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# FOLIATIONS DEFINED BY ONE FORM ON THE TORUS

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**Abstract.** In this work, we recall some important properties of foliations defined by a closed nonsingular one form on manifolds. We recall some results on the toral irrationnal flow and give some result on foliations defined by one form on the torus. We also give a construction of integrable homotopy between foliations, then we analyse some questions about transversely affine foliations and one forms.

## 1. Introduction

The foliations defined by a closed nonsingular one form have many interresting properties. This motivates us to recall the most important ones within a unified way. For example the leaves of a foliation defined by a closed nonsingular one form are diffeomorphic; they are closed or dense. It is an important investigation to study foliation defined by a one form on the torus $\mathbb{T}^2$. Before this we recall the classification of toral diffeomorphisms and give some properties of irrational flows on the torus. We construct an example of translation foliation and transversely affine foliation in the torus. In the sequel we construct an integrable homotopy between foliations on $\mathbb{T}^2 \times [0, 1]$ using families of closed one forms. We analyse in the end of our work the following problem: giving $n$ one forms on a manifold of dimension $m$ with $m \geq n$, such that any form defines a transversely affine foliation, is the foliation defined by the sum of these forms, if it exists, a tranversely affine foliation? We give some conditions for which this is true. We give a counterexample where the foliation is not transversely affine.

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## 2. Foliations on manifolds

**Definition 2.1.** *Let $M$ be a manifold of dimension $m$. A foliation of dimension $n$ on $M$ is an $C^r-$ atlas $\mathcal{F}$ on $M$ which is maximal and has the following properties:*
*i) If $(U,\phi)\in\mathcal{F}$ then $\phi(U)=U_1\times U_2\in\mathbb{R}^n\times\mathbb{R}^{m-n}$ where $U_1$ and $U_2$ are open discs.*
*ii) If $(U,\phi)$ and $(V,\psi)$ are in $\mathcal{F}$ and $U\cap V\neq\emptyset$ then the transition map:*

$$\psi\circ\phi^{-1}:\phi(U\cap V)\longrightarrow\psi(U\cap V) \text{ is of the form}$$
$$\psi\circ\phi^{-1}(x,y)=(h_1(x,y),h_2(y)) \text{ if } (x,y)\in\mathbb{R}^n\times\mathbb{R}^{m-n}$$

Let $(U,\phi)\in\mathcal{F}$ be a card of the foliation such that $\phi(U)=U_1\times U_2\in\mathbb{R}^n\times\mathbb{R}^{m-n}$, we call plaques of $\mathcal{F}$ the sets $\phi^{-1}(U_1\times\{c\})$, $c\in U_2$. As $M$ is covered by plaques we can define the equivalence relations on $M$: $p\sim q$ if and only if there exists $\alpha_1,...\alpha_n$ $n$ plaques such that $p\in\alpha_1$ and $q\in\alpha_n$ and $\forall i\in\{1,...,n\}$ on a $\alpha_i\cap\alpha_{i+1}\neq\emptyset$. The equivalence classes of $\sim$ are called leaves of the foliation $\mathcal{F}$.
We can also define a $C^r$ codimension $s$ foliation on $M$ by a maximal set of pairs $(U_i,f_i)$, $i\in I$, where $U_i$ are open subsets of $M$ and the $f_i:U_i\longrightarrow\mathbb{R}^s$ are submersions that verify:
$(i)$ $\bigcup_{i\in I}U_i=M$
$(ii)$ if $U_i\cap U_j\neq\emptyset$, there exists $g_{ij}$ diffeomorphism of $\mathbb{R}^s$ such that $f_i=g_{ij}\circ f_i$ on $U_i\cap U_j$
$f_i$ are distinguished and the plaques are connected components of the sets $f_i^{-1}(c)$ , $c\in\mathbb{R}^s$.

## 3. Properties of foliations defined by a nonsingular closed one form

Let $M$ be a manifold of dimension $n$ and $\mathcal{F}$ a foliation of $M$ defined by a nonsingular closed one form $\omega$ ($\mathcal{F}=Ker\omega$).

**First property**: The leaves of $\mathcal{F}$ are all diffeomorphic. They are all closed or dense.

**Proof of the first property**:
Let $X$ be the vector field dual of the form $\omega$ (relatively to a metric ). $\omega(X)\equiv 1$. Then we have
$L_X\omega=i_xd\omega+di_X\omega=0$
Let $(\phi_t)_{t\in\mathbb{R}}$ be the one parameter local group of diffeomorphisms genereted by $X$. As $L_X\omega=0$ then the foliation $\mathcal{F}$ is invariant by $(\phi_t)_{t\in\mathbb{R}}$. All the leaves are diifeomorphic and are exchaged transtively by $(\phi_t)_{t\in\mathbb{R}}$.
Let prove that the leaves are all closed or all dense.
Let:

$$\rho:\pi_1(M)\to\mathbb{R}$$
$$[\nu]\mapsto\int_\nu\omega$$

The global holonomy homeomorphism .
Set $\Gamma=\rho(\pi_1(M))\subset\mathbb{R}$ the global holonomy group of $\mathcal{F}$
Let the fibration $p:M\to\mathbb{R}/\Gamma$
• If $\Gamma$ is cyclic that is it is discreet $p:M\to\mathbb{R}/\overline{\Gamma}$ is the fibration of $M\longrightarrow S^1$ and all the leaves are closed.

• If $\Gamma$ is not cyclic that is it is dense, then the leaves are all dense.$\Box$

**Second property**: $\mathcal{F}$ is without holonomy in the sense of Ehresmann.

**Proof of the second property**:
As $\mathcal{F}$ is defined by $\omega$ we can find a regular cover $\mathcal{U} = (U_i)_{i \in I}$ of $M$ and some cards $\phi : U_i \to \mathbb{R}^n$ such that $\omega = \phi_i^*(dy_n)$ for all $i$. If $\{U_1, ..., U_k\}$ is a chain of open sets joining $x$ to $y$, two points on the same leaf $F$ of $\mathcal{F}$, and $T$ and $T'$ the transverse submanifolds to $\mathcal{F}$ passing to $x$ respectively to $y$ and $T_i$ the transverse submanifolds passing to $x_i = \gamma(t_i)$ for a subdivision of $[0, 1]$; $t_0 = 0 < t_1 < ... < t_k = 1$ and $\gamma[t_{i-1}, t_i] \subset U_i$ for a path $\nu : [0, 1] \to F$ such that $\nu(0) = x$ and $\nu(1) = y$.
Sliding along the plaques of the $U_i$ are translations and then the sliding along the leaves are translations and the foliation is without holonomy.$\Box$

**Third property**:
If $\mathcal{F}$ is defined by a nonsingular closed one form $\omega$ on a compact manifold $M$ then $H^1(M, \mathbb{R}) \neq \emptyset$.

**Proof of the third property**:
Suppose that $[\omega] = 0$ that means $\omega$ is exact. Then there exists a function $f$ on $M$ such that $\omega = df$. As $M$ is compact the function $f$ is bounded and reachs its limits in $M$ according to Weirstrass theorem generalized to manifolds. There exists $x_0 \in M$ such that $d_{x_0} f = 0$. Therefore $\omega(x_0) = 0$ and then $x_0$ is a singular point for $\omega$ which is a contradiction . $[\omega] \neq 0$ in $H^1(M, \mathbb{R})$.$\Box$
We recal the following theorem due to Tishler which gives us some information on the geometry of manifold that admit a nonsingular closed one form.

**Theorem 3.1.** [5] *A compact manifold $M$ of dimension $n$ which admit a nonsigular closed one form is fibered over the circle.*

3.1. **Gobillion-Vey invariant for codimension one foliations.** Let $\mathcal{F}$ be a codimension one foliation on a manifold $M$ of dimension $n$ defined by a one form $\omega$ such that $d\omega = \omega \wedge \omega_1$.
Then we have : $d(d\omega) = d\omega \wedge \omega_1 - \omega \wedge d\omega_1 = 0$
$\Rightarrow \omega \wedge d\omega_1 = 0 \Rightarrow d\omega_1 = \omega \wedge \omega_2$ where $\omega_2$ is a one form.
We get three one forms $\omega$ , $\omega_1$ and $\omega_2$
Set $\Omega = \omega \wedge \omega_1 \wedge \omega_2$. $[\Omega] \in H^3(M)$ is an invariant for the foliation $\mathcal{F}$ called the Gobillon-Vey invariant.
Now we'll prove that this is an invariant.

**Proof of the fact that the cohomology class of the three form $\Omega = \omega \wedge \omega_1 \wedge \omega_2$, $[\Omega] \in H^3(M)$, is an invariant for the foliation $\mathcal{F}$.**

We have $\mathcal{F} = ker\omega'$ then $\omega' = f\omega$ where $f$ is a function that does not vanish from $M$ to $\mathbb{R}$ then :

$$
\begin{aligned}
d\omega' &= d(f\omega) \\
&= f d\omega + df \wedge \omega \\
&= f\omega \wedge \omega_1 + df \wedge \omega \\
&= f\omega \wedge (\omega_1 - \frac{df}{f})
\end{aligned} \tag{1}
$$

Set $\omega'_1 = \omega_1 - \frac{df}{f}$
$d\omega' = \omega' \wedge \omega'_1$
$d(d\omega') = 0 \Leftrightarrow \omega' \wedge d\omega'_1 = 0 \Leftrightarrow d\omega'_1 = \omega' \wedge \omega'_2$
Set $\Omega' = \omega' \wedge \omega'_1 \wedge \omega'_2$.
We have the following equality: $d\omega'_1 = d\omega_1$

$$
\begin{aligned}
d\omega'_1 = d\omega_1 &\Longleftrightarrow \omega \wedge \omega_2 = \omega' \wedge \omega'_2 \\
&\Longleftrightarrow \omega \wedge \omega_2 = f\omega \wedge \omega'_2 \\
&\Longleftrightarrow \omega \wedge \omega_2 = \omega \wedge f\omega'_2 \\
&\Longleftrightarrow \omega'_2 = \frac{1}{f}\omega_2
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
d\Omega &= d(\omega \wedge \omega_1 \wedge \omega_2) \\
&= d\omega \wedge \omega_1 \wedge \omega_2 - \omega \wedge d\omega_1 \wedge \omega_2 + \omega \wedge \omega_1 \wedge \omega \\
&= \omega \wedge \omega_1 \wedge \omega_1 \wedge \omega_2 - \omega \wedge \omega \wedge \omega_2 \wedge \omega_2 + \omega \wedge \omega_1 \wedge d\omega_2 \\
&= \omega \wedge \omega_1 \wedge d\omega_2
\end{aligned} \tag{3}
$$

Let prove that $d\Omega = 0$
$d\omega_1 = \omega \wedge \omega_2 \Longrightarrow d(d\omega_1) = 0 \Longrightarrow d\omega \wedge \omega_2 = \omega \wedge d\omega_2$
Coming back to the expression of $d\Omega$ we have:
$d\Omega = -\omega_1 \wedge \omega \wedge \omega_2 = -\omega_1 \wedge d\omega \wedge \omega_2 = -\omega_1 \wedge \omega \wedge \omega_1 \wedge \omega_2 = 0$
Now we'll check the relation between $\Omega'$ and $\Omega$.

$$
\begin{aligned}
\Omega' = \omega' \wedge \omega'_1 \wedge \omega'_2 &= f\omega \wedge (\omega_1 - \frac{df}{f}) \wedge \frac{1}{f}\omega_2 \\
&= f\omega \wedge \omega_1 \wedge \frac{1}{f}\omega_2 - f\omega \wedge \frac{df}{f} \wedge \frac{1}{f}\omega_2 \\
&= \omega \wedge \omega_1 \wedge \omega_2 + \omega \wedge \omega_2 \wedge \frac{df}{f} \\
&= \Omega + d\omega_1 \wedge \frac{df}{f} \\
&= \Omega + d(\omega_1 \wedge \frac{df}{f})
\end{aligned} \tag{4}
$$

We have $\Omega' - \Omega = d(\omega_1 \wedge \frac{df}{f})$ and finaly $[\Omega'] = [\Omega]$ $\square$

## 4. Foliations on the torus

4.1. **Toral diffeomorphisms.** Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The homeomorphisms of the torus $\mathbb{T}^2$ correspond to the elements of the linear group $GL_2(\mathbb{Z})$ as any element $\alpha \in GL_2(\mathbb{Z})$ sends $\mathbb{Z}^2$ to itself and induces a continuous map $h_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$. the homeomorphism $h_\alpha$ has an inverse $h_{\alpha^{-1}}$ and $\alpha$ is the matrix of the map $(h_\alpha)^* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$.
The map $h_\alpha$ preserve the orientation if and only if $det(\alpha) = 1$ that means $\alpha \in SL_2(\mathbb{Z})$.

4.2. **Classification of toral diffeomorphisms.** If $\phi$ is a toral diffeomorphism, it is isotopic to an element of $SL_2(\mathbb{Z})$. For a such $\phi$ we have three cases:
$a$) the elliptic case where the eigenvalues are imaginar and $\phi$ is a finite order hyperbolic rotation;
$b$) the parabolic case where the eigenvalues are equal and have module equal to one and $\phi$ is a Dehn twist";
$c$) the hyperbolic case where the eigenvalues are real and distinct and $\phi$ is an Anosov diffeomorphism.

Indeed if $\alpha \in SL_2(\mathbb{Z})$ is repesented by a $2 \times 2$ matrix the characteristic polynomial is $t^2 - trace(\alpha)) + 1$ . The eigenvalues $\lambda$ and $\lambda^{-1}$ of $\alpha$ are:
$a$) complexe numbers, that means $trace(\alpha) = 0$ , 1 or $-1$ or ;
$b$) all $\neq \pm 1$, that means $trace(\alpha) = \pm 2$;
$c$) distinct and real, that means $\mid trace(\alpha) \mid > 2$.

4.3. **Irrational flows on the torus.** Let $\mathbb{T}^2 = S^1 \times S^1$ the torus of dimension 2. An element of $\mathbb{T}^2$ is defined as a pair of angles $(\theta^1, \theta^2)$ each one is determined modulo $2k\pi$, $k \in \mathbb{Z}$.

Let $(\theta_0^1, \theta_0^2)$ be an element of the torus $\mathbb{T}^2$. The square $]\theta_0^1 - \pi, \theta_0^1 + \pi[\times]\theta_0^2 - \pi, \theta_0^2 + \pi[$ of $\mathbb{R}^2$ corresponds to an open set of $\mathbb{T}^2$ on which $(\theta^1, \theta^2)$ is a local coordinates system.
The vector fields $\frac{\partial}{\partial\theta^1}$ and $\frac{\partial}{\partial\theta^2}$ are defined on all $\mathbb{T}^2$. Let $(\lambda^1, \lambda^2)$ be a pair of real numbers which are not both equal to zero.
The vector field $X = \lambda^1 \frac{\partial}{\partial\theta^1} + \lambda^2 \frac{\partial}{\partial\theta^2}$ is a nonsingular vector field. It determines a foliation $\mathcal{F}_{(\lambda^1,\lambda^2)}$ of dimension 1 on the torus $\mathbb{T}^2$.
Let $x_0 = (\theta_0^1, \theta_0^2) \in \mathbb{T}^2$. The leaf $F_{x_0}$ passing through $x_0$ is the trajectory of the vector field $X$ passing through $x_0$; it is the image of the immersion $\phi_{x_0} : \mathbb{R} \longrightarrow \mathbb{T}^2$ defined by $\phi_{x_0}(t) = (\theta_0^1 + t\lambda^1, \theta_0^2 + t\lambda^2)$; $\phi_{x_0}$ is an integral curve of the vector field $X$ obtained by integration of $X$ and $F_{x_0} = \{(\theta_0^1 + t\lambda^1, \theta_0^2 + t\lambda^2) t \in \mathbb{R}\}$.
To investigate the nature of the leaves of $\mathcal{F}_{(\lambda^1,\lambda^2)}$ we consider the following two cases according to the quotient $\frac{\lambda^1}{\lambda^2}$ is rationnal or irrationnal.
**First case**: $\frac{\lambda^1}{\lambda^2} \in \mathbb{Q}$
We can suppose that $\lambda^1 \in \mathbb{Z}$ and $\lambda^2 \in \mathbb{Z}$. Let $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, then we have: $\phi_{x_0}(t_0 + n) = (\theta_0^1 + t_0\lambda^1 + n\lambda^1, \theta_0^2 + t_0\lambda^2 + n\lambda^2)$ and
$\phi_{x_0}(t_0) = (\theta_0^1 + t_0\lambda^1, \theta_0^2 + t_0\lambda^2)$

According to the fact that $n\lambda^1$ and $n\lambda^2$ are in $\mathbb{Z}$ $\phi_{x_0}(t_0+n)$ and $\phi_{x_0}(t_0)$ are projecting in the same point of the leaf $F_{x_0}$. This means that the orbits of the vector field $X$ are periodic. The leaves are closed: they are circles.

**Second case** $\frac{\lambda^1}{\lambda^2} \notin \mathbb{Q}$

$a)$ Let prove that $\phi_{x_0}$ is injective.

Let $t_1$ and $t_2 \in \mathbb{R}$ with $t_1 \neq t_2$ and $\phi_{x_0}(t_1) = \phi_{x_0}(t_2)$. This means that there exists $n \in \mathbb{Z}$ such that $t_2 = n + t_1$

$\phi_{x_o}(t_2) = \phi_{x_0}(n+t_1) = (\theta_0^1 + t_1\lambda^1 + n\lambda^1, \theta_0^2 + t_1\lambda^2 + n\lambda^2)$

$\phi_{x_0}(t_2) = (\theta_0^1 + t_1\lambda^1, \theta_0^2 + t_1\lambda^2) \Rightarrow n\lambda^1 \in \mathbb{Z}$ and $n\lambda^2 \in \mathbb{Z}$ and $\frac{n\lambda^1}{n\lambda^2} \in \mathbb{Q} \Rightarrow \frac{\lambda^1}{\lambda^2} \in \mathbb{Q}$ what contradicts the hypothesis : $\phi_{x_0}$ is injective. The leaves are diffeomorphic to the real line $\mathbb{R}$.

$b)$ Let prove that the leaves are dense in $\mathbb{T}^2$

Let prove that $F_{x_0}$ is dense in $\mathbb{T}^2$; for this we have to prove just that it is dense in the open set $]\theta_0^1 - \pi, \theta_0^1 + \pi[\times]\theta_0^2 - \pi, \theta_0^2 + \pi[$. Changing the origin we can suppose that $\theta_0^1 = \theta_0^2 = 0$.

the plaques of the open set $]-\pi,\pi[\times]-\pi,\pi[$ are the traces of the parallèle lines directed by the vector with the component $(\lambda^1, \lambda^2)$ then they are lines with irrational slope. We have just to show that the trace of the leaf $F_{x_0}$ on the axis $\theta^2 = 0$ is dense in $]-\pi,\pi[$. The expression of $\phi_{x_0}$ becomes $\phi_{x_0}(t) = (t\lambda^1, t\lambda^2)$ $t \in \mathbb{R}$.

The point of $F_{x_0}$ on the axis $\theta^2 = 0$ are those such that $t\lambda^2 = 2k\pi \Rightarrow t = \frac{2k\pi}{\lambda^2}$ and then $\theta^1 = t\lambda^1 = \frac{2k\pi\lambda^1}{\lambda^2}$.

Or on the circle $S^1$ endowed with the group structure the point of the angle at the center $\frac{2k\pi\lambda^1}{\lambda^2}$ ($\frac{\lambda^1}{\lambda^2} \notin \mathbb{Q}$) is an infinite subgroup which is dense because of the density of $\mathbb{R} - \mathbb{Q}$. Then the leaf $F_{x_0}$ is dense in $\mathbb{T}^2$.

## 4.4. Foliations defined by one form on the torus.

**Definition 4.1.** *We say that a one form is affine if it defines a transversely affine foliation on a manifold. A closed nonsigular one form that defines et translation foliation is called a translation form.*

Recall that a codimension 1 transversely orientable foliation $\mathcal{F}$ on a manifold $M$ is transversely affine if there exists a cover $(U_i)_{i\in I}$ of $M$ and a family of submersions $f_i : U_i \to \mathbb{R}$ such that on a connected component of $U_i \cap U_j \neq \emptyset$ we have $f_i = \gamma_{ij} f_j$, where $\gamma_{ij}$ are affine transformations that preserve the orientation of $\mathbb{R}$ :

$$\gamma_{ij}(x) = a_{ij}x + b_{ij}$$

According to [3] (page 170), this definition is equivalent to the existence of a pair $(\omega, \omega_1)$ of $1-$forms such that:

- $d\omega = \omega \wedge \omega_1$.
- $d\omega_1 = 0$
- the foliation $\mathcal{F}$ is defined by $\omega$.

We now prove the following result:

**Theorem 4.1.** *There exist two one forms $\omega$ and $\beta$ in the torus such that $\omega$ is a translation form and $\alpha$ is an affine form.*

**Proof:**
Let $\mathbb{T}^2$ the torus seen as a revolution surface around the $z-axis$ in $\mathbb{R}^3$. It is parametrized by

$$\begin{cases} x(u,v) = (a + b\cos v)\cos u \\ y(u,v) = (a + b\cos v)\sin u \\ z(u,v) = b\sin v \end{cases}$$

where $0 < b < a$ and $(u,v) \in \mathbb{R}^2$. It follows that

$$ds^2 = b^2dv^2 + (a + b\cos v)^2du^2$$

We set $\omega_v = bdv$ and $\omega_u = (a + b\cos v)du$ two $1 - forms$ on the torus.

$d\omega_v = 0$ so $\omega_v$ is a closed one form on the torus. It defines a codimension one foliation on the torus. The foliation $\mathcal{F}_v$ defined by $\omega_v$ is a translation foliation So $\omega = \omega_v$ is a translation form.

$$\begin{aligned} d\omega_v &= -b\sin v dv \wedge du \\ &= \frac{-b\sin v}{a + b\cos v} dv \wedge (a + b\cos v)du \\ &= (a + b\cos v)du \wedge \frac{b\sin v}{a + b\cos v} dv \\ &= \omega_u \wedge \alpha \end{aligned} \tag{5}$$

with $\alpha = \frac{b\sin v}{a+b\cos v}dv$. We have $d\alpha = 0$, so $\omega_u$ is integrable. It defines a codimension one foliation on the torus $\mathbb{T}^2$.

The foliation $\mathcal{F}_u$ defined by $\omega_u$ is a transversely affine foliation. So $\beta = \omega_u$ is an affine form. □

**Remark 4.1.** *The Godbillon-Vey class $GV(\mathcal{F}_v) = 0$. The Godbillon-Vey invariant $GV(\mathcal{F}_u) = 0$.*

## 5. Integrable homotopy on $\mathbb{T}^2 \times [0,1]$

We recall the following definition of integrable homotopy.

**Definition 5.1.** *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be to codimension $q$ foliations on a differentiable manifold $M$. A class $C^r$ integrable homotopy between these foliations is a class $C^r$ codimension $q$ foliation $\mathcal{H}$ on $M \times [0,1]$ transverse to $M \times \{t\}$ for $0 \le t \le 1$ such that $\mathcal{H}|M \times \{0\} = \mathcal{F}_0$ and $\mathcal{H}|M \times \{1\} = \mathcal{F}_1$.*

*Here homotopy can be seen as a deformation of $\mathcal{F}_0 = \mathcal{H}_0$ and $\mathcal{F}_1 = \mathcal{H}_1$ throughout intermediate foliations $\mathcal{H}_t$ induced by $\mathcal{H}$ on $M \times \{t\}$.*

Let $(\omega_t)_{t\in[0,1]}$ be a family of closed one form on the torus. We suppose that any $t \in [0,1]$, $\omega_t$ defines a minimal foliation on the torus. Now consider the product $\mathbb{T}^2 \times [0,1]$ where every tore $\mathbb{T}^2 \times \{t\} = \mathbb{T}^2_t$ is endowed with the foliation $\mathcal{F}_t$ defined by the form $\omega_t$.

**Theorem 5.1.** *The family $(\mathcal{F}_t)_{t\in[0,1]}$ defines an integrable homotopy foliation on $\mathbb{T}^2 \times [0,1]$ between $\mathcal{F}_0$ and $\mathcal{F}_1$.*

**Proof:**
Let $\mathcal{H}$ be the foliation on $\mathbb{T}^2 \times [0,1]$ such that the induced foliation on $\mathbb{T}^2_t$ is equal to $\mathcal{F}_t$. It is not difficult to see that the foliation $\mathcal{H}$ is an integrable homotopy on $\mathbb{T}^2 \times [0,1]$ between $\mathcal{F}_0$ and $\mathcal{F}_1$. □

## 6. Transversely affine foliation and one form in a manifold

Now consider the following situation. Let $\omega = \sum_{i=1}^{n} \omega_i$ be a nonsigular one form on a manifold of dimension $m$ with $m \geq n$ such that the foliation defined by $\omega_i$, is a transversely affine foliation. Let $\mathcal{F}$ be the foliation defined by $\omega$ if it exists. We can ask the following questions:
- Is the foliation $\mathcal{F}$ transversely affine?
-If not in general, is there some condition on the $\omega_i$ such that the foliation $\mathcal{F}$ is transversely affine?
We give a condition for the foliation $\omega$ to be transversely affine. Because of the condition of integrability of a one form, the first question is equivalente to the integrability of the one form $\omega$. Indeed, a one form $\omega$ on a manifold is integrable, that is it defines a foliation on $M$ if $d\omega = \omega \wedge \alpha$ for $\alpha$ a one form on $M$. So if the form $\omega$ defines a transversely affine foliation so $d\alpha = 0$, in addition to the integrability condition. In the following result we give a neccesary condition for $\omega$ to define transversely affine foliation.

**Theorem 6.1.** *Let $M$ be a manifold of dimension $m$ and $\omega = \sum_{i=1}^{n} \omega_i$ where $\omega_i$ defines a transversely affine foliation on $M$. This means that for $i = 1, ..., n$ there exists a closed one form $\alpha_i$ such that $d\omega_i = \omega_i \wedge \alpha_i$. A necessary condition for the foliation defined by $\omega$ to be a transversely affine foliation is that $\omega_i \wedge \alpha_j = 0$ for $i \neq j$; $i = 1, ..., n$ and $j = 1, ..., n$.*

**Proof**
If $\omega = \sum_{i=1}^{n} \omega_i$ then $d\omega = \sum_{i=1}^{n} \omega_i \wedge \alpha_i = \omega \wedge \alpha$ where $\alpha = \sum_{i=1}^{n} \alpha_i$ if $\omega_i \wedge \alpha_j = 0$ for $i \neq j$; $i = 1, ..., n$ and $j = 1, ..., n$. □

In the following we give an example where the first question is not true. We consider the case of Anosov bundle. Let $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism of the torus $\mathbb{T}^2$ induced by a hyperbolic matrix $A$ (i.e $A \in SL(2, \mathbb{Z})$ and $|trA| > 2$). We get by suspension of $\psi$ the 3-manifold:

$$\mathbb{T}^3_A = \mathbb{T}^2 \times I/(x, 0) \sim (\psi(x), 1)$$

with $I = [0, 1]$. This manifold is a torus bundle over the circle. The matrix $A$ has two eigenvalues $\lambda$ and $1/\lambda$. The foliations on $\mathbb{R}^2$ by lines of slope $\lambda$ and $1/\lambda$ are invariant by the action of $\mathbb{Z}^2$ and induce on the torus $\mathbb{T}^2$ two foliations invariant by $\psi$.

By suspension of these foliations of $\mathbb{T}^2$ one obtains on $\mathbb{T}^3_A$ two foliations which we call model foliations.

**Theorem 6.2.** *On $\mathbb{T}^3_A$ there exists two one forms that define two transversely affine foliation and such that the foliation defined by the one form equal to the sum of the two form is not a transversely affine foliation.*

For the proof of this result we will use the two forms that define the model foliations on $\mathbb{T}^3_A$. We will use the following lemma which proves that model foliations are transversely affine.

**Lemma 6.1.** [1] *Any model foliation $\mathcal{F}$ is transversely affine.*

**Proof:** Let $\lambda$ be the eigenvalue corresponds to $\mathcal{F}$ and $y$ the second cordinate eingenvalue. Consider the one form $\omega = \lambda^t dy$ on $\mathbb{T}^2 \times \mathbb{R}$. It is invariant by the map $(m, t) \mapsto (\phi(m), t + 1)$. Indeed $\lambda^{t+1}dy = \lambda(\lambda^t dy) = \lambda\omega$. It passes to the quotient on the bundle $\mathbb{T}^3_A$ and in another part, for the vector $v$ tangent to a leaf defined by the eingenvalue $\lambda$, we have $\omega(v) = 0$ because $\omega$ has not a component on the direction defined by the foliation. $\omega = 0$ defines the model foliation. In other part, the form $\omega_1 = -log\lambda dt$ on $\mathbb{T}^2 \times \mathbb{R}$ is invariant by the map $(m, t) \mapsto (\phi(m), t+1)$ and passes to the quotient on $\mathbb{T}^3_A$.

$$
\begin{aligned}
d\omega = d(\lambda^t dy) &= d(e^{tlog\lambda}dy) \\
&= log\lambda e^{tlog\lambda}dt \wedge dy \\
&= -\lambda^t dy \wedge log\lambda dt = \lambda^t dy \wedge (-log\lambda dt) \\
&= \omega \wedge \omega_1.
\end{aligned} \tag{6}
$$

$d\omega_1 = d(-log\lambda dt) = -log\lambda d(dt) = 0.$ □

**Proof of the Theorem**

According to the previous lemma, we have two one forms $\omega_\lambda$ and $\omega_{\frac{1}{\lambda}}$ that define on $\mathbb{T}^3_A$ two transversely affine foliations. Now set $\omega = \omega_\lambda + \omega_{\frac{1}{\lambda}}$.

$$
\begin{aligned}
d\omega &= d(\lambda^t dy) + d((\frac{1}{\lambda})^t dx \\
&= \lambda^t dy \wedge (-log\lambda dt) + (\frac{1}{\lambda})^t dx \wedge (-\log\frac{1}{\lambda}dt) \\
&= (-\lambda^t dy + (\frac{1}{\lambda})^t dx) \wedge (\log\lambda dt) \\
&\neq \omega \wedge \alpha \qquad with \qquad d\alpha = 0
\end{aligned} \tag{7}
$$

We conclude that $\omega$ is not integrable so $\omega$ does not define a transversely affine foliation.□

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