.$$

This shows that the construction in section 2 is justified.

A similar construction has been used in [7, Construction 3] to find the incircle of an arbelos. In that construction we may readily identify the relevant line ϕ and the collinearty (in our notation) of the points $\{G, F, P\}$. We note that our justification for the construction in section 2 is somewhat lengthy. The emphasis has been on finding why the internal center of similarity F is on the line ϕ (See Property (VI)). We regard this as an intriguing feature.

4. CONSTRUCTION OF A SPECIAL LINE θ

We now proceed to our main goal of constructing the line θ.

The construction runs as follows (See Figure 7):

(A) Find the internal and external centers of similarity $\{F, I\}$ of the circles $\{\alpha, \beta\}$.
(B) Make the circle ψ tangent to $\{\alpha, \beta\}$ having its center collinear with their centers $\{A, C\}$.
(C) Construct the circle of antisimilitude λ, by first making tangents to ψ from I, and then using the distance from I to a tangent point as the radius of λ.
(D) Make circle η with diameter equal to the distance from the external center of similarity I, to the midpoint Λ, between the circle centers $\{A, C\}$. One intersection point between the circles $\{\lambda, \eta\}$ is P.
(E) Make line $\phi = PF$.

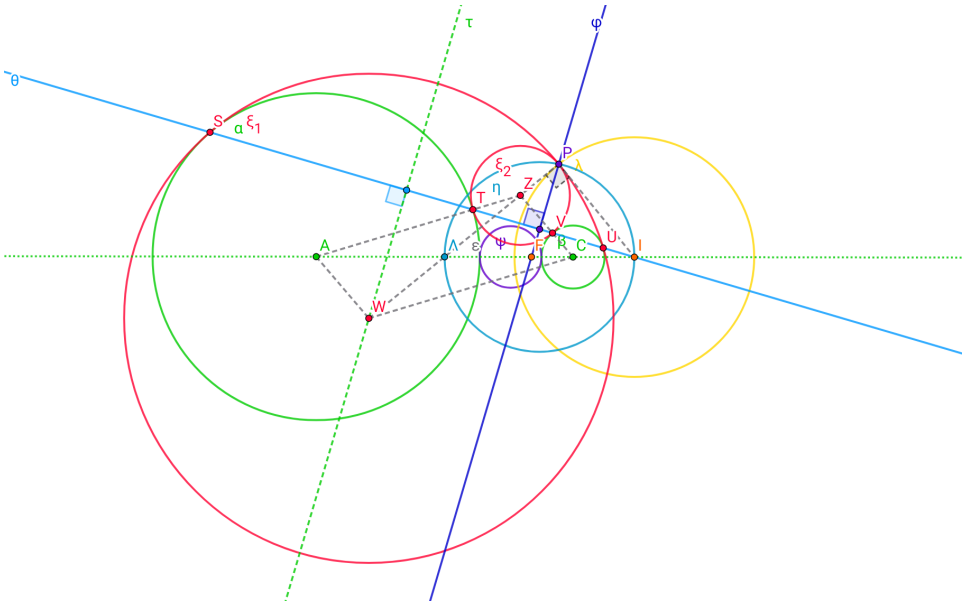

FIGURE 7. Construction of special line θ

(F) Make line θ through the external center of similarity I, perpendicular to ϕ. The intersection points between ϕ and the circles $\{\alpha, \beta\}$, are the two pairs of antihomologous points $\{S, U\}$ and $\{T, V\}$.

The circles $\{\xi_1, \xi_2\}$ tangent to $\{\alpha, \beta\}$ at the two pairs of antihomologous points, are also tangent to each other at the point P.

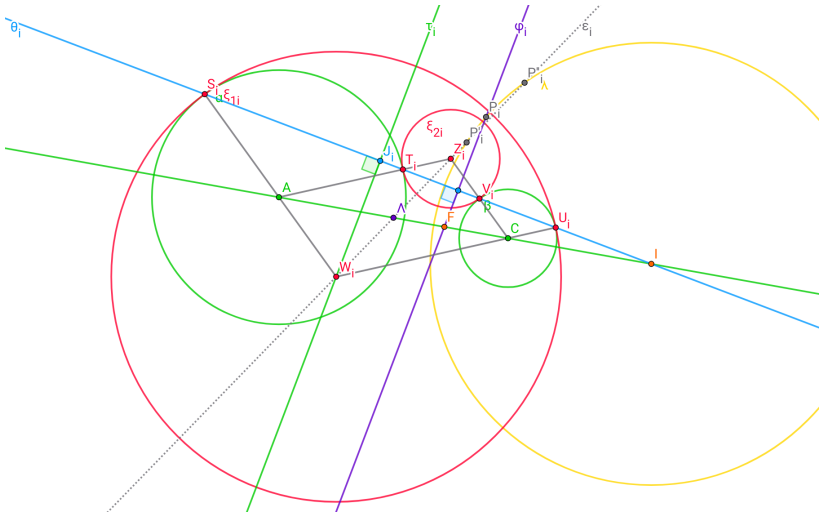

FIGURE 8. Justification of figure 7 (see text)

We seek to justify the construction in Figure 7.

We consider only the circles $\{\alpha, \beta, \lambda\}$, a variable line $S_iT_iV_iU_i = \theta_i$ through I, and a line $P_iF = \phi_i$ (through the fixed point F and parallel

to the perpendicular bisector of S_iU_i), where P_i is a point on the circle of antisimilitude λ and F is, as previously, the internal center of similarity for the given circles $\{\alpha, \lambda\}$ (See Figure 8).

When the line $P_iF = \phi_i$ is parallel to the perpendicular bisector of $S_iU_i = \tau_i$, the circle center $G_i \to \infty$ and the circle κ_i degenerates to the line $S_iT_iV_iU_i$ (See Figure 6).

All circle pairs $\{\xi_{1_i}, \xi_{2_i}\}$ tangent to $\{\alpha, \beta\}$ at antihomologous points $\{S_i, U_i\}$ and $\{T_i, V_i\}$ respectively, have pairs of parallel radii [6, p. 352]: $W_iS_i \parallel Z_iV_i$ and $W_iU_i \parallel Z_iT_i$. A parallelogram can be made by joining the circle centers $\{A, W_i, C, Z_i\}$. Its center Λ is the intersection of the diagonals at their midpoints ([6, p. 102]). It is clear that the circle centers $\{W_i, Z_i\}$ and Λ are collinear and this line, ϵ_i, intersects the circle λ at the points $\{P_i', P_i''\}$.

As $\angle S_iIA$ varies (all the while $P_iF = \phi_i \perp S_iU_i$), the point P_i will move on the circle λ (F being fixed), and it will be equal to P at one particular angle of $\angle S_iIA$. At precisely this angle:

$$\phi_i \to \phi, \quad S_i \to S, \quad T_i \to T, \quad V_i \to V, \quad U_i \to U.$$

We may bring to bear our previous findings (See Equation 1) and by analogy deduce :

$$\xi_{1_i} \to \xi_1, \quad \xi_{2_i} \to \xi_2.$$

Also, we may add:

$$W_i \to W, \quad Z_i \to Z \quad P_i' \to P, \quad P_i'' \to P, \quad \epsilon_i \to \epsilon, \quad \tau_i \to \tau.$$

The main observation is that the line $\epsilon = W\Lambda ZP$ will be tangent to the circle of antisimilitude λ at P, and $\angle \Lambda PI$ will then be a right angle. So we may construct the circle η with diameter equal to ΛI to find the point P (an unknown in this setting) from $\eta \cap \lambda$, giving the direction of the line $FP = \phi$. A perpendicular to ϕ through I establishes the special line θ, and the construction above is justified.
From the above we then have (See Figure 7):

Theorem 4.1. *Given two non-congruent circles*[4] $\{\alpha, \beta\}$*, a line* θ*, through their external center of similarity* I *exists, intersecting* $\{\alpha, \beta\}$ *in the pairs of antihomologous points* $\{(S, U), (T, V)\}$*. Two unique circles,* $\{\xi_1, \xi_2\}$*, are tangent at one pair each. If the line* θ *is perpendicular to the line* $FP = \phi$*, where* F *is the internal center of similarity of* $\{\alpha, \beta\}$*, and* FP *is parallel to the perpendicular bisector of* $SU = \tau$*, then* $\{\xi_1, \xi_2\}$ *are tangent at the point* P*.*

The construction works equally well when the two given circles either intersect or are externally tangent. When tangent, a different construction may be used which we expand upon in the following (See Figure 9).

[4]Three cases are valid:
1. Non-intersecting and mutually external. 2. Intersecting. 3. Externally tangent.

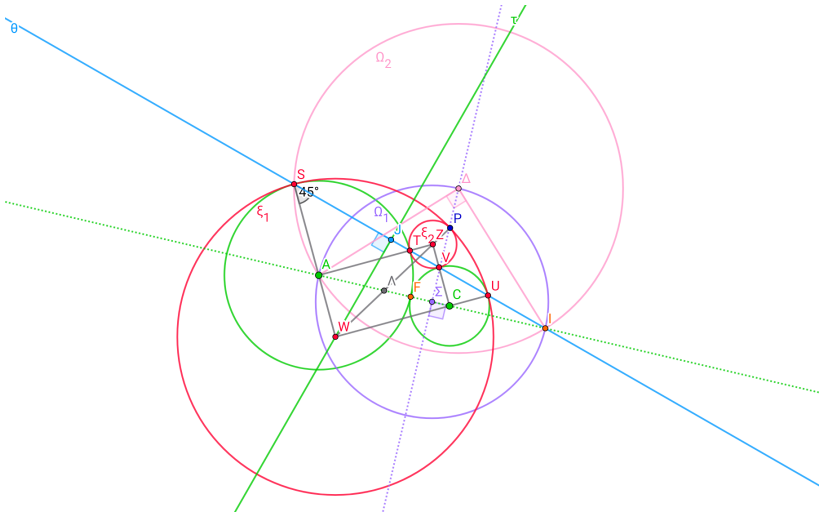

FIGURE 9. A different construction of the line θ when the given circles are externally tangent

The construction in Figure 9 runs as follows:

(A) Find I, the external center of similarity of $\{\alpha, \beta\}$.
(B) Make perpendicular bisector of the segment AI giving point Σ.
(C) Make circle Ω_1 with radius $A\Sigma = \Sigma I$ giving point Δ on the perpendicular bisector in (B).
(D) Make circle Ω_2 centered at Δ with radius $A\Delta = \Delta I$ giving $\Omega_2 \cap \alpha = S$, and we then have $IS = \theta$.

We seek to justify the construction in Figure 9. From previously, we know that the quadrilateral $AWCZ$ is a parallelogram.

Proposition 4.1. *The parallelogram $AWCZ$ is a rectangle $\Rightarrow \angle ISA = 45°$.*

Proof. The radii of the circles $\{\xi_1, \xi_2, \alpha, \beta\}$ are $\{R, r, a, b\}$ respectively. $WZ = R - r$ is a diagonal of the parallelogram $AWCZ$ and $AC = a + b$ is the other diagonal. Equality of the sides $AW = R - a = CZ = b + r$ gives $R - r = a + b$. Thus, the two diagonals are equal and as they are bisected by their intersection point, the parallelogram $AWCZ$ is a rectangle [6, p. 102, p. 119]. So, $\angle AWC = \angle SWU = 90°$, $WS = WU = R$ and triangle SWU is a right angled isosceles $\Rightarrow \angle USA = \angle ISA = 45°$.

We exploit this result in the construction by making the circle Ω_2 span the segment AI at its center Δ by an angle $= 90°$. The intersection between the circle Ω_1 and the perpendicular bisector of AI gives us Δ. The intersection between the circles $\{\alpha, \Omega_2\}$ gives us the point S and the inscribed angle $\angle ISA$ in Ω_2 spans the segment $AI \Rightarrow \angle ISA = 45°$. The construction is thus justified.

Finally we consider the constant product $IT \cdot IV = IS \cdot IU = PI^2 = IA \cdot IC - ab$, which, given two circles, is valid for every line through a center of similarity defining two antihomologous points, e.g. $\{T, V\}$ or $\{S, U\}$ (See

[6, p. 356] and Figure 7). We derive an expression for this product in terms of the radii $\{R, r, a, b\}$ (defined above), when this line is the special line θ. d is the diagonal AC of the parallelogram $AWCZ$. We use the parallelogram law and find:

$$\begin{aligned} 2(R-a)^2 + 2(R-b)^2 &= (R-r)^2 + d^2 \\ \Rightarrow 4R^2 - 4R(a+b) &= d^2 - (a-b)^2. \end{aligned} \tag{2}$$

By similar triangles we find (assuming $a > b$):

$$IA = \frac{ad}{a-b},$$
$$IC = \frac{bd}{a-b}.$$

This gives,

$$IT \cdot IV = IA \cdot IC - ab = d^2 \frac{ab}{(a-b)^2} - ab.$$

Using (2) and $R - r = a + b$, we thus have:

$$IT \cdot IV = \frac{ab(d^2 - (a-b)^2)}{(a-b)^2} = \frac{ab(4R^2 - 4R(a+b))}{(a-b)^2} = \frac{4Rrab}{(a-b)^2}. \tag{3}$$

When the given circles are externally tangent, (3) simplifies to:

$$IT \cdot IV = \left(\frac{2ab}{a-b}\right)^2 = \left(\frac{2Rr}{a-b}\right)^2. \tag{4}$$

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