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ON THE DETERMINATION OF A QUADRILATERAL

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Abstract. In this paper we prove some results about the determination of a quadrilateral.

1. Introduction

In this section we will recall some known results which we will use in the following (see [1]).

Theorem 1. *Let* a, b, c, d *be real numbers,* $a, b, c, d > 0$*. These numbers can be the lengths of the sides of a quadrilateral if and only if* $a < b + c + d$*,* $b < c + d + a$*,* $c < d + a + b$ *and* $d < a + b + c$*.*

Theorem 2. *Let* a, b, c, d *real numbers,* $a, b, c, d > 0$*. The following statements are equivalent:*
(i) *there exists a quadrilateral with sides of lengths* a, b, c *and* d*;*
(ii) *there exists a convex quadrilateral with sides lengths* a, b, c *and* d*.*

Theorem 3 (Sturm's Theorem)**.** *Let* $ABCD$ *be a quadrilater with sides lengths* a, b, c *and* d*. Then there exists a cyclic quadrilateral* $MNPQ$ *such that* $MN = a$*,* $NP = b$*,* $PQ = c$ *and* $QM = d$*.*

Remark 1. We consider a quadrilateral with "rigid sides" and constant lengths, its vertices being "mobile articulations". Then, the geometrical interpretation of Theorem 2 and Theorem 3 is that any quadrilateral can be "deformed" in order to obtain a convex quadrilateral, respectively a cyclic quadrilateral.

Remark 2. The angle $\widehat{NMQ}$ in Theorem 3 is equal

$$\widehat{NMQ} = \arccos \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}.$$

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2. MAIN RESULTS

We saw in the first part that four strictly positive numbers that verify the inequalities in Theorem 1 can be the lengths of the sides of a quadrilateral, but this is not uniquely determined. In this section we will prove a quadrilateral existence theorem. The result contains a relationship between the sides and an angle of the quadrilateral. Finally, we will make some remarks regarding this result. Taking into account Remark 2, we will "deform" the quadrilateral until it generates into a triangle, in order to find the extreme values of an angle of the quadrilateral depending on the sides of the quadrilateral.

Definition 1. We say that a quadrilateral is degenerate if and only if it has at least three collinear vertices.

Remark 3. If a quadrilateral is degenerate, then it has one angle equal to 0 or equal to π (see Fig. 1, respectively Fig. 2).

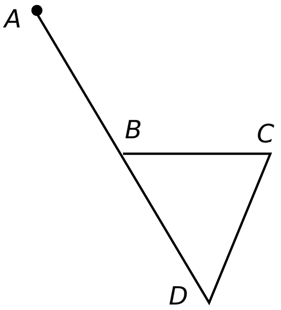

Figure 1

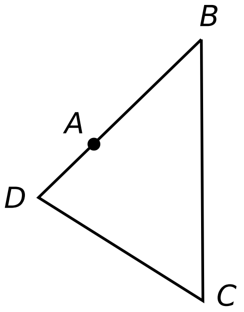

Figure 2

Theorem 4. *Let a, b, c, d be real numbers such that $a, b, c, d > 0$, $a < b + c + d$, $b < c + d + a$, $c < d + a + b$, $d < a + b + c$,*

$$\alpha = \frac{a^2 + d^2 - (b + c)^2}{2ad} \quad \textit{and} \quad \beta = \frac{a^2 + d^2 - (b - c)^2}{2ad}.$$

Then there exists a quadrilateral $ABCD$, which can also be degenerate, such that $AB = a$, $BC = b$, $CD = c$, $DA = d$ and $\widehat{BAD} = A$ if and only if

$$\arccos(\min\{1, \beta\}) \leq A \leq \arccos(\max\{-1, \alpha\}). \tag{1}$$

Proof. In triangle ABD (see Fig. 3) we have $BD^2 = a^2 + d^2 - 2ad \cos A$ and in triangle BCD we have $\cos C = \dfrac{b^2 + c^2 - BD^2}{2bc}$.

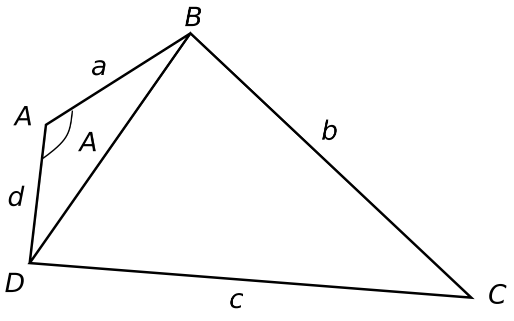

Figure 3

Substituting, we obtain $\cos C = \dfrac{b^2+c^2-a^2-d^2+2ad\cos A}{2bc}$ and taking that $0 \le C \le \pi$, it results that $-1 \le \cos C \le 1$,

or $-1 \le \dfrac{b^2+c^2-a^2-d^2+2ad\cos A}{2bc} \le 1$, from where

$$\alpha = \frac{a^2+d^2-(b+c)^2}{2ad} \le \cos A \le \frac{a^2+d^2-(b-c)^2}{2ad} = \beta. \tag{2}$$

We have that

$$\begin{aligned}\alpha &= 1 + \frac{(a-d)^2-(b+c)^2}{2ad} = 1 + \frac{(a-d-b-c)(a-d+b+c)}{2ad}\\ &= 1 - \frac{(-a+b+c+d)(a+b+c-d)}{2ad}\end{aligned}$$

and

$$\begin{aligned}\beta &= -1 + \frac{(a+d)^2-(b-c)^2}{2ad} = -1 + \frac{(a+d-b+c)(a+d+b-c)}{2ad}\\ &= -1 + \frac{(a-b+c+d)(a+b-c+d)}{2ad}.\end{aligned}$$

Taking into account the hypothesis in the theorem, it follows that $\alpha < 1$ and $\beta > -1$. Then, from (2) we have that

$$\max\{-1,\alpha\} \le \cos A \le \min\{1,\beta\}. \tag{3}$$

Because the function arccos is decreasing on $[-1,1]$, from (3) it follows (1).

Remark 4. Since $\alpha < \beta$, $\alpha < 1$, $\beta > -1$ and because $\max\{-1,\alpha\} < \min\{1,\beta\}$, it follows that the interval $\big[\max\{-1,\alpha\},\min\{1,\beta\}\big]$ is nonempty, from which it follows that $\big[\arccos\big(\min\{1,\beta\}\big),\arccos\big(\max\{-1,\alpha\}\big)\big]$ is an interval included in the interval $[-1,1]$. Then according to relation (1), there are an infinity of values of A for which there is a quadrilateral $ABCD$ from Theorem 4.

Corollary 1. *In conditions of Theorem* 4, *there exists a cyclyc quadrilateral* $ABCD$ *such that* $AB = a$, $BC = b$, $CD = c$, $DA = d$ *and* $\widehat{BAD} = A = \arccos \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)}$.

Proof. Taking Theorem 4 into account it is sufficient to show that

$$\arccos\big(\min\{1,\beta\}\big) \ < \ \arccos\frac{a^2+d^2-b^2-c^2}{2(ad+bc)} \ < \ \arccos\big(\max\{-1,\alpha\}\big),$$

equivalent to $\min\{1,\beta\} > \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)} > \max\{-1,\alpha\}$. The inequality $1 > \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)}$ is equivalent to $(b+c)^2-(a-d)^2 > 0$, equivalent to $(b+c-a+d)(b+c+a-d) > 0$, which is a true inequality. We have

$$\beta = \frac{a^2+d^2-(b-c)^2}{2ad} = \frac{a^2+d^2-b^2-c^2+2bc}{2ad} > \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$$

and then $\min\{1,\beta\} > \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)}$.
The inequality $\dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)} > -1$ is equivalent to $(a+d)^2-(b-c)^2>0$, equivalent to $(a+d-b+c)(a+d+b-c)>0$, which is a true inequality.
We have $\dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)} > \alpha = \dfrac{a^2+d^2-(b+c)^2}{2ad}$, equivalent to

$ad(a^2+d^2-b^2-c^2) > (ad+bc)(a^2+d^2-(b+c)^2)$, equivalent after calculus to $(b+c)^2-(a-d)^2>0$, equivalent to $(b+c-a+d)(b+c+a-d)>0$, which is a true inequality, so $\dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)} > \max(-1,\alpha)$. From the above it results that $\min\{1,\beta\} > \dfrac{a^2+d^2-b^2-c^2}{2(ad+bc)} > \max\{-1,\alpha\}$. Now we show that the quadrilateral $ABCD$ is cyclic.
In triangle ABD (see Fig. 3) we have

$$BD^2 = a^2+d^2-2ad\cos A = a^2+d^2-2ad\cos\left(\arccos\frac{a^2+d^2-b^2-c^2}{2(ad+bc)}\right)$$
$$= a^2+d^2-\frac{ad(a^2+d^2-b^2-c^2)}{ad+bc}.$$

In triangle BCD we have $\cos C = \dfrac{b^2+c^2-BD^2}{2BC}$ and replacing BD^2, we obtain

$$\cos C = \frac{b^2+c^2-a^2-d^2+\dfrac{ad(a^2+d^2-b^2-c^2)}{ad+bc}}{2bc} =$$
$$= \frac{(b^2+c^2-a^2-d^2)\left(1-\dfrac{ad}{ad+bc}\right)}{2bc} = -\frac{a^2+d^2-b^2-c^2}{2(ad+bc)},$$

so $\cos C = -\cos A$. Because $A\in(0,\pi)$ then $\pi-A\in(0,\pi)$ and we have $\cos C=\cos(\pi-A)$. Taking that $C,\pi-A\in(0,\pi)$ and the function cos is decreasing to $(0,\pi)$, it results that $C=\pi-A$, equivalent to $A+C=\pi$, so $ABCD$ is a cyclic quadrilateral.

We will determinate the position of α relative to -1 and β relative to 1. We have

$$\alpha-(-1)=\alpha+1=\frac{(a+d)^2-(b+c)^2}{2ad}=\frac{(a+d+b+c)(a+d-b-c)}{2ad},$$
so

$$\alpha \le -1 \iff a+d \le b+c, \tag{4}$$

$$\alpha > -1 \iff a+d > b+c, \tag{5}$$

with equality in (4) if and only if $a+d=b+c$.
On the other hand

$$\beta-1=\frac{(a-d)^2-(b-c)^2}{2ad}=\frac{(a-d+b-c)(a-d-b+c)}{2ad}, \text{ so}$$

$$\beta<1 \iff \left(\begin{cases} a+b<c+d \\ a+c>b+d\end{cases} \text{ or } \begin{cases} a+b>c+d \\ a+c<b+d\end{cases}\right), \tag{6}$$

$$\beta \geq 1 \Longleftrightarrow \left(\begin{cases} a+b \leq c+d \\ a+c \leq b+d \end{cases} \text{or} \quad \begin{cases} a+b \geq c+d \\ a+c \geq b+d \end{cases} \right), \tag{7}$$

with equality in (7) if and only if $\begin{cases} a+b=c+d \\ a+c=b+d \end{cases}$, equivalent to $\begin{cases} a=d \\ b=c \end{cases}$.

If $\alpha \leq -1$ and $\beta \geq 1$, taking (4) and (7) into account then

$$\begin{cases} a+d \leq b+c \\ a+b \leq c+d \\ a+c \leq b+d \end{cases} \tag{8}$$

or

$$\begin{cases} a+d \leq b+c \\ a+b \geq c+d \\ a+c \geq b+d. \end{cases} \tag{9}$$

From the first inequality from (8), by summing a and d to both members, we have $2a+d \leq a+b+c$ and $a+2d \leq b+c+d$.
Because $d < 2a+d$ and $a < a+2d$, from the above relations it follows that $d < a+b+c$ and $a < b+c+d$. By summing a in the second and in the third inequality we have $b < 2a+b \leq a+c+d$ and $c < 2a+c \leq a+b+d$, from where $b < a+c+d$ and $c < a+b+d$. So from (8) it results that $a < b+c+d$, $b < c+d+a$, $c < d+a+b$ and $d < a+b+c$. Similarly, it is shown that from (9) the above inequalities are obtained.

Remark 5. From (4) and (6) it does not follows that a, b, c, d can be the lengths of the sides a quadrilateral, for example $a = 2$, $b = 3$, $c = 9$ and $d = 3$. These values verify (4) and the first group of inequalities in (6), but $c > d + a + b$.
Similarly for (5) and (7).

Lemma 1. *If $a, b, c, d \in \mathbb{R}$, $a, b, c, d > 0$, $a < b+c+d$, $b < c+d+a$, $c < d+a+b$, $d < a+b+c$, then there exists a quadrilateral $ABCD$ such that $AB = a$, $BC = b$, $CD = c$ and $DA = d$. If in addition $a+d \leq b+c$, then $\widehat{BAD} < \pi$.*

Proof. We note $\widehat{BAD} = A$ and let the quadrilateral $ABCD$ (see Fig. 4).

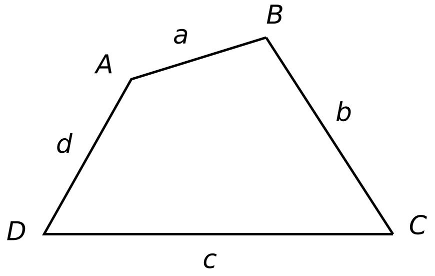

Figure 4

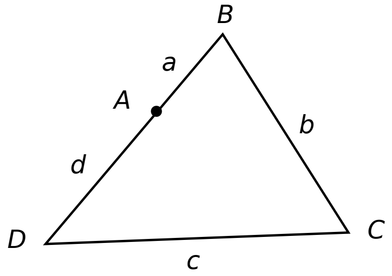

Figure 5

We "deform" this figure so that the points D, A, B are collinear (see Fig. 5). This is possible because, according to the hypothesis we have $a+d \leq b+c$. Then, the extreme value of A is π, so $A < \pi$.

Lemma 2. *If $a, b, c, d \in \mathbb{R}$, $a, b, c, d > 0$, $a < b+c+d$, $b < c+d+a$, $c < d+a+b$, $d < a+b+c$, then there exists a quadrilateral $ABCD$ such* *that* $AB = a$, $BC = b$, $CD = c$ *and* $DA = d$. *If in addition* $\begin{cases} a+b \leq c+d \\ a+c \leq b+d \end{cases}$ *or* $\begin{cases} a+b \geq c+d \\ a+c \geq b+d \end{cases}$, *then* $0 < \widehat{BAD}$.

Proof. If $\begin{cases} a+b \leq c+d \\ a+c \leq b+d \end{cases}$, by addition we have $a \leq d$.

If $a = d$, from the inequalities above we obtain that $\begin{cases} b \leq c \\ c \leq b \end{cases}$, so $b = c$. We deform the quadrilateral $ABCD$ so that A, B, C, D are collinear, with B and D coinciding (see Fig. 6).

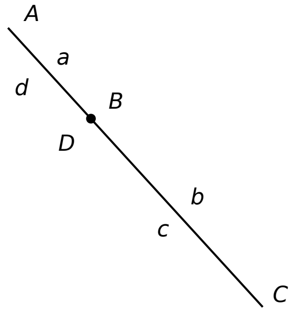

Figure 6

Now, we are discussing the case $a < d$ (see Fig. 7). Because $a \neq d$, at least one inequality $\begin{cases} a+b \leq c+d \\ a+c \leq b+d \end{cases}$ is strict, from where $b \leq c + (d-a)$, $c \leq b + (d-a)$ with at least one inequality is strict. From $d < a+b+c$ it results that $(d-a) < b+c$. We "deform" the quadrilateral $ABCD$ so that A, B, D are collinear (see Fig. 8) and this is possible because from the above inequalities there is triangle BDC, which can also be degenerate.

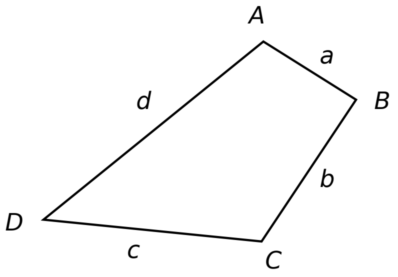

Figure 7

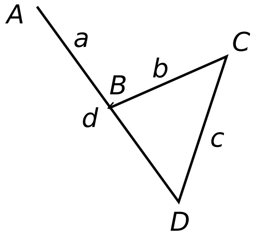

Figure 8

Then, the extreme value of A is 0, so $0 < A$.

Theorem 5. *Let* a, b, c, d *be real numbers such that* $a, b, c, d > 0$ *and the relationship* (8) *or* (9) *occurs. Then there exists a quadrilateral* $ABCD$ *such that* $AB = a$, $BC = b$, $CD = c$, $DA = d$ *and* $\widehat{BAD} = A$, *where* A *is arbitrary,* $A \in (0, \pi)$.

Proof. The proof from Theorem 4, Lemma 1 and Lemma 2.

Theorem 6. *Let* a, b, c, d *be real numbers such that* $a, b, c, d > 0$ *and one of the group of relationships*

$$\left\{\begin{array}{l} a+d>b+c \\ a+b<c+d \\ a+c>b+d \end{array}\right. \tag{10}$$

or

$$\left\{\begin{array}{l} a+d>b+c \\ a+b>c+d \\ a+c<b+d \end{array}\right. \tag{11}$$

take place. Then there exists a quadrilateral $ABCD$ *such that* $AB = a$, $BC = b$, $CD = c$, $DA = d$ *and* $\widehat{BAD} = A$, *where*

$$\arccos \frac{a^2+d^2-(b-c)^2}{2ad} \leq A \leq \arccos \frac{a^2+d^2-(b+c)^2}{2ad}. \tag{12}$$

Proof. It was shown above that from (8) and respectively (9) it follows that $a < b+c+d$, $b < c+d+a$, $c < d+a+b$ and $d < a+b+c$. Following the same idea, it is shown that from (10), respectively (11), the above inequalities result.
Taking (5), (6) and Theorem 4 into account, we obtain relation (12).

Theorem 7. *If* $a, b, c, d \in \mathbb{R}$, $a, b, c, d > 0$, $a < b+c+d$, $b < c+d+a$, $c < d+a+b$, $d < a+b+c$, *then there exists a quadrilateral* $ABCD$ *such that* $AB = a$, $BC = b$, $CD = c$ *and* $DA = d$. *If in addition take place inequalities in*

(*i*) (4) *and* (6), *then* $\arccos \dfrac{a^2+d^2-(b-c)^2}{2ad} \leq A < \pi$;

(*ii*) (5) *and* (7), *then* $0 < A \leq \arccos \dfrac{a^2+d^2-(b+c)^2}{2ad}$.

Proof. It results from Theorem 4 and the inequalities (4)–(7).

Remark 6. The problem of determining the quadrilateral can be also be posed under other conditions that depend on the number of given sides and angles of the quadrilateral.

References

[1] Pop, O.T., Minculete, N. and Bencze, M., *An introduction to quadrilateral geometry*, Ed. Didactică şi Pedagogică, Bucureşti, 2013.

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