



## QUADRILATERALS SATISFYING

$$T_a^n + T_c^n = T_b^n + T_d^n$$

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**Abstract.** We identify convex quadrilaterals  $ABCD$  satisfying the relation  $T_a^n + T_c^n = T_b^n + T_d^n$ , where  $T_a, T_b, T_c, T_d$  are the area of triangles  $ABP, BCP, CDP, DAP$  respectively,  $P$  is the diagonal intersection, and  $n$  is a non-zero integer or  $\pm\frac{1}{2}$ .

### 1. INTRODUCTION

In a convex quadrilateral  $ABCD$  with diagonal intersection  $P$ , let  $T_a, T_b, T_c, T_d$  be the area of the subtriangles  $ABP, BCP, CDP, DAP$  respectively. One problem at a 2001 Korean mathematics competition according to [1, pp. 92, 128] was to prove that the equality

$$T_a + T_c = T_b + T_d$$

is satisfied only in quadrilaterals where at least one diagonal bisects the other diagonal. At the Third Round of the 1980 Leningrad High School Olympiad for Grade 8 [5, p. 287], a question was to prove that if

$$T_a^2 + T_c^2 = T_b^2 + T_d^2,$$

then at least one diagonal bisects the other diagonal. Such quadrilaterals were called *bisect-diagonal quadrilaterals* in [2].

In this note we shall investigate the similar but more general equation

$$(1) \quad T_a^n + T_c^n = T_b^n + T_d^n$$

where  $n$  is a non-zero integer or  $\pm\frac{1}{2}$ , and determine in which convex quadrilaterals it holds. Such explorations will provide an excellent opportunity for students to generalize, and be a nice example of when certain algebraic identities are useful to solve a geometric problem.

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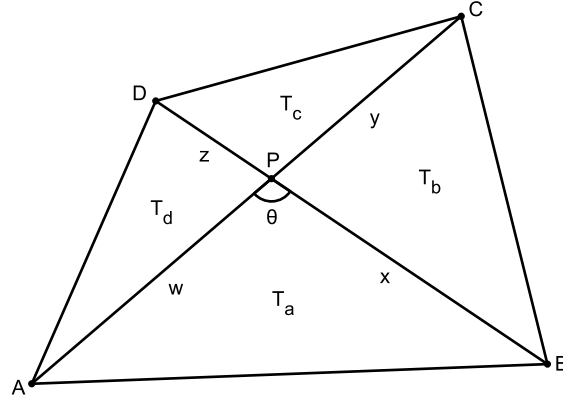


FIGURE 1. The semidiagonals and the subtriangle areas

2. THE CASES WHEN  $n = 1, 2, 3, 4$ 

Let  $w = AP$ ,  $x = BP$ ,  $y = CP$ ,  $z = DP$  be the semidiagonals and  $\theta = \angle APB$  (see Figure 1). Using the well-known trigonometric formula for the area of a triangle expressed in terms of two sides and the included angle,  $T_a = \frac{1}{2}wx \sin \theta$  and

$$T_a = \frac{1}{2}xy \sin(\pi - \theta) = \frac{1}{2}xy \sin \theta$$

and similar formulas hold for the other two triangle areas. Then we have that the Korean problem, after dividing with non-zero factors, is equivalent to  $wx + yz = xy + zw$ , that is

$$(w - y)(x - z) = 0.$$

Here equality holds if and only if at least one diagonal bisects the other diagonal. In the same way, the Leningrad problem yields

$$w^2x^2 + y^2z^2 - x^2y^2 - z^2w^2 = 0,$$

or equivalently,

$$(w - y)(w + y)(x - z)(x + z) = 0$$

with  $w = y$  or  $x = z$  as the only possible solutions to the geometric problem. Hence again we have that at least one diagonal bisects the other diagonal.

For the similar third degree problem

$$T_a^3 + T_c^3 = T_b^3 + T_d^3,$$

it is factored (using the semidiagonals) into

$$(w - y)(w^2 + wy + y^2)(x - z)(x^2 + xz + z^2) = 0.$$

Even here equality holds if and only if at least one diagonal bisects the other diagonal. A pattern is emerging. The four degree problem

$$T_a^4 + T_c^4 = T_b^4 + T_d^4$$

has the semidiagonal factorization

$$(w - y)(w + y)(w^2 + y^2)(x - z)(x + z)(x^2 + z^2) = 0$$

and the pattern is intact.

### 3. THE GENERAL CASE

Next we tackle the general problem. When  $n$  is a positive integer, we get that

$$T_a^n + T_c^n = T_b^n + T_d^n$$

with the help of the semidiagonals is equivalent to

$$w^n x^n + y^n z^n - x^n y^n - z^n w^n = 0,$$

that is,

$$(2) \quad (w^n - y^n)(x^n - z^n) = 0.$$

From the algebraic formula

$$w^n - y^n = (w - y)(w^{n-1} + w^{n-2}y + \dots + wy^{n-2} + y^{n-1}),$$

where the expression in the last parenthesis is positive in our problem, we get that (2) has only two possible solutions,  $w = y$  or  $x = z$ , so at least one diagonal bisects the other diagonal.

How about when  $n$  is a negative integer? The case when  $n = -1$  was considered in a Corollary in [2, p. 216]. We get that

$$T_a^{-1} + T_c^{-1} = T_b^{-1} + T_d^{-1}$$

is equivalent to

$$\frac{(w - y)(x - z)}{wxyz} = 0$$

so  $w = y$  or  $x = z$  as in the previous cases. The reciprocal second degree equation

$$T_a^{-2} + T_c^{-2} = T_b^{-2} + T_d^{-2}$$

is factored into

$$\frac{(w - y)(w + y)(x - z)(x + z)}{(wxyz)^2} = 0$$

with the same solutions. In general,

$$T_a^{-n} + T_c^{-n} = T_b^{-n} + T_d^{-n}$$

is factored as

$$\left(\frac{1}{w^n} - \frac{1}{y^n}\right) \left(\frac{1}{x^n} - \frac{1}{z^n}\right) = \frac{(y^n - w^n)(z^n - x^n)}{(wxyz)^n} = 0$$

which holds if and only if at least one diagonal bisects the other diagonal according to the conclusion for positive integers  $n$ .

### 4. THE CASES WHEN $n = \pm\frac{1}{2}$

Let us conclude with the cases when  $n = \pm\frac{1}{2}$ . Then we have

$$\sqrt{T_a} + \sqrt{T_c} = \sqrt{T_b} + \sqrt{T_d}$$

which, when using semidiagonals, is equivalent to

$$(\sqrt{w} - \sqrt{y})(\sqrt{x} - \sqrt{z}) = 0$$

and

$$\frac{1}{\sqrt{T_a}} + \frac{1}{\sqrt{T_c}} = \frac{1}{\sqrt{T_b}} + \frac{1}{\sqrt{T_d}}$$

that with the help of semidiagonals is factored into

$$\frac{(\sqrt{w} - \sqrt{y})(\sqrt{x} - \sqrt{z})}{\sqrt{wxyz}} = 0.$$

In both cases we have the only possible solutions that at least one diagonal bisects the other diagonal. Hence we have proved infinitely many new characterizations of bisect-diagonal quadrilaterals, that at least one diagonal in a convex quadrilateral is bisected by the other diagonal if and only if (1) holds, where  $n$  is any non-zero integer or  $n = \pm\frac{1}{2}$ .

## 5. CONCLUDING REMARKS

There are other interesting relations between the subtriangle areas  $T_a$ ,  $T_b$ ,  $T_c$ ,  $T_d$ . In Theorem 8.1 in [3] it was proved that each of the following are necessary and sufficient conditions for a convex quadrilateral to be a parallelogram:

- $T_a = T_b = T_c = T_d$
- $T_a = T_c$  and  $T_b = T_d$
- $T_b = T_d = \frac{1}{4}K$
- $T_b^2 = T_a T_c$  and  $2T_d = T_a + T_c$

where  $K$  is the area of the quadrilateral, and in Theorem 6.1 in [4] it was proved that each of the following formulas

- $K = 4\sqrt{T_a T_c}$  or  $K = 4\sqrt{T_b T_d}$
- $K = 4\sqrt[4]{T_a T_b T_c T_d}$
- $K = \frac{1}{4}(\sqrt{T_a} + \sqrt{T_b} + \sqrt{T_c} + \sqrt{T_d})^2$

also characterize parallelograms.

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