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CONJUGATE COORDINATES AND APPLICATION TO TWO GEOMETRIC LOCI

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Abstract. In this article, after a preliminary review of the rudiments of *"conjugate coordinates"*, we apply this technique to study two geometric loci.

The first locus describes the points having a constant sum of distances from the sides of a triangle. It is proved that the locus, for a sufficiently large sum of distances, is a hexagon with parallel opposite sides, which is inscribed in an ellipse homothetic to the in-ellipse of the triangle with center at $X(37)$. For smaller values of this sum it is proved, that the locus has a certain relation to *"Bevan center"* $X(40)$ of the triangle.

The second locus considers the vectors realizing the distances of a point to its projections on the sides of the triangle. It is proved that the points for which the sum of these vectors has a constant measure are homothetic ellipses centered at the symmedian point $X(6)$ of the triangle, and their axes coincide with the asymptotes of the Jerabek hyperbola of the triangle.

1. Introduction

The *"conjugate coordinates"* (X, Y) of the plane are related to a positively oriented Cartesian coordinate system (x, y) and denote the conjugate complex numbers $X = x + iy$ and $Y = \overline{X} = x - iy$. Every curve represented through a function $f(x, y) = 0$ generates a corresponding repre-

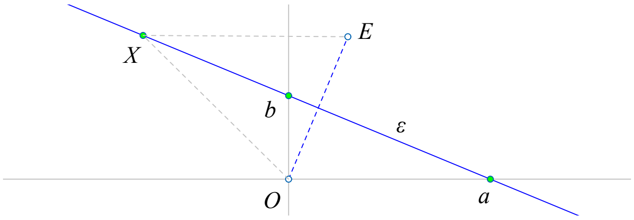

Figure 1. The *"reflex point"* E of the line ε

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sentation in these coordinates $F(X,Y)=0$, through the substitution in f of $x=\frac{1}{2}(X+Y)$ and $y=\frac{1}{2i}(X-Y)$. As an example, let $\varepsilon : \frac{x}{a}+\frac{y}{b}=1$, be a line (see Figure 1) not passing through the origin O. Under the aforementioned substitution, the equation transforms to

$$(1) \qquad \frac{X}{E}+\frac{Y}{\overline{E}} = 1 \quad \text{with} \quad E=\frac{2ab}{a^2+b^2}(b+ia) \ .$$

The point E is called *"reflex point"* of the line ε, and is indeed the reflection in the line of the origin O.

Conjugate coordinates were introduced by Frank Morley [1], who called them *"circular coordinates"* and used them in the context of replacing the real parameter of a rational curve of the plane with a complex unit number $t \in S^1$. Thus, he represented the curve as a map $f: \ S^1 \longrightarrow \mathbb{R}^2$, $X=f(t)$ of the unit circle S^1 into the plane. The unit numbers were called *"turns"* and f was given the name of a *"map equation"* of the corresponding rational curve.

In the case of the preceding example of a line (see Figure 1), a point $X \in \varepsilon$ is equidistant from O and E, consequently, there is a unit complex number t (a turn), producing a *"map equation"* of the line [2, p.152]:

$$(2) \qquad tX=(X-E) \quad \Rightarrow \quad X \ = \ \frac{E}{1-t} \quad \text{for} \quad t \in S^1 \ .$$

The present work was motivated by the study of papers [3], [4], [5], [6], [7], [8], [9], [10], [11], in which the authors used this technique in order to establish metric properties of triangles and curves related to triangles. For this they used triangles ABC with vertices at points of the unit circle $\{t_1,t_2,t_3\} \subset S^1$ (see Figure 2). It is easily seen that these unit complex numbers satisfy the cubic equation

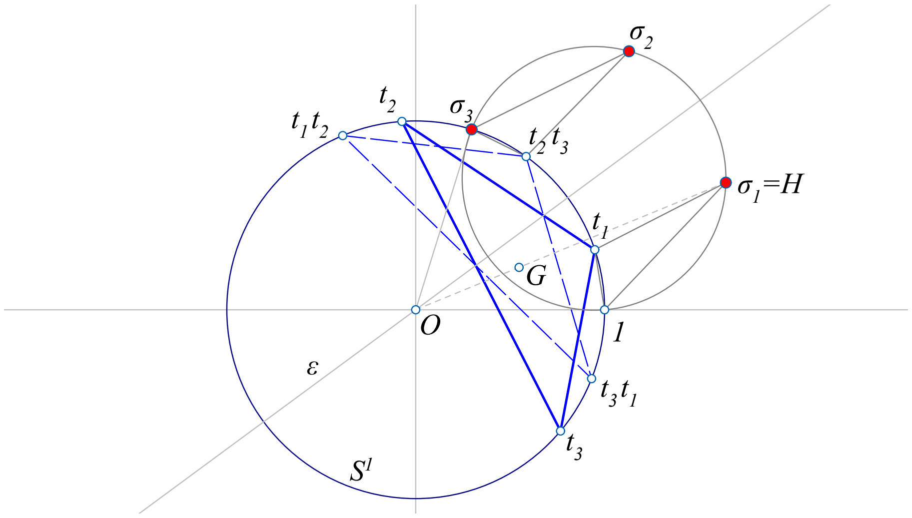


FIGURE 2. Triangle $A(t_1)B(t_2)C(t_3)$ and $\{\sigma_1 \ , \ \sigma_2 \ , \ \sigma_3\}$

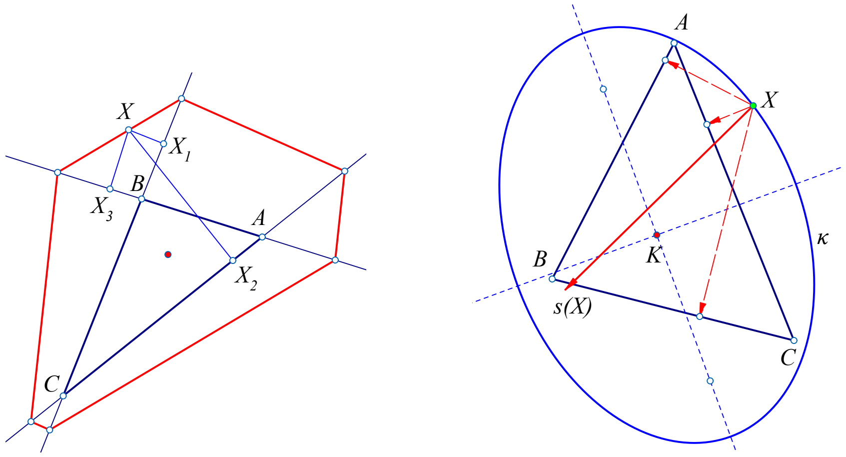
$$(3) \qquad t^3-\sigma_1 t^2+\sigma_2 t-\sigma_3 \ = \ 0 \ ,$$

$$(4) \qquad \text{with} \quad \sigma_1=t_1+t_2+t_3 \quad , \quad \sigma_2=t_1t_2+t_2t_3+t_3t_1 \quad , \quad \sigma_3=t_1t_2t_3 \ .$$

The figure suggests some, easy to prove, relations which is useful to have in mind: (i) The triangle with vertices $\{t_2t_3, t_3t_1, t_1t_2\}$ is the reflection of triangle $\{t_1t_2t_3\}$ in line ε, bisecting the angle $\widehat{1, O, \sigma_3}$. (ii) $\{\sigma_2\}$ is the reflection of σ_1 in ε. These properties result mainly from the fact that unit complex numbers are a closed set under multiplication, negation, rise to rational powers and conjugation, latter having $\overline{t} = \frac{1}{t}$.

In particular, we notice that $\sigma_1/3 = (t_1 + t_2 + t_3)/3$ is the centroid G, and $\sigma_1 = 3G$ is the orthocenter of the triangle. Consequently the segment (line) $[\sigma_1, t_1]$ is orthogonal to the side $[t_2, t_3]$, with analogous properties holding for the other vertices. By the aforementioned reflective symmetry w.r.t. the line ε, point σ_2 is the orthocenter of the triangle with vertices $\{t_2t_3, t_3t_1, t_1t_2\}$ and the segment (line) $[\sigma_2, t_2t_3]$ is orthogonal to the side $[t_3t_1, t_1t_2]$, with analogous properties holding for t_3t_1 and t_1t_2.

Figure 3 presents the subjects of our study. They are two geometric loci connected with the sum of distances of a point X from the sides of the triangle. The first (3-(1)) shows a locus of points X such that the sum of the distances from the sides $\sum |XX_i| = k$ is constant. The second (3-(2)) shows a locus of points X such that the sum of the vectors realizing the distances from the sides has a constant length, i.e. the vector $s(X) = \sum XX_i$ has a constant measure $|s(X)| = |\sum XX_i| = k$.



(1) Locus for $\sum |XX_i| = const.$ (2) Locus for $|\sum XX_i| = const.$

Figure 3. Geometric loci of distance-sums from the sides

Regarding the first problem, Elias Abboud, working with linear programming in the context of Viviani's theorem, showed in [12], among other important results, that the triangle can be divided in parallel segments, along which the sum of distances $\sum |XX_i|$ is constant. This is trivially true for points X in the inner domain of the triangle (remark 3.1) but not immediately seen for points X lying on the outer domains defined by the sides of the triangle. Obviously the sides of the polygonal locus in our figure 3-(1) are particular cases of his. Our treatment establishes a global view of the subject using geometric tools to depict the directions of these segments and their relations to the geometry of the triangle of reference.

Regarding the second problem, I should also mention a work by Abboud, in which he proves a result for a similar problem. In [13] namely, he proves a standard result, that the locus of points, for which the sum $\sum |XX_i|^2$ is constant, is an ellipse. This follows from the fact that $|XX_i|^2$ is the square of a linear function representing the sideline of the triangle. This is also quite different from our second problem, which deals with the *vectorial* sum and the loci of points satisfying the condition $|s(X)| = |\sum XX_i| = k$ is constant. Here again we explore the inner workings of the corresponding configurations and show that these loci, for various constants $\{k\}$, are homothetic ellipses centered at the *symmedian* point K of the triangle of reference. In figure 3-(2) the vector $s(X)$ has constant length, as point X glides on the ellipse κ. We show also that these ellipses bear a certain relation to the *Jerabek hyperbola* ([14]) of the triangle. The two problems are discussed below, after a review of the conjugate system of coordinates.

2. The conjugate system of coordinates

After the definition and the remarks, made in the introduction, we review some elementary properties needed in the work to follow. An extensive discussion with proofs of all properties listed below, as well as interesting applications, can be found in the article by Carver [4]. To familiarize us with the concepts, let us determine the line representation in the case the line ε passes through the origin.

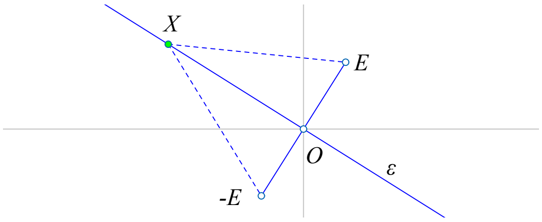


Figure 4. Map equation of line through O : $X = E\frac{1+t}{1-t}$

To find the map equation for a line ε through O, take an arbitrary point E on the line through O, which is orthogonal to ε, and consider ε as the medial line of the segment EE' with $E' = -E$, the reflected of E in ε (see Figure 4). For every point X on ε there is a turn $t \in S^1$ such that

$$(X - E) = t(X + E) \quad \Leftrightarrow \quad X = E\frac{1+t}{1-t}\,. \tag{5}$$

In particular, for points x of the x-axis we can take $E = i$ and the map equation becomes

$$x = i\frac{1+t}{1-t}\,. \tag{6}$$

For a parameterized curve $f(x)$ of the plane, replacing x with the right side $x(t)$ of this formula, we get a map equation $X = F(t) = f(x(t))$ of the curve. Eliminating t from $X = F(t)$ and its conjugate $Y = \overline{F(t)}$ we obtain the so called *"self-conjugate"* equation of the curve in the form $F_c(X,Y) = 0$.

As an example, the line through O, expressed through equation (5), has conjugate

$$(7) \qquad Y = \overline{X} = \overline{E}\frac{1+\frac{1}{t}}{1-\frac{1}{t}} = -\overline{E}\frac{1+t}{1-t} \quad \Rightarrow \quad \frac{X}{E} + \frac{Y}{\overline{E}} = 0 \ .$$

The unit complex number (turn) $s = \overline{E}/E \in S^1$ is called *"clinant"* of the line, and using it, equations (1) and (7) take respectively the form:

$$(8) \qquad sX + Y = sE \quad \text{and} \quad sX + Y = 0 \ .$$

Obviously parallel lines are characterized by the same clinant. This is the analogous of the *"slope"* of a line in a Cartesian system of coordinates. Figure 5 shows the relation of the clinant s to the reflex point E of the line ε. Clearly the reflex point E uniquely determines the line. The clinant s determines up to sign the normal of the line, consequently its direction and the line itself, if the line passes through the origin. The following properties are immediate consequences of the definition.

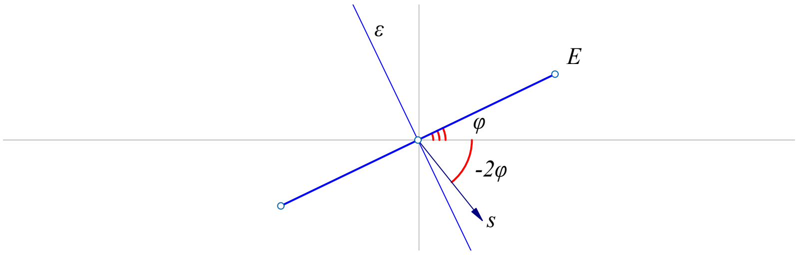


FIGURE 5. The clinant $s \in S^1$ of the line ε and its parallels

Lemma 2.1.
1. *Parallel lines are characterized by the same clinant.*
2. *Orthogonal lines are characterized by opposite clinants* $s' = -s$.
3. *The clinants of the bisectors of two lines with clinants* $\{s, s'\}$ *are* $\{\pm\sqrt{ss'}\}$.
4. *If the lines* $\{\varepsilon, \varepsilon'\}$ *have the clinants* $\{s, s'\}$, *then the angle* ψ *of the lines satisfies*

$$(9) \qquad e^{i\psi} = \sqrt{\frac{s}{s'}} \ .$$

5. *The two numbers* $\{s_{12}, s'_{12} = \pm\sqrt{t_1 t_2}\}$, *for* $\{t_1, t_2 \in S^1\}$, *are diametral points on the medial line of the chord* $[t_1, t_2]$ *of* S^1 *(Figure 6).*

The following properties ([4]) are also easily verified.

Lemma 2.2. (1) *The collinearity of three points* $(X, Y), (X', Y')$ *and* (X'', Y''), *equivalently, the line through points* $\{(X', Y'), (X'', Y'')\}$ *is expressed through the vanishing of the determinant:*

$$(10) \qquad \begin{vmatrix} X & Y & 1 \\ X' & Y' & 1 \\ X'' & Y'' & 1 \end{vmatrix} = 0 \ .$$

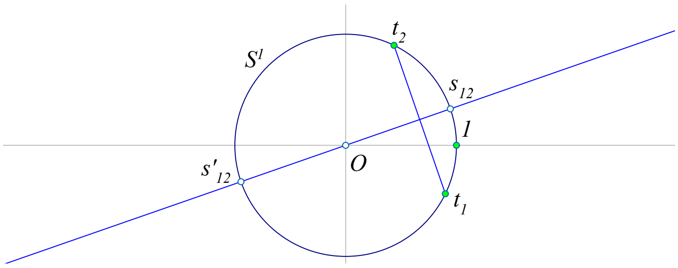


Figure 6. Meaning of $\{s_{12}, s'_{12} = \pm\sqrt{t_1t_2}\}$

(2) *The line through points* $\{(X', Y'), (X'', Y'')\}$ *is represented through equation:*

$$(Y' - Y'')X - (X' - X'')Y + X'Y'' - X''Y' = 0$$

with clinant: $s = -\dfrac{Y' - Y''}{X' - X''}$, *and reflex point:* $E = -\dfrac{X'Y'' - X''Y'}{Y' - Y''}$.

(3) *The line with clinant* s*, passing through* (X_1, Y_1) *is given by:*

$$sX + Y = sX_1 + Y_1 \ . \tag{11}$$

(4) *The line through* (X_1, Y_1) *orthogonal to line* $\varepsilon : \ sX + Y = sE$ *is given by:*

$$-sX + Y = -sX_1 + Y_1 \ . \tag{12}$$

(5) *The medial line of the segment with endpoints* $\{(X_1, Y_1), (X_2, Y_2)\}$ *is given by:*

$$sX + Y = s\frac{X_1 + X_2}{2} + \frac{Y_1 + Y_2}{2}, \quad \text{with clinant: } s = \frac{Y_1 - Y_2}{X_1 - X_2} \ . \tag{13}$$

(6) *The line through two turns* $\{t_1, t_2\}$ *has clinant* $\frac{1}{t_1t_2}$ *and is*

$$\frac{1}{t_1t_2}X + Y = \frac{t_1 + t_2}{t_1t_2} \ . \tag{14}$$

(7) *The intersection of lines* $\{s_1X + Y = s_1E_1 \ , \ s_2X + Y = s_2E_2\}$

$$(X_0, Y_0) = \frac{1}{s_1 - s_2}(\ s_1E_1 - s_2E_2 \ , \ -s_1s_2(E_1 - E_2)\ ) \ . \tag{15}$$

(8) *Points* $\{(X_1, Y_1), (X_2, Y_2)\}$ *are related by the reflection in the line* $\varepsilon : \ sX + Y = sE$ *if they satisfy:*

$$sX_1 + Y_2 = sE \ . \tag{16}$$

(9) *The reflected of the point* (X_1, Y_1) *in the line* $\varepsilon : \ sX + Y = sE$ *is the point* (X_2, Y_2) *with*

$$Y_2 = s(E - X_1) \quad \Rightarrow \quad X_2 = \overline{Y_2} = \overline{s}(\overline{E} - Y_1) \ . \tag{17}$$

(10) *The projection of the point* (X_1, Y_1) *on the line* $\varepsilon : \ sX + Y = sE$ *is the point*

$$(X_2, Y_2) \ = \ \frac{1}{2}(X_1 + \overline{s}(\overline{E} - Y_1) \ , \ Y_1 + s(E - X_1)) \ . \tag{18}$$

The Wallace-Simson line (WS line) of the point $t \in S^1$ w.r.t. the triangle $t_1t_2t_3$ inscribed in S^1 is the line passing through the projections of P on the sides of the triangle.

Lemma 2.3. *The WS line of t relative to the triangle $t_1t_2t_3$ for points (turns) $\{t, t_1, t_2, t_3 \in S^1\}$ is represented in complex conjugate coordinates through the self conjugate equation ([4, p.84]):*

$$-\frac{t}{\sigma_3}X + Y = \frac{1}{2t} + \frac{\sigma_2}{2\sigma_3} - \frac{\sigma_1}{2\sigma_3}t - \frac{1}{2\sigma_3}t^2 \quad \Leftrightarrow \tag{19}$$

$$-\frac{t}{\sigma_3}X + Y = -\frac{t}{\sigma_3}E \quad \textit{with} \quad E = \frac{t^3 + \sigma_1 t^2 - \sigma_2 t - \sigma_3}{2t^2}\,. \tag{20}$$

Remark 2.1. *Below we'll work with triangles $\{A(t_1), B(t_2), C(t_3)\}$ inscribed in the unit circle S^1 with vertices at the points $\{t_1, t_2, t_3\}$ of S^1. This does not mean any loss of generality, since the properties we are interested in, are invariant under similarities. Thus, using a homothety mapping the circumcircle of the arbitrary triangle onto S^1, we can reduce the arbitrary configuration to a more convenient for the calculations involved. In fact, we can also turn the triangle inscribed in S^1 about its circumcenter to a new position, so that the corresponding constant $\sigma_3 = t_1t_2t_3$ coincides with the unit $1 \in S^1$. This simplifies further the calculations and the corresponding formulas. Figure 7 shows a triangle brought to this, so to say, "normal position", according to the preceding clarification. The fact that in this case, with $\sigma_3 = 1$, the quantities $\{\sigma_1, \sigma_2\}$ lie symmetrically w.r.t. to the x-axis, thus representing two complex conjugate numbers, follows as special case of the axial symmetry noticed in the remarks accompanying figure 2. This remark implies also the following properties (see Figure 7):*

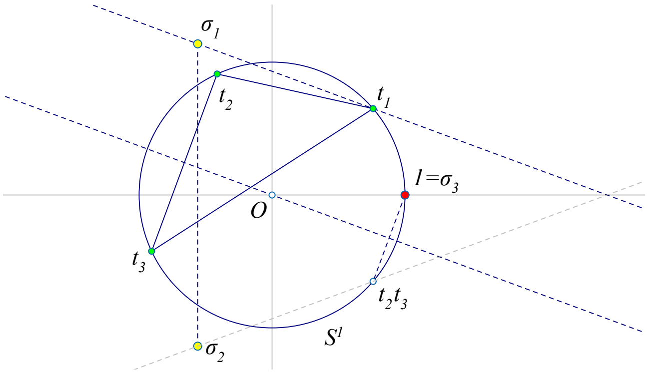


Figure 7. Triangle brought to *"normal position"*

Lemma 2.4. *For a triangle $\{t_1t_2t_3\}$ inscribed in the unit circle S^1 with $\sigma_3 = t_1t_2t_3 = 1$, with fixed the position of t_1 and varying the position of $t_2 \in S^1$ hold the following properties:*

1. *$t_2t_3 \in S^1$ remains constant and is the reflection of t_1 on the x-axis.*
2. *Line $[t_2, t_3]$ varies remaining parallel to the segment $[1, t_2t_3]$.*
3. *The orthocenter $\sigma_1 = t_1 + t_2 + t_3$ of the triangle $t_1t_2t_3$ varies on the line through t_1 which is parallel to the medial line of the segment $[1, t_2t_3]$.*

3. THE SUM OF DISTANCES

The sum of distances of a point X from the sides of the triangle ABC is half the sum of distances of X from its reflections on the sides. For pairs of indices (i,j) taking the values $\{(1,2),(2,3),(1,2)\}$, the side $[t_i,t_j]$ is represented through

$$\frac{X}{t_i+t_j}+\frac{Y}{\frac{1}{t_i}+\frac{1}{t_j}}=1 \quad \Leftrightarrow \quad X+Yt_it_j-(t_i+t_j)=0\ , \tag{21}$$

the reflected X_{ij} of X on that side satisfies, for $Y_{ij}=\overline{X}_{ij}$ (lemma 2.2-8):

$$X+Y_{ij}t_it_j-(t_i+t_j)=0 \quad \Rightarrow$$

$$Y_{ij}=\frac{(t_i+t_j)-X}{t_it_j} \quad \Rightarrow \quad X_{ij}=t_i+t_j-Yt_it_j \quad \Rightarrow$$

$$X_{ij}-X \ = \ (t_i+t_j-Yt_it_j-X) \ = \ k_{ij}\sqrt{t_it_j} \quad \text{with} \quad k_{ij}\in\mathbb{R}\ . \tag{22}$$

Thus, with a proper choice s_{ij} of one of the two roots $\sqrt{t_it_j}$ of t_it_j, which is a complex unit, the sum of distances $s(X)$ of X from the sides of the triangle is

(23)

$$s(X)=\frac{1}{2}\sum k_{ij} \ = \ \frac{1}{2}\sum\frac{t_i+t_j-Yt_it_j-X}{\sqrt{t_it_j}} \ = \ a-bY-cX \quad \text{with}$$

$$a=\frac{1}{2}\sum\frac{t_i+t_j}{\sqrt{t_it_j}}\ , \quad b=\frac{1}{2}\sum\sqrt{t_it_j}\ , \quad c=\frac{1}{2}\sum\frac{1}{\sqrt{t_it_j}}\ . \tag{24}$$

Suppose now that X varies on a line $X=\frac{E}{1-t}$, $t\in S^1$ and seek those E for which the corresponding sum of distances $s(X)$ remains constant along the line:

$$X=\frac{E}{1-t} \quad \Rightarrow \quad Y=-\frac{\overline{E}t}{1-t} \quad \Rightarrow$$

$$s(X)=s\left(\frac{E}{1-t}\right)=a+b\frac{\overline{E}t}{1-t}-c\frac{E}{1-t}=$$

$$a+\frac{b\overline{E}t-cE}{1-t}=a+\frac{b\overline{E}t-b\overline{E}+b\overline{E}-cE}{1-t}=a-b\overline{E}+\frac{b\overline{E}-cE}{1-t}\ .$$

Requiring from $s(X)$ to be constant along this line, and equal to k_E say, implies the vanishing $b\overline{E}-cE=0$, and the equality for the constant: $k_E=a-b\overline{E}$. The first equation, since $\overline{b}=c$, implies that $b\overline{E}=cE=r\in\mathbb{R}$ is real. From this and the second equation we obtain

$$k_E=a-r \ \Leftrightarrow \ k_E=\sum\frac{t_i+t_j}{2\sqrt{t_it_j}}-r=\frac{1}{2}\sum\left(\sqrt{\frac{t_i}{t_j}}+\sqrt{\frac{t_j}{t_i}}\right)-r. \tag{25}$$

Notice that the sum consists of real numbers, since each parenthesis is the sum of a complex unit and its conjugate. From the last equation we see that

$$E=\frac{r}{c}=\frac{a-k_E}{c}=\frac{\frac{1}{2}\sum\frac{t_i+t_j}{\sqrt{t_it_j}}-k_E}{\frac{1}{2}\sum\frac{1}{\sqrt{t_it_j}}}\ . \tag{26}$$

Lemma 3.1. *For a proper choice* s_{ij} *of one of the roots* $\{\sqrt{t_i t_j}\}$, *the vector* E *of equation* (26) *is a multiple of the orthocenter* $H = \sum s_{ij}$ *of the triangle with vertices* $\{s_{12}, s_{23}, s_{31}\}$ *and the lines along which the sum* $s(X) = k_E$ *is constant are orthogonal to the direction of* E.

Proof. The claim about the orthocenter follows from a short calculation, showing that the following equation has always a real solution: $E = \lambda H =$

$$\lambda \sum s_{ij} = \lambda \sum \sqrt{t_i t_j} \quad \Leftrightarrow \quad \frac{\frac{1}{2}\sum \frac{t_i+t_j}{\sqrt{t_i t_j}} - k_E}{\frac{1}{2}\sum \frac{1}{\sqrt{t_i t_j}}} = \lambda \sum \sqrt{t_i t_j} \quad \Leftrightarrow$$

$$\frac{1}{2}\sum \frac{t_i + t_j}{\sqrt{t_i t_j}} - k_E = \lambda \left(\sum \sqrt{t_i t_j}\right)\left(\frac{1}{2}\sum \frac{1}{\sqrt{t_i t_j}}\right) = \frac{\lambda}{2}\left|\sum \sqrt{t_i t_j}\right|^2 .$$

Since the first quantity from the left on the last line is a real number, λ is a real positive or negative number.

The second claim follows from the fact that the line under consideration is, per definition, orthogonal to its reflex vector E.

Figure 8 shows the geometric locus of points X whose sum of distances from the sides of the triangle $A(t_1)B(t_2)C(t_3)$ is a constant k, such that no interior point of the triangle realizes this sum. It is a hexagon with vertices on the sidelines of the triangle and parallel opposite sides. Its sides are segments of lines orthogonal to orthocenter vectors H of triangles with vertices appropriate quadratic roots $\{s_{ij} = \sqrt{t_i t_j}\}$. For each pair (i, j) there are two such (opposite to each other) roots and the selection of the appropriate one is dictated by the requirement to produce positive values for the constants

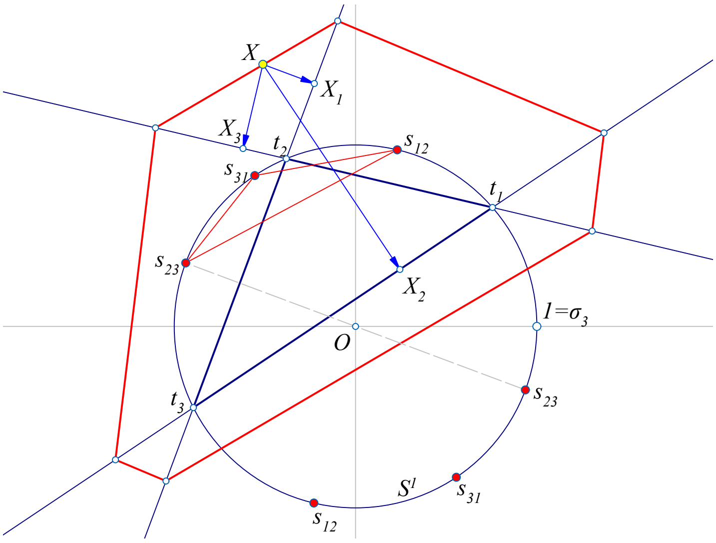


FIGURE 8. $X : \ |XX_1| + |XX_2| + |XX_3| = k : $ constant

$\{k_{ij}\}$ in equations (22). These, in turn, depend on the location of X in one of the seven regions defined by the side-lines of the triangle ABC. We notice that the direction of E, and consequently of the sides of the polygon does not depend on the constant k_E.

Figure 9 shows another instance (for another value of $k = |s(X)|$), in which the locus has points in the interior of the triangle and the polygon becomes a quadrilateral with one side running inside the triangle. In the figure appears also the appropriate triangle $\{s_{12}s_{23}s_{31}\}$ and the direction OH to its orthocenter, to which this side is orthogonal. We recapitulate the preceding discussion in the following theorem.

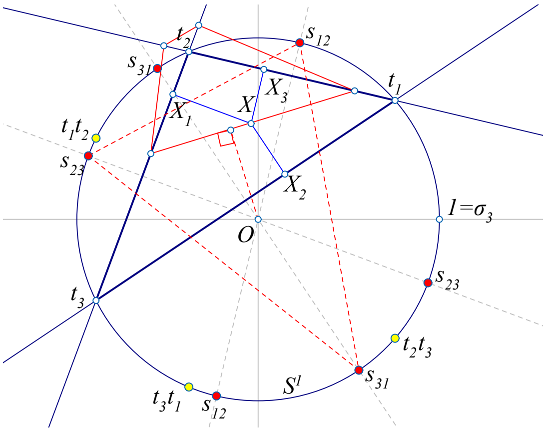


FIGURE 9. X with $|XX_1| + |XX_2| + |XX_3| = k$: constant

Theorem 3.1. *The geometric locus of points X, for which the sum of the distances $s(X) = \sum |XX_i|$ from the sides of the triangle $A(t_1)B(t_2)C(t_3)$ inscribed in the unit circle S^1 is equal to a constant k, is a polygon with vertices at the sidelines of the triangle. The direction of each side of this polygon is orthogonal to the vector OH_i, where H_i is the orthocenter of a triangle with vertices an appropriate selection of the double-valued roots $\{\sqrt{t_1t_2}, \sqrt{t_2t_3}, \sqrt{t_3t_1}\}$. Latter are the intersection points of the circumcircle S^1 with the diameters orthogonal to the sides of the triangle.*

Figure 10 illustrates the preceding discussion showing also the configuration for the generic triangle ABC, resulting from the corresponding similar triangle inscribed in the unit circle. We use again the symbols $\{s_{12}, s_{23}, s_{31}, s'_{12}, s'_{23}, s'_{31}\}$ to denote the points on its circumcircle, which are defined by diameters orthogonal to the sides and correspond to the roots $\{\sqrt{t_it_j}\}$ on the unit circle. They define four independent triangles and their orthocenters $\{H_0, H_1, H_2, H_3\}$ plus four other triangles which are the symmetric to the indicated in the figure w.r.t. to the circumcenter O.

The vertices $\{s_{12}, s_{23}, s_{31}\}$ of one of the triangles, which we call *"central"* (in the figure the one containing the circumcenter O) are selected in the following way. s_{23} lies on the medial line of the side BC and on the same side of BC with the point A. Analogously s_{31} is on the same side of CA with B and s_{12} on the same side of AB with C.

The points $\{s'_{12}, s'_{23}, s'_{31}\}$ are diametral to the preceding ones. Each of them, together with a side of the central, defines a triangle, which we call a *"flanc"* of the central triangle. The figure shows the three flanks $\{s'_{23}s_{12}s_{31},\ s'_{31}s_{23}s_{12},\ s'_{12}s_{31}s_{23}\}$, the four orthocenters $\{H_0, H_1, H_2, H_3\}$ of the central triangle and its flanks, and the directions $\{v_0, v_1, v_2, v_3\}$ which are orthogonal to the orthocenters. These are the directions of line segments along which the sum $s(X)=\sum |XX_i|$ can be constant.

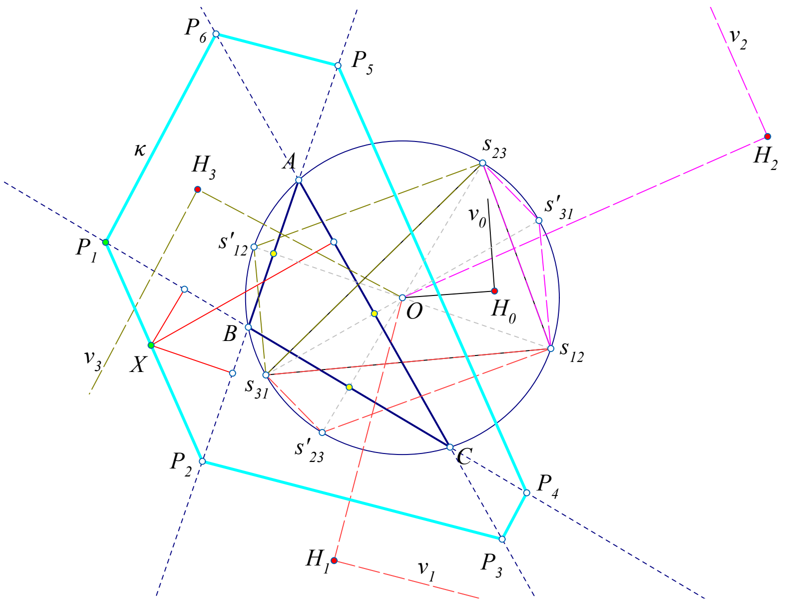


FIGURE 10. Points X with constant $s(X)=\sum |XX_i|$

The symmetry in O defines four congruent to the former triangles with corresponding orthocenters lying symmetrically w.r.t. to O and orthogonal to them directions which are correspondingly the same with the former. From this symmetry follows also the parallelism of the opposite sides of the hexagon κ shown, which describes a locus of points $s(X)=k$ not containing inner points of the triangle ABC.

Figure 11 illustrates an other aspect of the preceding figure, showing the central triangle relabeled $A'B'C'$, its symmetric $A''B''C''$ in O and the pairwise symmetrically lying flanks of the two triangles. The following theorem lists some of the properties of this figure, which are easy to prove and will be discussed in a forthcoming note.

Theorem 3.2. *With the definitions and notation of this section, the following are valid properties.*

(1) *The orthocenter H_0 of the central triangle $A'B'C'$ is the "Bevan center" $X(40)$ of the triangle of reference ABC, and the orthocenter H_0' of its symmetric triangle $A''B''C''$ is the incenter of ABC. Thus, the Incenter and the Bevan center are symmetric w.r.t. the circumcenter.*

(2) *The lines $\{AA', BB', CC'\}$ define the triangle $A_1B_1C_1$ coinciding with the excentral (or tritangent) triangle of ABC, the three points $\{A_1, B_1, C_1\}$ being the excenters of ABC. The lines AA_1, BB_1 and CC_1 are altitudes of $A_1B_1C_1$, the triangle ABC being its orthic. The orthocenter of $A_1B_1C_1$ coincides with H_0' and its Euler circle coincides with the circumcircle of ABC.*

(3) *The area of the central triangle $A'B'C'$ is equal to the sum of areas of its flanks.*

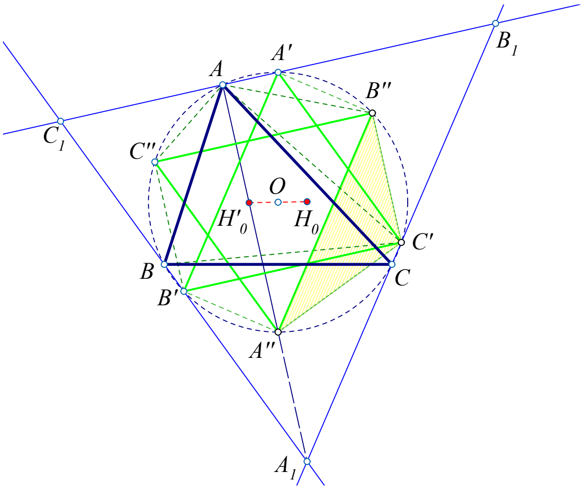


FIGURE 11. Triangles $\{A'B'C', A''B''C''\}$ and their flanks

Corollary 3.1. *The points X of a segment PQ, running inside the triangle ABC, have a constant sum $s(X) = \sum |XX_i|$ of distances from the sides of the triangle, if and only if PQ is orthogonal to the line $OX(40)$ passing through the circumcenter O and the Bevan center $X(40)$ of the triangle (see Figure 12).*

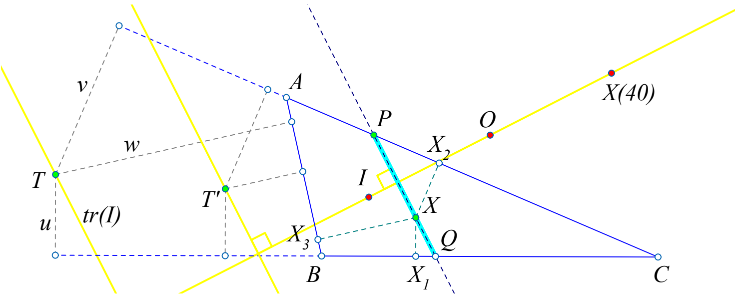


FIGURE 12. $PQ \ni X \;\Rightarrow\; \sum |XX_i|$: constant

Remark 3.1. *It is easily seen that the "trilinear polar" $tr(I)$ of the incenter is orthogonal to the line OI containing the Bevan center. Thus, these segments of constancy PQ, are parallel to the trilinear polar $tr(I)$ of the incenter. Notice that the points T of $tr(I)$ expressed in trilinear coordinates $(u : v : w)$ w.r.t. ABC satisfy the equation $u + v + w = 0$ and the points T', on a line parallel to $tr(I)$, satisfy $u + v + w = t'$ for a constant t'. While the distances $\{|XX_i|\}$ are non-negative, the trilinear coordinates $(u : v : w)$ coincide only in absolute value with the $\{|XX_i|\}$, being strictly positive and coinciding with the $\{|XX_i|\}$ only for points X of the inner domain of the triangle. In other words, the segments of constancy PQ are parts of parallels to the trilinear polar $tr(I)$ of the incenter, falling inside the inner domain of the triangle.*

In view of this remark, we could handle the case of inner points of the triangle in a simpler way, using the properties of $tr(I)$. Our treatment however is more general, handling the properties of points in a unified way, independently of their location relative to the triangle of reference.

4. Ellipse circumscribing the polygon

The polygon in figure 10, describing the locus of points X with a constant sum $s(X) = \sum |XX_i| = k$, for a sufficiently large k, is a hexagon, with parallel opposite sides, which intersect at the line at infinity. By the inverse of Pascal's *"hexagrammum mysticum theorem"* [15, p.65], there is a conic κ circumscribing this hexagon. The following propositions relate this conic to the geometry of the triangle of reference ABC.

First, we prove that the conic is a central one and locate its center. The existence of the center follows from the fact that the hexagon has parallel sides. Hence joining the middles of two such sides we obtain the diameter of the circumscribing conic conjugate to the direction of the parallel sides. The center will occur at the intersection of two such diameters. The parallelism of these sides implies also that the diameter will pass through one vertex of the triangle of reference ABC. Figure 13 shows two such diameters along the lines $\{AM, BN\}$, points $\{M, N\}$ being the middles of the sides $\{P_2P_3, P_4P_5\}$ of the hexagon $p = P_1..P_6$.

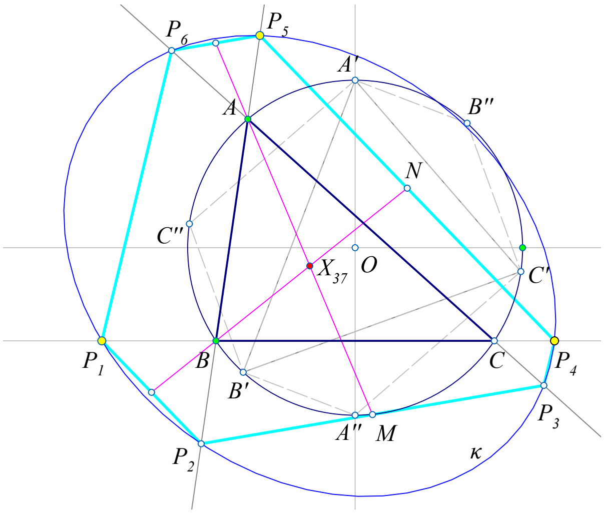


Figure 13. Conic circumscribing the polygon $p = P_1..P_6$

We'll work with Cartesian coordinates adapted to the triangle of reference and such that the coordinates of the points are respectively

$$A(a_1, a_2)\ ,\quad B(b_1, b_2)\ ,\quad C(-b_1, b_2),$$

thus, line BC being parallel to the x-axis, the y-axis being the medial line of BC. The Cartesian coordinates of the vertices $\{P_i\}$ result from the requirement to their distances from the sides of ABC to have the constant sum k. Thus, for P_1, denoting its projections on AC and AB respectively by X_2 and X_3, we have

$$P_1X_2 = P_1C\sin(\widehat{C}) \ , \quad P_1X_3 = P_1B\sin(\widehat{B}) \quad \text{with} \quad P_1X_2 + P_1X_3 = k$$
$$\Leftrightarrow \quad P_1C\sin(\widehat{C}) + P_1B\sin(\widehat{B}) = k \quad \text{and} \quad P_1C - P_1B = a \ ,$$

where $\{a = |BC|, b = |CA|, c = |AB|\}$ denote the side-lengths of the triangle. Solving the two last equations w.r.t. to $\{P_1B, P_1C\}$ we get

$$\begin{cases} P_1B = \dfrac{k - a\sin(\widehat{C})}{\sin(\widehat{B}) + \sin(\widehat{C})} = \dfrac{2Rk - ac}{b+c} \ , \\ P_1C = \dfrac{k + a\sin(\widehat{B})}{\sin(\widehat{B}) + \sin(\widehat{C})} = \dfrac{2Rk + ab}{b+c} \ , \end{cases} \tag{27}$$

vvv v bwhere R denotes the circumradius of the triangle ABC. From these, writing P_1 in the form $P_1 = (1-\lambda)B + \lambda C$, we obtain the coordinates of P_1, and analogously working, also the coordinates of the other vertices of the hexagon:

$$P_1 = \frac{1}{a(b+c)}\left(2Rk(B-C) + a(bB+cC)\right) \ , \tag{28}$$
$$P_2 = \frac{1}{c(a+b)}\left(2Rk(B-A) + c(aA+bB)\right) \ , \tag{29}$$
$$P_3 = \frac{1}{b(a+c)}\left(2Rk(C-A) + b(aA+cC)\right) \ , \tag{30}$$
$$P_4 = \frac{1}{a(b+c)}\left(2Rk(C-B) + a(bB+cC)\right) \ , \tag{31}$$
$$P_5 = \frac{1}{c(a+b)}\left(2Rk(A-B) + c(aA+bB)\right) \ , \tag{32}$$
$$P_6 = \frac{1}{b(a+c)}\left(2Rk(A-C) + b(aA+cC)\right) \ . \tag{33}$$

Next step involves a tedious calculation of the lines $\{AM, BN\}$ and their intersection X, which I omit and write only the result:

$$X = \frac{1}{2(bc+ca+ab)}\left(a(b+c)A + b(c+a)B + c(a+b)C\right) \ . \tag{34}$$

This identifies X with the triangle center $X(37)$, which expressed in barycentrics ([16]), has precisely the coordinates $(a(b+c) : b(c+a) : c(a+b))$. Thus, we arrive at the following theorem.

Theorem 4.1. *With the definitions and notation of this section, the center of the conic κ circumscribing the hexagon $p = P_1..P_6$ is the center $X(37)$ of the triangle ABC.*

The conics κ depend on the constant k creating the hexagonal locus $p = P_1..P_6$ for sufficiently large k. In fact the function $s(X) = \sum |XX_i|$ is bounded in the compact domain defined by the triangle and obtains there a maximum value k_0. The arguments used above, for the construction of p and its circumscribing ellipse, are valid for all $k > k_0$. Our next task is to show that the corresponding conics κ for all these k are pairwise homothetic, with homotheties centered at $X(37)$. To show this we'll use the well known ellipse κ_0 inscribed in the triangle of reference ABC and centered at the point $X(37)$ of triangle ABC (see Figure 14). In fact, we'll show that every conic κ for a $k > k_0$ is homothetic to κ_0 w.r.t. the point $X(37)$ of ABC.

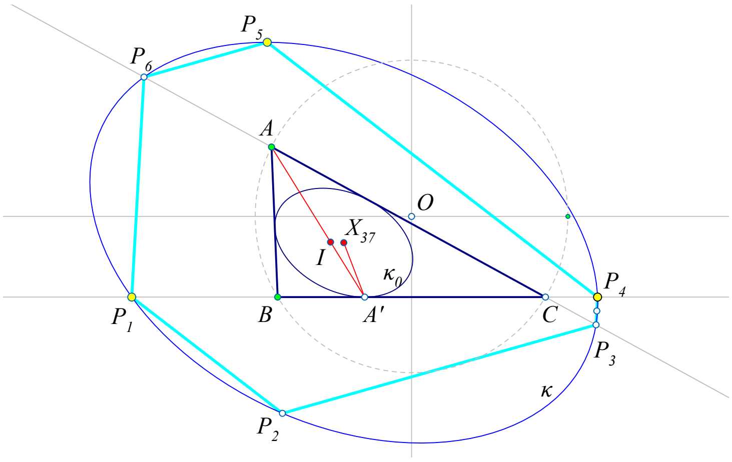


FIGURE 14. The in-ellipse κ_0 with center at $X(37)$

It is well known and easy to prove, that κ_0 is the triangle conic with *"perspector"* the incenter I of ABC. This means, that its contact points with the sides of the triangle are the traces of the bisectors on the sides of the triangle, like the point A' shown in the figure. This implies that the directions of the lines $\{X(37)A', BC\}$ are conjugate w.r.t. the conic κ_0. On the other hand, the middle of the chord P_1P_4 of the conic κ is, according to formulas (28) and (31),

$$\frac{1}{2}(P_1 + P_4) \;=\; \frac{1}{b+c}(bB + cC)\ ,$$

which is precisely the expression of A' in terms of $\{B, C\}$. This implies that the directions of the lines $\{X(37)A', BC\}$ are also conjugate w.r.t. to the conic κ. The same arguments can be used also for the other sides of the triangle ABC and the traces $\{B', C'\}$ of the other bisectors, showing analogously, that $\{(X(37)A', BC), (X(37)B', CA), (X(37)C', AB)\}$ define three pairs of conjugate directions common for the two conics κ_0 and κ. This leads to the following theorem.

Theorem 4.2. *With the definitions and notation of this section, the conics* κ_0 *and* κ *for* $k > k_0$ *have a common center and three pairs of conjugate directions correspondingly identical. This condition implies that the two conics are homothetic, consequently all conics* κ *satisfying this condition are ellipses homothetic to the ellipse* κ_0 *w.r.t. their common center, which is the point* $X(37)$ *of the triangle of reference* ABC.

Proof. The first part of the theorem has been settled by the preceding discussion. The homothecy question is answered by the following lemma.

Lemma 4.1. *Two conics having the same center* K *and three pairs of conjugate directions in common have all their pairs of conjugate directions in common and are homothetic w.r.t. their common center.*

Proof. The conjugation of directions is an involutive homography on the pencil K^* of lines through the center K ([17]). Since involutive homographies are completely determined by prescribing their values at two specific lines, the coincidence assumption implies that all pairs of conjugate diameters are common for the two conics. Latter implies the aforementioned homothecy of the two conics.

As a matter of fact, this is a special case of a more general property, according to which, *"two conics $\{\kappa, \kappa'\}$ having all their pairs of conjugate diameters correspondingly parallel, are homothetic"*.

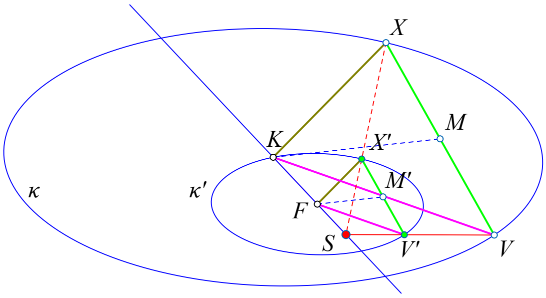


Figure 15. Conics with parallel conjugate diameters

To see this, consider two parallel half diameters $\{KV, FV'\}$ and two parallel chords $\{VX, V'X'\}$ (see Figure 15). These define the parallel conjugate diameters $\{KM, FM'\}$, with $\{M, M'\}$ the corresponding middles of the chords. It follows that triangles $\{KVM, FV'M'\}$ and consequently also triangles $\{KVX, FV'X'\}$ are homothetic w.r.t. to a point $S \in KF$. Fixing V and varying $X \in \kappa$ we see that the conics are homothetic w.r.t. S.

5. CONSTANT SUM OF DISTANCE VECTORS

We turn now to the second problem, concerning the vectors $\{XX_i\}$ realizing the distances of a point X from the sides of the triangle ABC. We work again with complex conjugate coordinates, assuming the triangle $A(t_1)B(t_2)C(t_3)$ to be inscribed in the unit circle S^1. We assume also that the triangle has been turned inside S^1 in the *"normal position"*, so that the constant $\sigma_3 = 1$ (see Figure 7).

The distance vectors $\{XX_i\}$ of X to the sides of the triangle can be expressed using the self conjugate equations of the side-lines of the triangle:

$$\frac{1}{t_i t_j} X + Y \;=\; \frac{1}{t_i t_j}(t_i + t_j) \quad \text{for} \quad i \neq j, i, j = 1..3 \;.$$

By lemma 2.2-10, the directed segments XX_k from X to the sides are then

$$XX_1 = X_1 - X \;=\; \frac{1}{2}(t_2 + t_3 - X - t_2 t_3 Y), \tag{35}$$

with analogous expressions for $X_2 - X$ and $X_3 - X$.

Setting the sum $s(X) = \sum XX_i = 0$ we obtain:

$$2\sigma_1 - 3X - \sigma_2 Y \;=\; 0 \;, \tag{36}$$

which characterizes the symmedian point of the triangle. This, because the symmedian point X is the only one whose distances from the sides are proportional to these sides and the triangle formed by the segments $\{XX_1, XX_2, XX_3\}$ closes (this is expressed through $s(X) = 0$), to a triangle similar to the triangle of reference ABC. Solving equation (36) w.r.t. X (taking into account $Y = \overline{X}$), we find the formula expressing the symmedian center K of the triangle $A(t_1)B(t_2)C(t_3)$ ([3]) :

$$X_K = \frac{6\sigma_1\sigma_3 - 2\sigma_2^2}{9\sigma_3 - \sigma_1\sigma_2} . \tag{37}$$

From equation (35) we see that the squared length of the vector is

$$|s(X)|^2 = \frac{1}{4}(2\overline{\sigma}_1 - 3Y - \overline{\sigma}_2 X)(2\sigma_1 - 3X - \sigma_2 Y) ,$$ which, under the condition $\sigma_3 = 1$, implying $\overline{\sigma_1} = \sigma_2$, becomes :

$$\begin{aligned} |s(X)|^2 = \frac{1}{4}\big(&3\sigma_1 X^2 + 3\overline{\sigma}_1 Y^2 + (|\sigma_1|^2 + 9)XY \\ &-2(\sigma_1^2 + 3\overline{\sigma}_1)X - 2(\overline{\sigma}_1^2 + 3\sigma_1)Y + 4|\sigma_1|^2\big) . \end{aligned}$$

Setting $\sigma_1 = u + iv$ and transforming the equation $|s(X)|^2 = k^2$, from conjugate to Cartesian coordinates, we find the equation of the conic:

$$F(x, y) = ax^2 + by^2 + c + 2hxy + 2gx + 2fy = 0 \quad \text{with}$$

$$a = \frac{u^2 + v^2 + 6u + 9}{16} , \quad b = \frac{u^2 + v^2 - 6u + 9}{16} , \quad c = \frac{u^2 + v^2}{4} - k^2 ,$$

$$h = -\frac{3v}{8} , \quad g = \frac{v^2 - u^2 - 3u}{8} , \quad f = \frac{2uv - 3v}{8} .$$

Using well known formulas for conics w.r.t. Cartesians, the *"invariants"* ([18]) of the conic are seen to be:

$$J_1 = a + b = \frac{u^2 + v^2 + 9}{8} , \qquad J_2 = ab - h^2 = \frac{(u^2 + v^2 - 9)^2}{256} ,$$

$$J_3 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = -k^2 \frac{(u^2 + v^2 - 9)^2}{256} .$$

showing that the conic is an ellipse. Besides the center, given by the well known formula is

$$x_0 = \frac{hf - bg}{ab - h^2} , \quad y_0 = \frac{gh - af}{ab - h^2} \quad \Rightarrow$$

$$x_0 = -\frac{2(v^2 - u^2 + 3u)}{v^2 + u^2 - 9} , \quad y_0 = -\frac{2(2u + 3)v}{v^2 + u^2 - 9} . \tag{38}$$

The expressions of (x_0, y_0) can be easily seen to be equivalent to equation (36) characterizing the symmedian point of the triangle in complex conjugate coordinates. Using standard material for quadratic equations of the plane ([18]), we see that the rotation angle ϕ of the coordinate axes to those of the ellipse is given by the formula

$$\tan(2\phi) = \frac{2h}{a - b} = -\frac{v}{u} .$$

Using this and the formula for the WS lines of a point $T \in S^1$ (lemma 2.3), we see that the axes of the ellipse are parallel to the WS lines $\{w_I, w_J\}$ of the endpoints $\{I, J\}$ of the diameter of S^1 which passes through the orthocenter

$H = \sigma_1$ of the triangle, i.e. the diameter determined by the Euler line of the triangle. These lines are the asymptotes of the *"Jerabek hyperbola"* ν of the triangle and intersect at the triangle center ([19]) $X(125)$, lying on the Euler circle (see Figure 16).

The first two coefficients of the canonical form of the ellipse

$$Ax^2 + By^2 + C = 0 \ ,$$

can be found solving the quadratic equation

$$x^2 - J_1 x + J_2 = 0 \quad \Leftrightarrow \quad x^2 - \frac{u^2+v^2+9}{8} - k^2\frac{(u^2+v^2-9)^2}{256} \ = \ 0 \quad \Leftrightarrow$$

$$x_1, x_2 \ = \ \frac{u^2+v^2+9 \pm 6\sqrt{u^2+v^2}}{16}$$

whereas the constant term results from c by adding $F(x_0, y_0) = -k^2$. From these we see that the ratio q of the axes of the ellipse κ is independent of the constant k involved in the definition of κ :

$$q \ = \ \sqrt{\frac{u^2+v^2+9+6\sqrt{u^2+v^2}}{u^2+v^2+9-6\sqrt{u^2+v^2}}} \ . \tag{39}$$

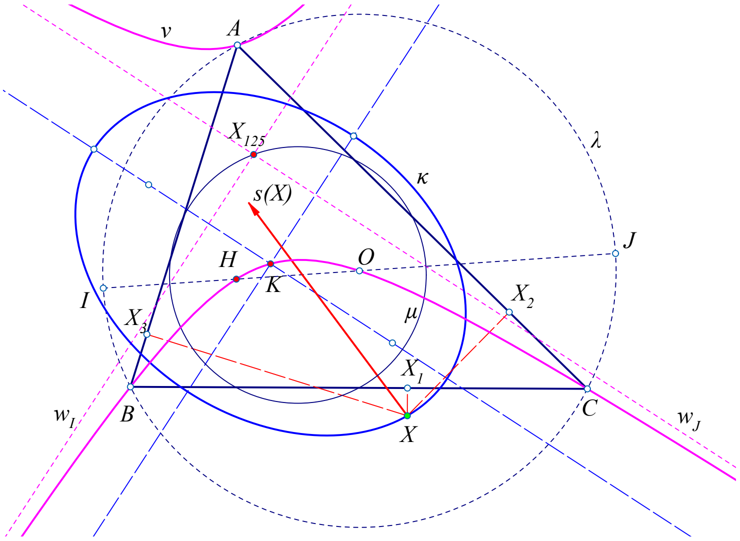


FIGURE 16. Ellipse $\kappa \ni X: \ |s(X)| = |\sum XX_i| \ = \text{k}$

We formulate the preceding results in the form of a theorem.

Theorem 5.1. *Consider the vector* $s(X) = \sum XX_i$, *where the three vectors* $\{XX_i\}$ *realize the distances of the point* X *to the sides of the triangle* ABC. *The condition* $|s(X)| = 0$ *characterizes the symmedian point* K *of the triangle. For a constant* $k > 0$, *the geometric locus of points* X, *for which* $|s(X)| = k$, *is an ellipse centered at* K. *The ellipses resulting for various values of* k *are homothetic w.r.t. the symmedian point* K *and their axes are parallel to the WS lines of the diametral points of the Euler line of* ABC, *these lines coinciding with the asymptotes of the Jerabek hyperbola of the triangle.*

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