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# ALGEBRAIC POINTS ON SOME QUOTIENTS OF THE FERMAT CURVE OF DEGREE 11

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**Abstract.** The aim of this work is to give a parametrization of the set of algebraic points of degree $d \in \{4, 5, 6, 7, 8\}$ on the subfamily of quotients of the Fermat curve of degree 11 of affine equation $\mathcal{C}_r : y^{11} = x^r (x-1)^r$ where $r \in \{1, 2, 3, 4, 5\}$. Our essential tools are the Mordell-Weil group, the Abel-Jacobi theorem, linear systems, and the birational morphisms.

## 1. Introduction

Arithmetic questions about the number of points of degree $d$ on a smooth algebraic curve over a number field $K$ lead to geometric questions about the curve.

The theorem of Faltings states that an irreducible curve $\mathcal{C}$ of geometric genus $g \geq 2$ defined over a number field $K$ has only a finite $K$-rational points. But since on an algebraic closure $\overline{K}$ of $K$ the number of $K$-rational points on $\overline{K}$ has the same cardinal as $K$, we are led to ask how the number of points increases with the degree $d$ of an extension $L$ of $K$.

For any integer $d \geq 1$, let us denote the set of algebraic points of degree at most $d$ on $\mathcal{C}$ over the number field $K$ by $\mathcal{C}^{(d)}(K)$ :

$$\mathcal{C}^{(d)}(K) = \left\{P \in \mathcal{C}(\overline{K}) \,\middle|\, [K(P) : K] \leq d\right\}$$

The set $\mathcal{C}^{(d)}(K)$ may be finite or infinite. Indeed, Faltings showed that the rational points of a variety are distributed over a finite number of translates of abelian sub-varieties contained in the variety. This statement can even be used to prove qualitative results about points of degree at most $d$ on a curve, see for example the work of Debarre and Klassen [3]. If we consider a curve $\mathcal{C}$ which admits a non-constant morphism $\mathcal{C} \to \mathbb{P}_1$ defined on $K$ of degree $d$, then $\mathcal{C}^{(d)}(K)$ is infinite.

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Currently, one of the most important problems concerning the classification of algebraic points on a curve $\mathcal{C}$ is the determination of the set $\mathcal{C}^{(d)}(K)$. In fact, there is no general algorithm for determining this set or deciding if it is empty.
However, our work consists of studying special cases of curves where we can give a parametrization of algebraic points of given degree $d$. It seems that a necessary condition is that the Mordell-Weil group of the Jacobian $J(\mathbb{Q})$ of $\mathcal{C}$ is finite.
Consider the family curves $\mathcal{C}_r$ $(1 \leq r \leq 5)$ with affine equation

$$\mathcal{C}_r : y^{11} = x^r (x-1)^r$$

This family is a special case of quotients of Fermat curves with affine equations

$$\mathcal{C}_{r,s}(p) : y^p = x^r (x-1)^s$$

where $1 \leq r, s, r+s \leq p-1$ and $p \geq 5$ is a positive prime.
The curves $\mathcal{C}_{r,s}(p)$ are of genus $g = \frac{p-1}{2}$ and are called quotients of the Fermat curve of affine equation

$$F_p : u^p + v^p = 1.$$

These curves are frequently mentioned in the literature (see [1], [6], [2] and [5]). The Mordell-Weil group of $\mathcal{C}_{r,s}(p)$ is finite in the following cases: for $p = 5$ or 7, and for $p = 11$ and $r = s$ (see [5]).
Gross and Rohrlich, in [5], proved firstly that the Mordell-Weil group $J_r(\mathbb{Q})$ of the family $\mathcal{C}_r$ is finite and given by

$$J_r(\mathbb{Q}) = \frac{\mathbb{Z}}{11\mathbb{Z}}.$$

They also showed that the set $\mathcal{C}_{r,s}(p)(\mathbb{Q})$ is formed by the following three rational points

$$P_0 = (0,0,1), \quad P_1 = (1,0,1), \quad P_\infty = (1,0,0).$$

In [2], Coly and Sall determined the set $\mathcal{C}_2^{(3)}(\mathbb{Q})$ of algebraic points of degree at most 3 on the curve $\mathcal{C}_2$.
Balde, Diallo and Sall have given a parametrization of algebraic points of given degree $l$ with $l \geq 9$ in [1].
Let denote by $\mathcal{G}_d^r$ the set of algebraic points of degree $d$ on $\mathcal{C}_r$ , i.e.

$$\mathcal{G}_d^r = \left\{ P \in \mathcal{C}_r\left(\overline{\mathbb{Q}}\right) \mid [\mathbb{Q}(P) : \mathbb{Q}] = d \right\}.$$

The main objective of this work is to determine the set $\mathcal{C}_r^{(8)}(\mathbb{Q})$ of algebraic points of degree at most 8 on $\mathbb{Q}$, more precisely the set

$$\bigcup_{4 \leq d \leq 8} \mathcal{G}_d^r \qquad \forall\, r \in \{1,2,3,4,5\}.$$

Our main result is the Theorem 3.1

## 2. Preliminaries

### 2.1. **Algebraic extension.**

A complex number $\alpha \in \mathbb{C}$ is algebraic if there exists a non-zero polynomial $f \in \mathbb{Q}[X]$ with $f(\alpha) = 0$. The algebraic closure of $\mathbb{Q}$ is the set

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ algebraic}\}.$$

$\overline{\mathbb{Q}}$ is called the algebraic number fields.
Let $\theta \in \overline{\mathbb{Q}}$, then the smallest subfield of $\overline{\mathbb{Q}}$ that contains $\mathbb{Q}$ and $\theta$ is commonly denoted $\mathbb{Q}(\theta)$. In this case $\mathbb{Q}(\theta)$ is an algebraic extension of $\mathbb{Q}$ of finite degree over $\mathbb{Q}$. We note that $deg(\theta)$ :

$$deg(\theta) = [\mathbb{Q}(\theta)) : \mathbb{Q}].$$

**Definition 2.1.** *Let $\mathcal{C}$ be an algebraic plane curve defined on $\mathbb{Q}$. The degree of an algebraic point $P \in \mathcal{C}$ is the degree of its field of definition over $\mathbb{Q}$. In other words, if we denote by $deg(P)$ the degree of $P$ over $\mathbb{Q}$, then we have*

$$deg(P) = [\mathbb{Q}(P) : \mathbb{Q}] = d.$$

*The point $P$ is called algebraic point of degree $d$ over $\mathbb{Q}$.*

More specifically:

- If $deg(P) = 1$, then $P$ is a rational point.
- If $deg(P) = 2$, then $P$ is a quadratic point.

### 2.2. **Divisors and rational functions.**

Let $\mathcal{C}$ be a smooth plane curve defined on a number field $K$.

**Definition 2.2.** *A divisor $D$ of $\mathcal{C}$ is a formal finite sum of distinct points of $\mathcal{C}$ :*

$$D = \sum_{P \in \mathcal{C}} n_P P$$

*where the $n_P \in \mathbb{Z}$ are almost all zero. The degree of $D$ is the sum defined by*

$$deg\left(\sum_{P \in \mathcal{C}} n_P P\right) = \sum_{P \in \mathcal{C}} n_P deg\,(P).$$

The set of divisors is an abelian group, where the law of the group is the formal addition of points. This group is denoted by $Div(\mathcal{C})$.

**Definition 2.3.** *A rational function on an algebraic curve $\mathcal{C}$ is a function $f : \mathcal{C} \longrightarrow \mathbb{P}^1$, defined by polynomials, which has only a finite number of poles.*

Let $K(\mathcal{C})$ denote the field of all rational functions on $\mathcal{C}$ defined on $K$, then there is a natural morphism

$$K(\mathcal{C})^* \longrightarrow Div(\mathcal{C})$$

that associates to a rational function $f$ its divisor

$$div(f) = \sum_{P \in \mathcal{C}} ord_P(f) P$$

where $ord_P(f)$ is the order of vanishing of $f$ at $P$. It is a standard fact in the theory of algebraic curves that if $f$ is a non-zero rational function, then the number of poles of $f$ equals the number of zeros of $f$ (see [4]).

Here are some properties of the divisors of rational functions given by the following proposition.

**Proposition 2.1.** *(see [4]) Let $\psi$ and $\varphi$ be two rational functions of $K(\mathcal{C})$, then:*

(1) $div(\psi\varphi) = div(\psi) + div(\varphi)$;

(2) $div\left(\dfrac{\psi}{\varphi}\right) = div(\psi) - div(\varphi)$.

**Definition 2.4.** *Let $\mathcal{C}$ be a smooth curve and let $D \in Div(\mathcal{C})$. It is associated with the set of functions*

$$\mathcal{L}(D) = \{f \in K(\mathcal{C})^* \mid div(f) + D \geq 0\} \cup \{0\}.$$

*$\mathcal{L}(D)$ is called linear system. It is a finite-dimensional vector space over $K$. We denote by $l(D)$ the $K$-dimension of $\mathcal{L}(D)$.*

The following theorem classifies the curves according to their genus. It is called Riemann-Roch theorem.

**Theorem 2.1.** *(see [4]) Let $\mathcal{C}$ be a smooth curve. Then there exists a divisor $K_{\mathcal{C}}$ called the canonical divisor and an integer $g \geq 0$ called the genus of $\mathcal{C}$ such that for any divisor $D \in Div(\mathcal{C})$ we have :*

$$l(D) = deg(D) + 1 - g + l(K_{\mathcal{C}} - D).$$

In particular, using the previous notation, we have :

(i) $l(K_{\mathcal{C}}) = g$ and $deg(K_{\mathcal{C}}) = 2g - 2$.

(ii) If $deg(D) > 2g - 2$, then $l(D) = deg(D) + 1 - g$.

### 2.3. Jacobian map.

Let $J(\mathcal{C})$ be the jacobian of the curve $\mathcal{C}$ and $P_\infty$ be a $K$-rational point of $\mathcal{C}$, then we defined the map

$$\begin{array}{cccc} j: & \mathcal{C} & \longrightarrow & J(\mathcal{C}) \\ & P & \longmapsto & [P - P_\infty] \end{array}$$

where $j(P) = [P - P_\infty]$ is the class of $P - P_\infty$.
Let $Div(\mathcal{C})$ be the group of all divisors on $\mathcal{C}$ and $Div^0(\mathcal{C})$ denote the subgroup of divisors of degree 0. The map $j$ extends by linearity to $Div^0(\mathcal{C})$ and we note this extension again by $j$:

$$\begin{array}{cccc} j: & Div^0(\mathcal{C}) & \longrightarrow & J(\mathcal{C}) \\ & D & \longmapsto & [D - deg(D)P_\infty] \end{array}$$

$j$ is called the Abel-Jacobi map.
We have the following classical theorem :

**Theorem 2.2.** *(Abel-Jacobi) (See [4]). The map $j$ is surjective and its kernel consists of the divisors of functions on $\mathcal{C}$. In other words, the kernel of $j$ is formed by the divisors of rational functions.*

### 2.4. **Normalization of the curve $\mathcal{C}_r$.**

We know that in general for any curve

$$\mathcal{C}_r : y^{11} = x^r (x-1)^r ,$$

we can describe the associated curve

$$\mathcal{V}_r : y^{11} = x^r (x-1)^r \quad \text{with} \quad x(x-1) \neq 0.$$

The curve $\mathcal{V}_r$ is a smooth projective curve. There are then three points $P_0$, $P_1$ and $P_\infty$ such that

$$\mathcal{C}_a - \mathcal{V}_a = \{P_0,\, P_1,\, P_\infty\} .$$

Let $\mathcal{H}_r$ be the projective equation of $\mathcal{C}_r$ :

$$\mathcal{H}_r : Y^{11} = Z^{11-2r} X^r (X-Z)^r$$

The curve $\mathcal{H}_r$ is the Zariski closure of $\mathcal{V}_r \subset \mathbb{A}^2 \subset \mathbb{P}^2$ which is smooth except (perhaps) at three points

$$P_0' = (0,0,1), \qquad P_1' = (1,0,1), \qquad P_\infty' = (1,0,0).$$

More precisely $P_\infty'$, $P_0'$ and $P_1'$ are singular unless $r = 1$.
Let $\nu_r$ be the normalization map defined by

$$\nu_r : \mathcal{C}_r \longrightarrow \mathcal{H}_r$$

Then $\nu_r$ is bijective and we have :

$$\nu_r^{-1}\left(P_\infty'\right) = P_\infty, \quad \nu_r^{-1}\left(P_0'\right) = P_0, \quad \nu_r^{-1}\left(P_1'\right) = P_1.$$

The curves $\mathcal{C}_r$ are birationally equivalent to the curve $\mathcal{C}_1$. Thus we have the lemma :

**Lemma 2.1.** *For any $r \in \{1,2,3,4,5\}$, the curve $\mathcal{C}_r$ is birationally equivalent to the curve $\mathcal{C}_1$.*

**Proof.** Let the morphism $\varphi_r$ be defined by

$$\begin{array}{rccc} \varphi_r : & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_r \\ & (x,y) & \longmapsto & (x,y^r) \end{array}$$

$$\begin{aligned} (x,y^r) \in \mathcal{C}_r \iff & (y^r)^{11} - x^r (x-1)^r = 0 \\ \iff & \left(y^{11}\right)^r - (x(x-1))^r = 0 \\ \iff & \left(y^{11} - x(x-1)\right) \left( \sum_{0 \le k \le r-1} x^{r-1-k} (x-1)^{a-1-k} y^{11k} \right) = 0 \\ \iff & y^{11} - x(x-1) = 0 \\ \iff & (x,y) \in \mathcal{C}_1. \end{aligned}$$

### 2.5. **Geometric lemmas.**

Let $x$ and $y$ be the functions defined on $\mathcal{C}_r$ by $x = \dfrac{X}{Z}$ and $y = \dfrac{Y}{Z}$.

**Lemma 2.2.** *For all $r \in \{1,2,3,4,5\}$, we have :*

(a) *$div(x) = 11P_0 - 11P_\infty$ and $div(x-1) = 11P_1 - 11P_\infty$;*
(b) *$div(y) = rP_0 + rP_1 - 2rP_\infty$;*
(c) *$11j(P_0) = 0$, $11j(P_1) = 0$ and $j(P_0) + j(P_1) = 0$.*

**Proof.** See [5].

**Lemma 2.3.**

f) *The Mordell-Weil group of* $\mathcal{C}_r$ *is given by:*

$$J_r(\mathbb{Q}) \cong \frac{\mathbb{Z}}{11\mathbb{Z}} = \left\{ mj(P_1) \mid 0 \leq m \leq 10 \right\} = \left\{ mj(P_0) \mid 0 \leq m \leq 6 \right\}.$$

**Proof.** See [6].

If $m$ is a non-zero natural number then $mP_\infty$ is a divisor on $\mathcal{C}_r$. We have the lemma :

**Lemma 2.4.** *A* $\mathbb{Q}$*-basis of* $\mathcal{L}(mP_\infty)$ *is given by*

(1) *For* $0 \leq m \leq 10$*, we have :*
   (a) $\mathcal{L}(P_\infty) = \langle 1 \rangle$
   (b) $\mathcal{L}(2P_\infty) = \langle 1,\ y \rangle = \mathcal{L}(3P_\infty)$
   (c) $\mathcal{L}(4P_\infty) = \left\langle 1,\ y,\ y^2 \right\rangle = \mathcal{L}(5P_\infty)$
   (d) $\mathcal{L}(6P_\infty) = \left\langle 1,\ y,\ y^2,\ y^3 \right\rangle = \mathcal{L}(7P_\infty)$
   (e) $\mathcal{L}(8P_\infty) = \left\langle 1,\ y,\ y^2,\ y^3 \right\rangle = \mathcal{L}(9P_\infty)$
   (f) $\mathcal{L}(10P_\infty) = \left\langle 1,\ y,\ y^2,\ y^3,\ y^4,\ y^5 \right\rangle$

(2) *For* $m \geq 11$*, a* $\mathbb{Q}$*-basis of* $\mathcal{L}(mP_\infty)$ *is given by :*

$$\mathcal{B}_m = \left\{ y^i \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq \left[\frac{m}{2}\right] \right\} \cup \left\{ xy^j \mid j \in \mathbb{N} \text{ and } 0 \leq j \leq \left[\frac{m-11}{2}\right] \right\}.$$

**Proof**. The rational functions in $\mathcal{B}_m$ are of the form $P_{rs}(x,y) = x^r y^s$ and verify

$$div\left(P_{rs}(x,y)\right) = div\left(x^r y^s\right) = r div(x) + s div(y) \Longrightarrow 0 \leq 7r + 2s \leq m$$

The reasoning is divided into two cases:

**Case 1:** $1 \leq m \leq 10$. We have

$$11r + 2s \leq m \text{ with } r, s \in \mathbb{N} \Longrightarrow r = 0 \text{ and } 0 \leq s \leq m.$$

So we obtain (1).

**Case 2:** $m \geq 11$. It is clear that $m \geq 2g - 1 = 9$.

a) According to the Riemann-Roch theorem,

$$dim\left(\mathcal{L}(mP_\infty)\right) = m - g + 1 = m - 4.$$

It remains to show that $\#(\mathcal{B}_m) = m - g + 1$.

i) Suppose $m$ is even, and let $m = 2h$. Then we have

$$i \leq \left[\frac{m}{2}\right] = h \quad and \quad j \leq \left[\frac{m-11}{2}\right] = h - g - 1.$$

So we get $\mathcal{B}_m = \left\{1,\ y,\ \ldots,\ y^h\right\} \cup \left\{x,\ xy,\ \ldots,\ xy^{h-g-1}\right\}$. Therefore $\#\left(\mathcal{B}_m\right) = h + 1 + (h - g - 1 + 1) = m + 1 - g = dim\left(\mathcal{L}(mP_\infty)\right)$.

ii) Suppose $m$ is odd, and let $m = 2h + 1$. Then we have

$$i \leq \left[\frac{m}{2}\right] = h \quad and \quad j \leq \left[\frac{m-11}{2}\right] = h - g.$$

So we get $\mathcal{B}_m = \left\{1,\ y,\ \ldots,\ y^h\right\} \cup \left\{x,\ xy,\ \ldots,\ xy^{h-g}\right\}$. Hence $\#\left(\mathcal{B}_m\right) = h + 1 + (h - g + 1) = m + 1 - g = dim\left(\mathcal{L}(mP_\infty)\right)$.

b) Let us show that the family $\mathcal{B}_m$ is free.

| $P_{rs}(x,y)$ | 1 | $y$ | $y^2$ | ... | $y^{[\frac{m}{2}]}$ | $x$ | $xy$ | $xy^2$ | ... | $xy^{[\frac{m-11}{2}]}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of poles | 0 | 2 | 4 | ... | $2[\frac{m}{2}]$ | 11 | 13 | 15 | ... | $11+2[\frac{m-11}{2}]$ |

The multiplicities of the poles of the elements of $\mathcal{B}_m$ are all different. Therefore, the family is free. So $\mathcal{B}_m$ is a basis of $\mathcal{L}(mP_\infty)$.

## 3. Algebraic points on $C_r$

### 3.1. Main theorem.

The main result of this manuscript is the following theorem

**Theorem 3.1.** *Let $\mathcal{G}_d^r$ be the set of algebraic points of degree $d$ of the curve $\mathcal{C}_r : y^{11} = x^r(x-1)^r$ with $r \in \{1,2,3,4,5\}$. Then we have :*

(1) *The set $\mathcal{G}_4^r$ is given by*

$$\mathcal{G}_{4,1} = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}},\ y^r \right) \;\middle|\; [\mathbb{Q}(y):\mathbb{Q}] = 2 \right\}$$

(2) *The set $\mathcal{G}_5^r$ is empty $\mathcal{G}_5^r = \emptyset$*

(3) *The set $\mathcal{G}_6^r$ is given by $\mathcal{G}_{6,1} \cup \mathcal{G}_{6,2}$ with*

$$\mathcal{G}_{6,1} = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}},\ y^r \right) \;\middle|\; [\mathbb{Q}(y):\mathbb{Q}] = 3 \right\}$$

$$\mathcal{G}_{6,2} = \left\{ (\alpha y^m, y^r) \;\middle|\; \begin{array}{c} y^{11-m} - \alpha^2 y^m + \alpha = 0, \\ \alpha \in \mathbb{Q}^*,\ m \in \{5,6\} \end{array} \right\}$$

(4) *The set $\mathcal{G}_7^r$ is given by $\mathcal{G}_{7,1} \cup \mathcal{G}_{7,2}$ with*

$$\mathcal{G}_{7,1} = \left\{ (y^m(\lambda+\beta y), y^r) \;\middle|\; \begin{array}{c} y^{11-m} - y^m(\lambda+\beta y)^2 + \lambda + \beta y = 0, \\ \beta, \lambda \in \mathbb{Q}^*,\ m \in \{4,5\} \end{array} \right\}$$

$$\mathcal{G}_{7,2} = \left\{ \left( \frac{y^m}{\lambda+\beta y}, y^r \right) \;\middle|\; \begin{array}{c} y^{11-m}(\lambda+\beta y)^2 - y^m + \lambda + \beta y = 0, \\ \beta, \lambda \in \mathbb{Q}^*,\ m \in \{6,7\} \end{array} \right\}$$

(5) *The set $\mathcal{G}_8^r$ is given by $\mathcal{G}_{8,1} \cup \mathcal{G}_{8,2} \cup \mathcal{G}_{8,3} \cup \mathcal{G}_{8,4}$ with*

$$\mathcal{G}_{8,1} = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}},\ y^r \right) \;\middle|\; [\mathbb{Q}(y):\mathbb{Q}] = 4 \right\}$$

$$\mathcal{G}_{8,2} = \left\{ \begin{array}{c} (y^m(\lambda+\beta y+\gamma y^2), y^r) \;\Big|\; \beta,\gamma,\lambda \in \mathbb{Q},\ \lambda\gamma \neq 0,\ m \in \{3,4\} \\ y^{11-m} - y^m(\lambda+\beta y+\gamma y^2)^2 + \lambda + \beta y + \gamma y^2 = 0 \end{array} \right\}$$

$$\mathcal{G}_{8,3} = \left\{ \begin{array}{c} \left( \frac{y^m(1+\alpha y)}{\lambda+\beta y}, y^r \right) \Big|\; \alpha,\beta,\lambda \in \mathbb{Q}^*,\ m \in \{5,6\} \\ y^{11-m}(\lambda+\beta y)^2 - y^m(1+\alpha y)^2 + (1+\alpha y)(\lambda+\beta y) = 0 \end{array} \right\}$$

$$\mathcal{G}_{8,4} = \left\{ \begin{array}{c} \left( \frac{y^m}{\lambda+\beta y+\gamma y^2}, y^r \right) \Big|\; \beta,\gamma,\lambda, \in \mathbb{Q},\ \gamma\lambda \neq 0\, m \in \{7,8\} \\ y^{11-m}(\lambda+\beta y+\gamma y^2)^2 - y^m + \lambda + \beta y + \gamma y^2) = 0 \end{array} \right\}$$

3.2. **Proof of the main theorem.**

According to Lemma 2.1, the curves $\mathcal{C}_r$ are birationally equivalent to $\mathcal{C}_1$.
If the point $(u,v) \in \mathcal{C}_1$, then $(u,v^r) \in \mathcal{C}_r$. It suffices to prove the theorem on $\mathcal{C}_1$.
Let $R \in \mathcal{C}\left(\overline{\mathbb{Q}}\right)$ be an algebraic point of degree $d$ over the field $\mathbb{Q}$ i.e. $[\mathbb{Q}(R):\mathbb{Q}] = d$ and $R_1,\ldots,R_d$ the Galois conjugates of $R$. If $d \leq 3$, these points are described by Coly and Sall ([2]); so we can assume that $4 \leq d \leq 8$.
By the Lemma 2.3, we have

$$[R_1 + R_2 + \ldots + R_d - dP_\infty] \in J_1(\mathbb{Q}).$$

So the previous relation can be written as

$$[R_1 + R_2 + \ldots + R_d - dP_\infty] = -m\,[P_0 - P_\infty] \quad \text{with} \quad 0 \leq m \leq 10.$$

The linearity of Abel-Jacobi's map gives

$$[R_1 + R_2 + \ldots + R_d + mP_0 - (d+m)P_\infty] = 0 \quad \text{with} \quad 0 \leq m \leq 10.$$

Abel-Jacobi's theorem states that there exists a rational function $\psi$ such that

$$div(\psi) = R_1 + R_2 + \ldots + R_d + mP_0 - (d+m)P_\infty.$$

The Definition 2.4 of linear systems on the curve $\mathcal{C}$ gives us the relation

$$\psi \in \mathcal{L}((d+m)P_\infty) \quad \text{and} \quad ord_{P_0}(\psi) = m.$$

For the following

$$k = \left[\frac{m+d}{2}\right] \quad \text{et} \quad l = \left[\frac{m+d-11}{2}\right].$$

The Lemma 2.4 allows us to write $\psi$ in the form

$$\psi(x,y) = \sum_{i=0}^{k} a_i y^i + x \sum_{j=0}^{l} b_j y^j \quad \text{where} \quad a_i, b_j \in \mathbb{Q}.$$

For $1 \leq i \leq d$, we must have $\psi(R_i) = 0$ and $\psi(P_0) = 0$ (or $\psi(P_1) = 0$).

The reasoning is subdivided into 5 cases depending on the degree $d$ of the algebraic points.

3.2.1. **Algebraic points of degree $d = 4$.**

Depending on the values of $m$, the reasoning is subdivided into the following sub-cases :

(1) $m = 0$. The function $\psi \in \mathcal{L}(4P_\infty)$, so

$$\psi(x,y) = a_0 + a_1 y + a_2 y^2 = 0.$$

For $[\mathbb{Q}(x,y):\mathbb{Q}] = 4$, it is necessary and sufficient that $a_0 + a_1 y + a_2 y^2$ is irreducible in $\mathbb{Q}[Y]$. So $[\mathbb{Q}(y):\mathbb{Q}] = 2$. Hence

$$\mathcal{G}_4 = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}},\ ; y^r \right) \;\middle|\; [\mathbb{Q}(y):\mathbb{Q}] = 2 \right\}.$$

(2) $m \in \{1,2,5\}$. The function $\psi \in \mathcal{L}((4+m)P_\infty)$. Let

$$\mathcal{L}((4+m)P_\infty) = \mathcal{L}((3+m)P_\infty).$$

One of the points $R_i = P_0$; therefore $[\mathbb{Q}(x,y):\mathbb{Q}] < 4$.

(3) $m \in \{2,4,6\}$. The function $\psi \in \mathcal{L}((4+m)P_\infty)$ and $ord_{P_0}(\psi) = m$.

$$\psi(x,y) = a_m y^m + \ldots + a_k y^k = y^m\left(a_m + \ldots + a_k y^{k-m}\right) = 0.$$

Since $k - m = 1$, then $[\mathbb{Q}(x,y):\mathbb{Q}] < 4$.

(4) $m \in \{7,8,9,10\}$. The function $\psi \in \mathcal{L}((4+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. the integers $m$ and $l$ satisfy $k < m$ and $l \le 1$, so

$$\psi(x,y) = b_0 x + b_1 xy = x(b_0 + b_1 y) = 0.$$

Hence $[\mathbb{Q}(x,y):\mathbb{Q}] < 4$.

In conclusion, the set of points of degree 4 is $\mathcal{G}_4$.

3.2.2. **Algebraic points of degree $d = 5$.**

Depending on the values of $m$, the reasoning is subdivided into the following sub-cases:

(1) $m \in \{0,2,4\}$. The function $\psi \in \mathcal{L}((4+m)P_\infty)$. Or

$$\mathcal{L}((5+m)P_\infty) = \mathcal{L}((4+m)P_\infty).$$

One of the points $R_i = P_0$; therefore $[\mathbb{Q}(x,y):\mathbb{Q}] < 5$.

(2) $m \in \{1,3,5\}$. The function $\psi \in \mathcal{L}((5+m)P_\infty)$ and $ord_{P_0}(\psi) = m$.

$$\psi(x,y) = a_m y^m + \ldots + a_k y^k = y^m\left(a_m + \ldots + a_k y^{k-m}\right) = 0.$$

Since $k - m \le 2$, then $[\mathbb{Q}(x,y):\mathbb{Q}] < 5$.

(3) $m \in \{6,7,8,9,10\}$. The function $\psi \in \mathcal{L}((5+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k < m$ and $l \le 3$, so $k < m$ and $l \le 2$, so

$$\psi(x,y) = b_0 x + b_1 xy = x\left(b_0 + b_1 y + b_1 y^2\right) = 0.$$

Hence $[\mathbb{Q}(x,y):\mathbb{Q}] < 5$.

In conclusion, the set $\mathcal{G}_5$ of algebraic points of degree 5 is empty: $\mathcal{G}_5 = \emptyset$.

3.2.3. **Algebraic points of degree $d = 6$.**

Depending on the values of $m$, the reasoning is subdivided into the following sub-cases:

(1) $m = 0$. The function $\psi \in \mathcal{L}(6P_\infty)$, so

$$\psi(x,y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 = 0.$$

For $[\mathbb{Q}(x,y):\mathbb{Q}]=6$, it is necessary and sufficient that $a_0+a_1y+a_2y^2+a_3y^3$ is irreducible in $\mathbb{Q}[Y]$. Therefore $[\mathbb{Q}(y):\mathbb{Q}]=3$.
We obtain the family

$$\mathcal{G}_{6,1}=\left\{\left(\frac{1}{2}\pm\sqrt{y^{11}+\frac{1}{4}},\ y^r\right)\ \middle|\ [\mathbb{Q}(y):\mathbb{Q}]=3\right\}.$$

(2) $m\in\{1,3\}$. The function $\psi\in\mathcal{L}((6+m)P_\infty)$. Or

$$\mathcal{L}((6+m)P_\infty)=\mathcal{L}((5+m)P_\infty).$$

One of the points $R_i=P_0$, therefore $[\mathbb{Q}(x,y):\mathbb{Q}]<6$.

(3) $m\in\{2,4\}$. The function $\psi\in\mathcal{L}((6+m)P_\infty)$ and $ord_{P_0}(\psi)=m$. Therefore

$$\psi(x,y)=a_my^m+\ldots+a_ky^k=y^m\left(a_m+\ldots+a_ky^{k-m}\right)=0.$$

Since $k-m\leq 2$, then $[\mathbb{Q}(x,y):\mathbb{Q}]<6$.

(4) $m\in\{5,6\}$. The function $\psi\in\mathcal{L}((6+m)P_\infty)$ and $ord_{P_0}(\psi)=m$. We have $k=m$ and $l=0$, so

$$\psi(x,y)=a_my^m+b_0x=0\Leftrightarrow x=\alpha y^m,\quad \alpha\in\mathbb{Q}^*$$

The affine equation of the curve gives

$$y^{11}=\alpha y^m(\alpha y^m-1)\Longleftrightarrow y^{11-m}-\alpha^2y^m+\alpha=0.$$

We obtain the family

$$\mathcal{G}_{6,2}=\left\{(\alpha y^m,y^r)\ \middle|\ \begin{array}{c} y^{11-m}-\alpha^2y^m+\alpha=0,\\ \alpha\in\mathbb{Q}^*,\ m\in\{5,6\}\end{array}\right\}$$

In conclusion, the set $\mathcal{G}_6$ is given by $\mathcal{G}_{6,1}\cup\mathcal{G}_{6,2}$.

### 3.2.4. **Algebraic points of degree $d=7$.**

Depending on the values of $m$, the reasoning is subdivided into the following sub-cases:

(1) $m\in\{0,2\}$. The function $\psi\in\mathcal{L}((7+m)P_\infty)$. Or

$$\mathcal{L}((7+m)P_\infty)=\mathcal{L}((6+m)P_\infty)$$

One of the points $R_i=P_0$; therefore $[\mathbb{Q}(x,y):\mathbb{Q}]<7$.

(2) $m\in\{1,3\}$. The function $\psi\in\mathcal{L}((7+m)P_\infty)$ and $ord_{P_0}(\psi)=m$, so

$$\psi(x,y)=a_my^m+\ldots+a_ky^k=y^m\left(a_m+\ldots+a_ky^{k-m}\right)=0.$$

Since $k-m\leq 3$, then $[\mathbb{Q}(x,y):\mathbb{Q}]<7$.

(3) $m \in \{4,5\}$. The function $\psi \in \mathcal{L}((7+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k < m$ and $l \leq 3$, so $k = m+1$ and $l = 0$, hence

$$\psi(x,y) = a_m y^m + a_{m+1} y^{m+1} + b_0 x = 0.$$

The value of $x$ can be written as:

$$x = y^m(\lambda + \beta y) \quad \text{with} \quad \lambda,\, \beta \in \mathbb{Q}^*.$$

The affine equation of the curve gives

$$y^{11} = y^m(\lambda + \beta y)(y^m(\lambda + \beta y) - 1).$$

This equality gives the following equation

$$y^{11-m} - (\lambda + \beta y)^2 y^m + \lambda + \beta y = 0.$$

We get the family

$$\mathcal{G}_{7,1} = \left\{ (y^m(\lambda+\beta y), y^r) \;\middle|\; \begin{array}{c} y^{11-m} - y^m(\lambda+\beta y)^2 + \lambda + \beta y = 0, \\ \lambda, \beta \in \mathbb{Q}^*,\, m \in \{4,5\} \end{array} \right\}.$$

(4) $m \in \{6,7\}$. The function $\psi \in \mathcal{L}((7+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k < m$ and $l \leq 3$, so $k = m$ and $l = 1$, so

$$\psi(x,y) = a_m y^m + b_0 x + b_1 xy = 0.$$

The value of $x$ can be written as:

$$x = y^m(\lambda + \beta y)^{-1} \quad \text{with} \quad \lambda,\, \beta \in \mathbb{Q}^*.$$

The affine equation of the curve gives

$$y^{11} = y^m(\lambda + \beta y)^{-1}\left(y^m(\lambda + \beta y)^{-1} - 1\right).$$

This gives the following equation

$$y^{11-m}(\lambda + \beta y)^2 - y^m + \lambda + \beta y = 0.$$

We obtain the family points

$$\mathcal{G}_{7,2} = \left\{ \left( \frac{y^m}{\lambda+\beta y}, y^r \right) \;\middle|\; \begin{array}{c} y^{11-m}(\lambda+\beta y)^2 - y^m + \lambda + \beta y = 0, \\ \mathbb{Q}^*,\, m \in \{6,7\} \end{array} \right\}.$$

(5) $m \in \{8,9,10\}$. The function $\psi \in \mathcal{L}((7+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k < m$ and $l \leq 3$, so

$$\psi(x,y) = b_0 x + b_1 xy + b_2 xy^2 + b_3 xy^3 = 0.$$

The factored polynomial is

$$x\left(b_0 + b_1 y + b_2 y^2 + b_3 y^3\right) = 0.$$

Hence $[\mathbb{Q}(x,y) : \mathbb{Q}] < 7$.

Therefore $\mathcal{G}_7$ is given by $\mathcal{G}_{7,1} \cup \mathcal{G}_{7,2}$.

3.2.5. **Algebraic points of degree $d = 8$.**

Depending on the values of $m$, the reasoning is subdivided into the following sub-cases:

(1) $m = 0$. The function $\psi \in \mathcal{L}(8P_\infty)$, so

$$\psi\left(x, y\right) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 = 0.$$

For $[\mathbb{Q}(x, y) : \mathbb{Q}] = 8$, it is necessary and sufficient that the rational polynomial $a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4$ is irreducible in $\mathbb{Q}[Y]$. Therefore $[\mathbb{Q}(y) : \mathbb{Q}] = 4$. We obtain the family

$$\mathcal{G}_{8,1} = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}},\ y^r \right) \;\middle|\; [\mathbb{Q}(y) : \mathbb{Q}] = 4 \right\}.$$

(2) $m \in \{1\}$. The function $\psi \in \mathcal{L}(9P_\infty)$. Or

$$\mathcal{L}(9P_\infty) = \mathcal{L}(8P_\infty)$$

One of the points $R_i = P_0$, therefore $[\mathbb{Q}(x, y) : \mathbb{Q}] < 8$.

(3) $m \in \{2\}$. The function $\psi \in \mathcal{L}(10P_\infty)$ and $ord_{P_0}\left(\psi\right) = 2$, so

$$\psi\left(x, y\right) = a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 = 0.$$

The factorization of the polynomial gives

$$y^2 \left(a_2 + a_3 y + a_4 y^2 + a_5 y^3\right) = 0.$$

So $[\mathbb{Q}(x, y) : \mathbb{Q}] < 8$.

(4) $m \in \{3, 4\}$. The function $\psi \in \mathcal{L}((8 + m)P_\infty)$ and $ord_{P_0}\left(\psi\right) = m$. The integers $m$ and $l$ satisfy $k = m + 2$ and $l = 0$, so

$$\psi\left(x, y\right) = a_m y^m + a_{m+1} y^{m+1} + a_{m+2} y^{m+2} + b_0 x = 0.$$

The variable $x$ is given by

$$x = y^m \left(\lambda + \beta y + \gamma y^2\right) \quad \text{with} \quad \beta,\ \gamma,\ \lambda \in \mathbb{Q},\ \lambda\gamma \neq 0.$$

The affine equation of the curve gives

$$y^{11} = y^m \left(\lambda + \beta y + \gamma y^2\right) \left(y^m \left(\lambda + \beta y + \gamma y^2\right) - 1\right).$$

The reduction of the polynomial gives the equation

$$y^{11-m} - \left(\lambda + \beta y + \gamma y^2\right)^2 y^m + \lambda + \beta y + \gamma y^2 = 0.$$

We get the family of points

$$\mathcal{G}_{8,2} = \left\{ \begin{array}{c|c} \left(y^m(\lambda + \beta y + \gamma y^2), y^r\right) & \beta, \gamma, \lambda \in \mathbb{Q},\ \lambda\gamma \neq 0,\ m \in \{3, 4\} \\ \multicolumn{2}{c}{y^{11-m} - y^m(\lambda + \beta y + \gamma y^2)^2 + \lambda + \beta y + \gamma y^2 = 0} \end{array} \right\}$$

(5) $m \in \{5, 6\}$. The function $\psi \in \mathcal{L}((8+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k = m+1$ and $l = 1$, so

$$\psi(x,y) = a_m y^m + a_{m+1} y^{m+1} + b_0 x + b_1 xy = 0.$$

The variable $x$ is given by

$$x = \frac{y^m(1+\alpha y)}{\lambda + \beta y} \quad \text{with} \quad \alpha,\, \beta,\, \lambda \in \mathbb{Q}^*.$$

The affine equation of the curve gives

$$y^{11} = \frac{y^m(1+\alpha y)}{\lambda + \beta y}\left(\frac{y^m(1+\alpha y)}{\lambda + \beta y} - 1\right).$$

The reduction gives the equation

$$y^{11-m}(\lambda + \beta y)^2 - y^m(1+\alpha y)^2 + (1+\alpha y)(\lambda + \beta y) = 0.$$

We get the family of points

$$\mathcal{G}_{8,3} = \left\{ \begin{array}{l|l} \left(\dfrac{y^m(1+\alpha y)}{\lambda+\beta y}, y^r\right) & \alpha, \beta, \lambda \in \mathbb{Q}^*,\ m \in \{5,6\} \\ \multicolumn{2}{c}{y^{11-m}(\lambda+\beta y)^2 - y^m(1+\alpha y)^2 + (1+\alpha y)(\lambda+\beta y) = 0} \end{array} \right\}$$

(6) $m \in \{7, 8\}$. The function $\psi \in \mathcal{L}((8+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k = m$ and $l = 2$, so

$$\psi(x,y) = a_m y^m + b_0 x + b_1 xy + b_2 y^2 = 0.$$

The variable $x$ is given by

$$x = \frac{y^m}{\lambda + \beta y + \gamma y^2} \quad \text{with} \quad \beta,\, \gamma,\, \lambda \in \mathbb{Q},\ \lambda\gamma \neq 0.$$

The affine equation of the curve gives

$$y^{11} = \frac{y^m}{\lambda + \beta y + \gamma y^2}\left(\frac{y^m}{\lambda + \beta y + \gamma y^2} - 1\right).$$

The reduction gives the equation

$$y^{11-m}(\lambda + \beta y + \gamma y^2)^2 - y^m + \lambda + \beta y + \gamma y^2 = 0.$$

We obtain the family of points

$$\mathcal{G}_{8,4} = \left\{ \begin{array}{l|l} \left(\dfrac{y^m}{\lambda+\beta y+\gamma y^2}, y^r\right) & \beta, \gamma, \lambda, \in \mathbb{Q},\ \gamma\lambda \neq 0\, m \in \{7,8\} \\ \multicolumn{2}{c}{y^{11-m}(\lambda+\beta y+\gamma y^2)^2 - y^m + \lambda + \beta y + \gamma y^2) = 0} \end{array} \right\}$$

(7) $m \in \{9, 10\}$. The function $\psi \in \mathcal{L}((8+m)P_\infty)$ and $ord_{P_0}(\psi) = m$. The integers $m$ and $l$ satisfy $k < m$ and $l \leq 3$, so

$$\psi(x,y) = b_0 x + b_1 xy + b_2 xy^2 + b_3 xy^3 = 0.$$

The factorization gives

$$x\left(b_0 + b_1 y + b_2 y^2 + b_3 y^3\right) = 0.$$

Hence $[\mathbb{Q}(x,y) : \mathbb{Q}] < 8$.

In summary, the set $\mathcal{G}_8$ is given by $\mathcal{G}_{8,1} \cup \mathcal{G}_{8,2} \cup \mathcal{G}_{8,3} \cup \mathcal{G}_{8,4}$.

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