



200 CHARACTERIZATIONS OF PARALLELOGRAMS

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Abstract. This is the fourth part in an extensive project to collect, categorize and prove both old and new necessary and sufficient conditions for when a convex quadrilateral is a parallelogram. With an additional 41, we achieve a total of 200 compiled characterizations of parallelograms.

1. INTRODUCTION

There has been a rapid development the last few years regarding the number of known characterizations of certain convex quadrilaterals. In 2021, we concluded a study on tangential quadrilaterals by compiling a list of 100 characterizations of such quadrilaterals [43], and in 2025 the same task regarding squares achieved a total of 150 characterizations [47].

When the focus was shifted to parallelograms, we never thought they would have an even larger total number of characterizations, but already the previous three parts [45, 46, 48] (which we refer to as Part 1, Part 2, Part 3 respectively) accumulate 159 necessary and sufficient conditions. In this **Part 4**, we add 41 more and thus get a huge collection of 200 characterizations of parallelograms! About half of the sufficient conditions in this part were discovered by the authors, most of the rest were found in various mathematical journals or books from different time periods and in several different languages.

At the end we have compiled a list with all 200 characterizations in chronological order regarding the oldest source for each sufficient condition that we know of. About a dozen updates to the corresponding list in [46] have been made, where we found older publications of those characterizations, and this new list is twice as long as the previous one that only contained 103 necessary and sufficient conditions.

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2. BISECTORS AND BIMEDIANS

We begin by proving five characterizations of parallelograms that are about different bisectors or bimedians. They were all discovered by the second named author. A *bimedian* is a line segment that connects the mid-points of opposite sides in a quadrilateral.

Theorem 2.1. *A convex quadrilateral $ABCD$ satisfies any one of:*

- (a) *it has two pairs of opposite parallel perpendicular bisectors*
- (b) *any bimedian divide the quadrilateral into two congruent quadrilaterals with the same orientation*
- (c) *one diagonal bisects the perimeter and one bimedian bisects the area*
- (d) *one diagonal and one bimedian bisect the perimeter*
- (e) *$FH \parallel AB$ and $EG \parallel BC$, where the interior angle bisectors at A and B , B and C , C and D , D and A intersect at E , F , G , H respectively*

if and only if it's a parallelogram.

Proof. (a) Two pairs of opposite parallel perpendicular bisectors is equivalent to two pairs of opposite parallel sides, which characterize parallelograms.

(b) Any bimedian divide a parallelogram into two congruent parallelograms (see Figure 1).

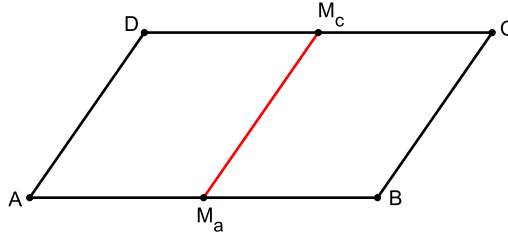


FIGURE 1. A bimedian $M_a M_c$

Conversely, if a bimedian divide a convex quadrilateral into two congruent quadrilaterals with the same orientation, then opposite sides have the same length. This confirms that it is a parallelogram.

(c) One bimedian bisects the area if and only if the quadrilateral is a trapezoid according to Proposition 4 in [38]. One diagonal bisects the perimeter of a trapezoid if and only if it is a parallelogram according to Theorem 6.1 (c) in Part 1.

(d) In a convex quadrilateral with consecutive sides a , b , c , d , it holds that one bimedian bisects the perimeter if and only if two opposite sides are congruent, and one diagonal bisects the perimeter if and only if $a + b = c + d$ or $a + d = b + c$. A quadrilateral where $a = c$ or $b = d$ and $a + b = c + d$ or $a + d = b + c$ clearly satisfies that both pairs of opposite sides are equal. This is equivalent to that it is a parallelogram according to Theorem 2.1 (a) in Part 1.

(e) We use notations as in Figure 2. Two pairs of consecutive angle bisectors intersect at F and H . By congruent triangles (AAS), we have $FL = FM = FN$ and $HI = HJ = HK$. Drawing $FO \parallel LI$ and $FQ \parallel NJ$ yields the two congruent triangles FOH and FQH (RHS), since $HO = HQ$.

We get that FH is parallel to AB if and only if the three points O, H, Q coincide, which is equivalent to that AB and DC are parallel. In the same way, EG is parallel to BC if and only if BC and AD are parallel. That both pairs of opposite sides are parallel is the definition of a parallelogram. \square

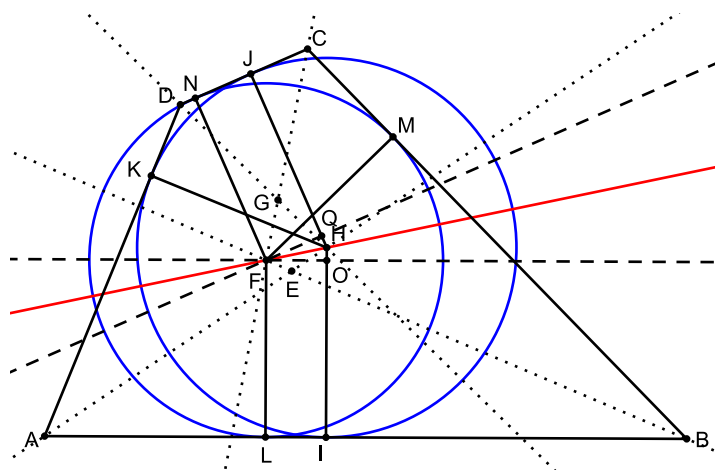


FIGURE 2. Tangency points for two circles centered at F and H

3. DOUBLE TYPE QUADRILATERALS

In the proof of the fourth condition in the next theorem, we will need the following preliminary result, which we will use one more time later in this paper.

Lemma 3.1. *If two triangles have a common base and one of them fits inside the other, then the interior triangle has a shorter perimeter than the exterior triangle.*

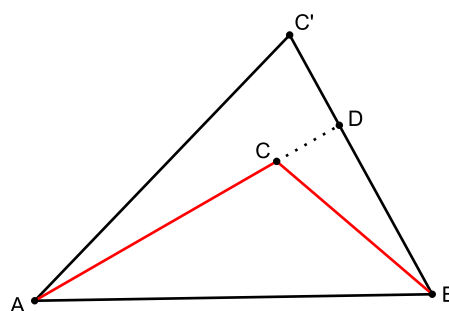


FIGURE 3. A triangle inside another triangle

Proof. This can be interpreted as: If C is a point inside triangle ABC' , then $AC + BC < AC' + BC'$. We start by extending AC to intersect BC' at a point D (see Figure 3). Applying the Triangle Inequality, we get $AC' + C'D > AD$ and $CD + BD > BC$. Adding them yields

$$AC' + BC' + CD > AD + BC$$

which we rewrite as

$$AC' + BC' > AD - CD + BC = AC + BC.$$

This proves the inequality $AC + BC < AC' + BC'$, and adding the common side AB results in the inequality for the triangle perimeters. \square

In the next theorem we study five different pairs of convex quadrilaterals that characterize parallelograms. The first is from [40, p. 10], (d) is from [91, p. 22], and Martín Guerra's proof of (e), that was given in [15], is included here with a few more details added than in the original publication.

An *extangential quadrilateral* is characterized by satisfying at least one of $a + b = c + d$ or $a + d = b + c$ (see [36]), a *quasi-isosceles quadrilateral* is defined as having at least one pair of equal opposite sides (see [44]), and a *Pythagorean quadrilateral* is defined as satisfying at least one of $a^2 + b^2 = c^2 + d^2$ or $a^2 + d^2 = b^2 + c^2$ (see [39]).

Extangential quadrilaterals can also be defined as having at least one diagonal that bisects the perimeter. Quadrilaterals where at least one diagonal bisects the other diagonal were called bisect-diagonal quadrilaterals in [41], so the fourth characterization is a combination of extangential and bisect-diagonal quadrilaterals. Quadrilaterals satisfying $a' + b' = c' + d'$, where a', b', c', d' are the length of the diagonal parts (the semidiagonals), have been named *semidiagonal quadrilaterals* in [15].

Theorem 3.1. *A convex quadrilateral with consecutive sides a, b, c, d and semidiagonals $a' = AP, b' = BP, c' = CP, d' = DP$, where the diagonals intersect at P , satisfies any one of:*

- (a) *it is both extangential and quasi-isosceles*
- (b) *it is both Pythagorean and quasi-isosceles*
- (c) *$a + d = b + c$ and $a^2 + b^2 = c^2 + d^2$*
- (d) *one diagonal bisects the perimeter and the other diagonal bisects the first diagonal*
- (e) *$a + b = c + d$ and $a' + b' = c' + d'$*

if and only if it's a parallelogram.

Proof. (a) According to the proof of Theorem 2.1 (d), this is just another way of stating that characterization.

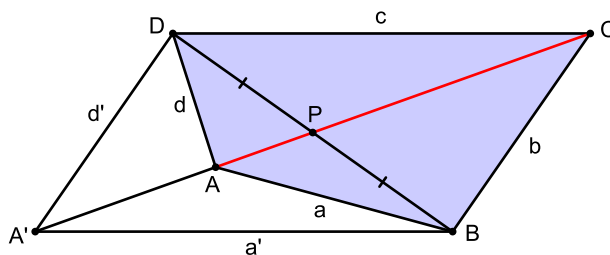
(b) A quadrilateral where $a^2 + b^2 = c^2 + d^2$ or $a^2 + d^2 = b^2 + c^2$ and $a = c$ or $b = d$ satisfies that $a = c$ and $b = d$, characterizing it is a parallelogram according to Theorem 2.1 (a) in Part 1.

(c) We rewrite the two given conditions as

$$\begin{cases} (a + c)(a - c) = (d + b)(d - b) \\ a - c = b - d. \end{cases}$$

Substitution and factorization yields $(a + c + d + b)(b - d) = 0$, with only one possible solution $b = d$ and hence also $a = c$. Equal opposite sides characterize parallelograms according to Theorem 2.1 (a) in Part 1.

(d) Without loss of generality, we only study the case when $BP = DP$ and $a + d = b + c$, where P is the point of intersection of the diagonals and we have subtracted the diagonal length BD from both sides of the second equality. Suppose for the sake of contradiction that $AP < CP$. We expand

FIGURE 4. Construction of parallelogram $A'BCD$

AP to a point A' such that $A'P = CP$, making $A'BCD$ a parallelogram since its diagonals bisect each other (see Figure 4). Its sides satisfy $d' = b$ and $a' = c$, where $d' = A'D$ and $a' = A'B$. From $a + d = b + c$, we then get $a + d = d' + a'$. But A is an interior point in triangle $A'BD$, so it holds that $a + d < a' + d'$ according to the Lemma. Hence we have reached a contradiction, and in the same way, the other case $AP > CP$ yields a contradiction. The only remaining possibility is $AP = CP$, and together with the assumption $BP = DP$, we conclude that $ABCD$ is a parallelogram.

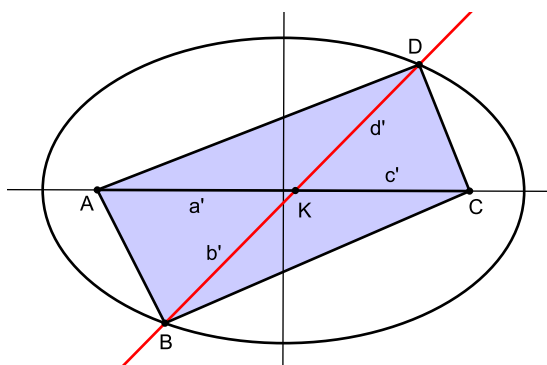


FIGURE 5. Extangential and semidiagonal quadrilateral

(e) All points on an ellipse satisfy the condition $AB + BC = CD + DA$, so an extangential quadrilateral (which is characterized by the same equality) can fit inside an ellipse such that its two focal points are at A and C , and the other two vertices belong to the ellipse (see Figure 5). The equation for an ellipse with focal points at $A = (-c, 0)$ and $C = (c, 0)$ has the equation

$$b^2x^2 + a^2y^2 - a^2b^2 = 0,$$

whose major axis and minor axis have length $2a$ and $2b$ respectively. Note that in this proof we use a , b , c as properties of the ellipse, *not* as lengths of the sides of the quadrilateral. We will consider a line with equation $y = m(x - k)$ passing through point $K = (k, 0)$ where $k > 0$ that intersects the ellipse at B and D . The proof amounts to showing that the extangential quadrilateral is also semidiagonal, that is $AK + KB = CK + KD$, if and only if $k = 0$, which is equivalent to that the diagonals bisect each other at the origin.

Inserting the equation for the line into the equation for the ellipse and simplifying, we get

$$(b^2 + a^2m^2)x^2 - 2ka^2m^2x + a^2m^2k^2 - a^2b^2 = 0$$

which has the solutions

$$x = \frac{ka^2m^2 \pm ab\sqrt{a^2m^2 + b^2 - m^2k^2}}{a^2m^2 + b^2}.$$

Then

$$y = m(x - k) = \frac{m(-b^2k \pm ab\sqrt{a^2m^2 + b^2 - m^2k^2})}{a^2m^2 + b^2}$$

and we get the coordinates for $B = (x_B, y_B)$ and D as

$$B = \left(\frac{ka^2m^2 - ab\sqrt{a^2m^2 + b^2 - m^2k^2}}{a^2m^2 + b^2}, \frac{m(-b^2k - ab\sqrt{a^2m^2 + b^2 - m^2k^2})}{a^2m^2 + b^2} \right)$$

$$D = \left(\frac{ka^2m^2 + ab\sqrt{a^2m^2 + b^2 - m^2k^2}}{a^2m^2 + b^2}, \frac{m(-b^2k + ab\sqrt{a^2m^2 + b^2 - m^2k^2})}{a^2m^2 + b^2} \right)$$

We calculate BK and DK with the distance formula, where

$$|BK|^2 = (k - x_B)^2 + (0 - y_B)^2 = (1 + m^2) \left(\frac{kb^2 + ab\sqrt{a^2m^2 + b^2 - m^2k^2}}{a^2m^2 + b^2} \right)^2$$

and a similar formula holds for DK . Hence

$$AK + KB = c + k + \frac{b(bk + a\sqrt{a^2m^2 + b^2 - m^2k^2})\sqrt{1 + m^2}}{a^2m^2 + b^2}$$

and

$$CK + KD = c - k + \frac{b(-bk + a\sqrt{a^2m^2 + b^2 - m^2k^2})\sqrt{1 + m^2}}{a^2m^2 + b^2}.$$

Finally

$$AK + KB = CK + KD \Leftrightarrow k \left(2 + \frac{2b^2\sqrt{1 + m^2}}{a^2m^2 + b^2} \right) = 0 \Leftrightarrow k = 0$$

so a convex quadrilateral $ABCD$ where $AB + BC = CD + DA$ also satisfies $AK + KB = CK + KD$ if and only if it is a parallelogram, where the diagonals intersect (and bisect) each other at K . \square

We note that (c) is a combination of extangential and Pythagorean quadrilateral, but it cannot be any combination. Considering when $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$, we get $a - c = d - b$ and $a^2 - c^2 = d^2 - b^2$, and via factorization and substitution yields

$$(a - c)(a + c - d - b) = 0.$$

Here there are two possible cases: $a = c$ implying $b = d$, so a parallelogram, but also $a + c = b + d$ (a tangential quadrilateral). The two conditions $a + b = c + d$ and $a + c = b + d$ imply $a = d$ and $b = c$ (a kite). Hence this combination of extangential and Pythagorean quadrilateral is not a characterization of parallelograms.

4. TILTED KITES

A *tilted kite* is a quadrilateral where at least two opposite angles are equal [42]. In the following theorem we prove four necessary and sufficient conditions for when a tilted kite is a parallelogram. The fourth is from [58].

Theorem 4.1. *A convex tilted kite $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ satisfies any one of:*

- (a) $\angle A = \angle C$ and $h_B = h_D$, where h_B and h_D are the heights in triangles ABC and ADC from B and D respectively
- (b) $\angle A = \angle C$ and $a^2 + b^2 = c^2 + d^2$
- (c) $\angle A = \angle C$ and $m_B = m_D$, where m_B and m_D are the medians in triangles ABC and ADC from B and D respectively
- (d) $\angle B = \angle D$, angles ACD and ACB are acute, and $AM = CM$ where M is the midpoint of diagonal BD

if and only if it's a parallelogram.

Proof. (a) Triangles $BH_B P$ and $DH_D P$ are similar (AA), where P is the intersection of the diagonals and H_B and H_D are the projections of B and D respectively on AC (see Figure 6). Hence

$$\frac{h_B}{BP} = \frac{h_D}{DP}$$

and we get that $h_B = h_D$ if and only if $BP = DP$. The two conditions $\angle A = \angle C$ and $BP = DP$ characterize parallelograms according to Theorem 7.1 (f) in Part 1.

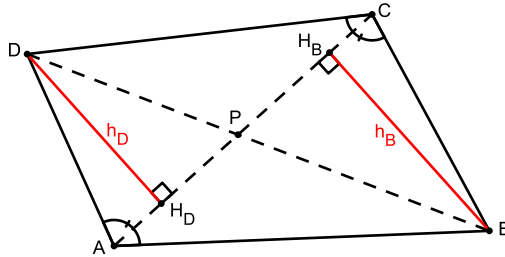
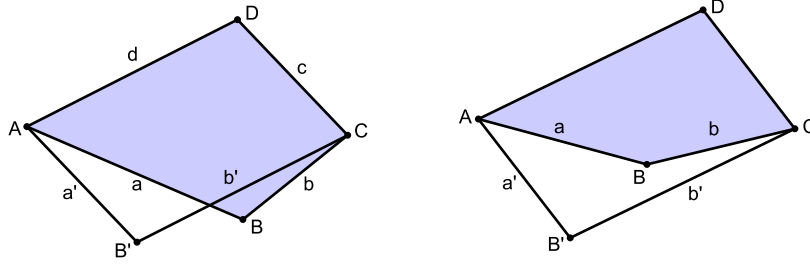


FIGURE 6. $\triangle BH_B P \sim \triangle DH_D P$

(b) The assumptions are trivially true in a parallelogram since $a = c$ and $b = d$, so let us consider the converse.

We begin by constructing a point B' such that $AB' := a' = c$ and $B'C := b' = d$. Then $AB'CD$ is a parallelogram, so $\angle DAB' = \angle DCB'$, implying that $\angle B'AB = -\angle B'CB$ (see Figure 7, left), which is impossible, or $\angle B'AB = \angle B'CB$ (see Figure 7, right). The latter case means that B lies inside of $AB'CD$, so $\angle B > \angle B'$. We also get $a'^2 + b'^2 = c^2 + d^2$, and using $a^2 + b^2 = c^2 + d^2$ this yields $a^2 + b^2 = a'^2 + b'^2$. Together with the Law of Cosines, it follows that $ab \cos B = a'b' \cos B'$. Since $\angle B > \angle B'$, we get $\cos B < \cos B'$. Hence $ab > a'b'$. Then

$$\begin{cases} a'^2 + b'^2 = a^2 + b^2 \\ 2a'b' < 2ab \end{cases} \Rightarrow (a' + b')^2 < (a + b)^2.$$

FIGURE 7. Different (im)possible positions for vertex B

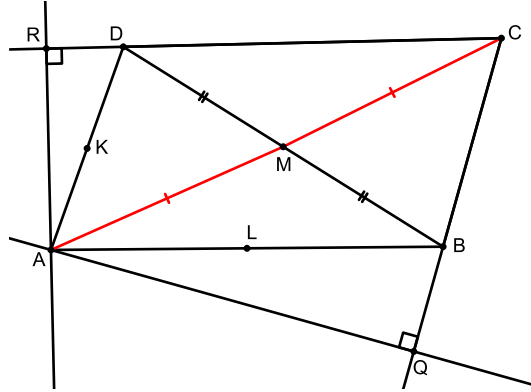
Therefore $a' + b' < a + b$ since both sides of this one-dimensional inequality are positive. But this latter inequality cannot be true, since it must hold $a' + b' > a + b$ according to the Lemma i Section 3. We have thus proved that B can neither be located outside of $B'C$ (and thus neither outside AB') nor inside $AB'CD$. B cannot be on $B'C$ or AB' either, since then $\angle A = \angle C$ would not be satisfied. So we only have the remaining possibility that $B = B'$. Then $ABCD$ is a parallelogram.

(c) Using the formula for the length of a median in a triangle, we get that $4m_B^2 = 4m_D^2$ is equivalent to

$$2(a^2 + b^2) - AC^2 = 2(c^2 + d^2) - AC^2$$

which simplifies to $a^2 + b^2 = c^2 + d^2$, so this condition is a characterization of parallelograms according to (b).

(d) These conditions are satisfied in a parallelogram since the diagonals bisect each other, so we prove the converse.

FIGURE 8. M is the midpoint of BD but maybe not a point on AC

Let R and Q be projections of A on the extensions of the sides DC and BC , and K and L be the midpoints of DA and BA respectively (see Figure 8). Then K and L are the circumcenters of triangles ARD and AQB respectively, so

$$LQ = LB = \frac{1}{2}AB \quad \text{and} \quad KR = KD = \frac{1}{2}AD.$$

Since $KM = \frac{1}{2}AB$ and $LM = \frac{1}{2}AD$, it follows that $LM = KR$ and $KM = LQ$. Quadrilateral $LMKA$ is a parallelogram, so $\angle MLA = \angle MKA$. Using

that triangles KRD and LQB are isosceles, we get

$$\begin{aligned}\angle RKD &= 180^\circ - 2\angle KDR = 180^\circ - 2(180^\circ - \angle ADC) \\ &= 2\angle ADC - 180^\circ = 2\angle ABC - 180^\circ \\ &= 2(180^\circ - \angle LBQ) - 180^\circ = 180^\circ - 2\angle LBQ \\ &= \angle BLQ\end{aligned}$$

and also

$$\begin{aligned}\angle MLQ &= 180^\circ - \angle ALM + \angle BLQ \\ &= 180^\circ - \angle AKM + \angle DKR \\ &= \angle MKR.\end{aligned}$$

This implies that triangles LMQ and KRM are congruent (SAS), so $RM = QM$. Since angles ACD and ACB are acute by assumption, RQ intersects AC at one point, and the perpendicular bisectors to these line segments are not parallel, so they also intersect at one point. This point must be M since $RM = QM$ and $AM = MC$. By construction we have that $\angle AQC = \angle ARC = 90^\circ$ so $AQCR$ is a cyclic quadrilateral. The perpendicular bisectors to its diagonals intersect at its circumcenter, so we conclude that M is the circumcenter of $AQCR$. Since $\angle AQC = 90^\circ$, M is also the midpoint of AC . This proves that both diagonals AC and BD bisect each other at M , confirming that $ABCD$ is a parallelogram. \square

5. SYMMETRY AND TRANSFORMATIONS

Here we prove nine characterizations of parallelograms that are about symmetry or transformations. Condition (a) is from [73, p. 191] and to prove the sufficient part in (b) was Problem 3 for Grade 8 on the 1989 Leningrad Mathematical Olympiad [23]. These two are closely related to Theorem 2.1 (d) in Part 2.

The concept of *sloping symmetry*, that is part of (c), (d) and (e), was described by Günter Graumann in [30] as a reflection, not necessarily orthogonal to the axis, which maps the quadrilateral onto itself. For such a sloping reflection the connection of one point and its picture is bisected by the axis and all connection lines between a point and its picture are parallel to each other.

The transformation centering and (f) were introduced in [77] and (g) is very similar to a theorem proved in [94, p. 4]. To prove the sufficient part in (i) was given as Problem 687 by Sidney Kung in [57] and solved via a different argument than ours by Michael Goldberg the same year.

Theorem 5.1. *A convex quadrilateral $ABCD$, where the diagonals intersect at P and $\{X_1, X_5\} \in AB$, $X_2 \in BC$, $\{X_3, X_6\} \in CD$, $X_4 \in DA$ such that $P \in \{X_1X_3, X_2X_4, X_5X_6\}$ and $X_1 \neq X_5$, satisfies any one of:*

- (a) P is the midpoint of X_1X_3 and X_5X_6
- (b) P is the midpoint of X_1X_3 and X_2X_4
- (c) both bimedians are axes of sloping symmetry
- (d) both diagonals are axes of sloping symmetry
- (e) one diagonal and one bimedian are axes of sloping symmetry

- (f) it is centered to a non-trivial quadrilateral with the same angles as $ABCD$
- (g) no spiral similarity can send AB to DC or AD to BC
- (h) $AE = CF$ for all $E \in \{AB, AD\}$ and $F \in \{CD, CB\}$ such that $P \in EF$
- (i) all straight lines through a fixed point are perimeter bisectors if and only if it's a parallelogram.

Proof. (a) Any transversal, for example X_1X_3 , between two opposite sides in a parallelogram that goes through P is bisected by P . This is a direct consequence of two congruent triangles X_1BP and X_3DP (ASA), so $X_1P = X_3P$. Thus we proceed to the converse.

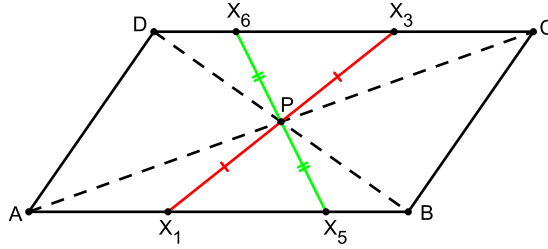


FIGURE 9. Two line segments bisected by P

In a convex quadrilateral where $X_1P = X_3P$ and $X_5P = X_6P$, triangles X_1PX_5 and X_3PX_6 are congruent (SAS), so $\angle PX_1X_5 = \angle PX_3X_6$, making AB and DC parallel (see Figure 9). Then $\angle BX_5P = \angle DX_6P$, so triangles BX_5P and DX_6P are congruent (ASA), resulting in $BP = DP$. Hence triangles ABP and CDP are also congruent (AAS), so $AP = CP$. Since the diagonals bisect each other, $ABCD$ is a parallelogram.

(b) As in (a), any transversal between two opposite sides in a parallelogram that goes through P is bisected by P .

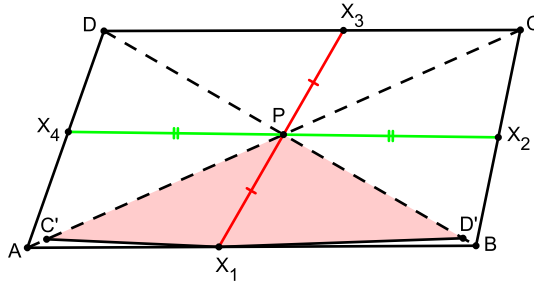


FIGURE 10. $CP = C'P$ and $DP = D'P$

Conversely, assume first that $CP \leq AP$ and $DP \leq BP$. Since $\angle CPD = \angle APB$, we can rotate triangle CPD 180° around P so that it fits inside triangle APB (see Figure 10). Then C maps to C' , D maps to D' and X_3 maps to X_1 . But we must have $X_3P = X_1P$ and CX_3D is a straight line segment. The only way for these two conditions to be fulfilled is if $C'P = AP$

and $D'P = BP$, otherwise $C'X_1D'$ will be a broken line segment. Hence the diagonals bisect each other, so $ABCD$ is a parallelogram.

If the assumed inequalities are not satisfied, then we can instead rotate one of triangles BCP or DAP onto the other and the same argument yields that the diagonals bisect each other.

(c) Both bimedians in a parallelogram are obviously axes of sloping symmetry since a bimedian divides a parallelogram into two congruent parallelograms.

Conversely, any convex quadrilateral having one bimedian as an axis of sloping symmetry must be a trapezoid since

$$\begin{cases} S(A) = B \\ S(B) = A \end{cases} \quad \text{and} \quad \begin{cases} S(D) = C \\ S(C) = D \end{cases}$$

where S is the function of the sloping reflection. Hence DC is parallel to AB . Consequently a convex quadrilateral having two bimedians as axes of sloping symmetry must have both pairs of opposite sides parallel, so it is a parallelogram by definition.

(d) In a convex quadrilateral having one diagonal as an axis of sloping symmetry (see Figure 11), we get by similarity that

$$\frac{EF}{AP} = \frac{DF}{DP} = \frac{FG}{PC}$$

so $EF = FG$ is equivalent to $AP = PC$. Therefore this is possible only in a quadrilateral where one diagonal bisects the other diagonal (called a bisect-diagonal quadrilateral in [41]). Hence both diagonals are axes of sloping symmetry if and only if both diagonals bisect each other, which characterize parallelograms according to Theorem 3.1 (a) in Part 1.

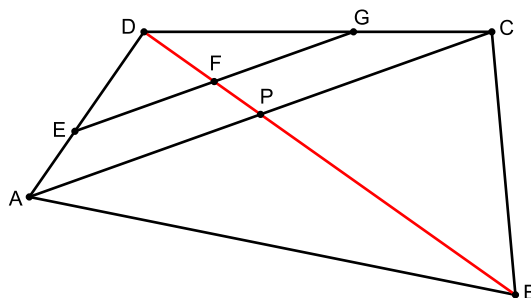


FIGURE 11. Diagonal BD is an axis of sloping symmetry

(e) One diagonal and one bimedian are axes of sloping symmetry if and only if the quadrilateral is a trapezoid where one diagonal bisects the other diagonal according to conclusions in the proofs of (c) and (d). This is equivalent to the quadrilateral being a parallelogram according to Theorem 7.1 (a) in Part 1.

(f) The transformation *centering* was defined in [77] in such a way that a vertex of a quadrilateral is transformed to the center of the circumcircle surrounding the remaining three vertices. Hence the vertices of a cyclic

quadrilateral are all transformed to the common circumcenter. A fundamental property of centering is that each vertex angle of a quadrilateral $ABCD$ is transformed into the supplementary angle of the opposite corner ([77, p. 26]). If we call the centered quadrilateral $A'B'C'D'$, then we have

$$\angle A' = \pi - \angle C, \quad \angle B' = \pi - \angle D, \quad \angle C' = \pi - \angle A, \quad \angle D' = \pi - \angle B.$$

A parallelogram is characterized by having any pair of adjacent angles supplementary, for instance

$$\begin{cases} \angle A = \pi - \angle B = \angle D' \\ \angle B = \pi - \angle C = \angle A' \\ \angle C = \pi - \angle D = \angle B' \\ \angle D = \pi - \angle A = \angle C'. \end{cases}$$

There are only two types of quadrilaterals that have supplementary angles: cyclic quadrilaterals where opposite angles are supplementary, and parallelograms where adjacent angles are supplementary. But cyclic quadrilaterals are centered to a point and are thus excluded. Hence a convex quadrilateral $ABCD$ is centered to a non-trivial quadrilateral $A'B'C'D'$ with the same vertex angles if and only if it is a parallelogram.

(g) In an Argand plane, we let the vertices of $ABCD$ be represented by the complex numbers A, B, C, D . A *spiral similarity* has the form

$$\sigma(z) = z_0 + \xi(z - z_0),$$

where z_0 is the center of the spiral similarity and ξ is a complex number determining the rotation and dilation. There are two cases to consider, but the only difference between them is that $B \leftrightarrow D$, which does not effect the outcome. The transformations $\sigma(A) = D$ and $\sigma(B) = C$ yield

$$\begin{cases} z_0 + \xi(A - z_0) = D \\ z_0 + \xi(B - z_0) = C \end{cases}$$

and this system of equations has the solution

$$\xi = \frac{D - C}{A - B}, \quad z_0 = \frac{AC - BD}{A - B + C - D}.$$

Here $A \neq B$ since AB is a side in the quadrilateral. Hence the spiral similarity is possible if and only if $A - B + C - D \neq 0$, that is, for all convex quadrilaterals except for parallelograms according to Theorem 3.1 (e) in Part 2.

(h) In a parallelogram, triangles AEP and CFP are congruent (ASA), so we directly get $AE = CF$.

Conversely, suppose without loss of generality there are two pairs of points $\{E, E'\} \in AB$ and $\{F, F'\} \in CD$ such that $AE = CF$ and $AE' = CF'$. Then $EE' = FF'$. Let us denote $\alpha := \angle BAP$, $\gamma := \angle DCP$, $v := \angle APE$, $u := \angle EPE'$ and $w := v + u$ (see Figure 12). By the Law of Sines, we get

$$\frac{\sin \alpha}{EP} = \frac{\sin v}{AE}, \quad \frac{\sin \gamma}{FP} = \frac{\sin v}{CF}$$

so by the assumption,

$$\frac{\sin \alpha}{\sin \gamma} = \frac{EP}{FP}.$$

In the same way

$$\frac{\sin(\pi - (\alpha + w))}{EP} = \frac{\sin u}{EE'}, \quad \frac{\sin(\pi - (\gamma + w))}{FP} = \frac{\sin u}{FF'}$$

which yields

$$\frac{\sin(\alpha + w)}{\sin(\gamma + w)} = \frac{EP}{FP}$$

Therefore

$$\frac{\sin \alpha}{\sin \gamma} = \frac{\sin(\alpha + w)}{\sin(\gamma + w)}$$

which by cross multiplication and applying the addition formula for sines simplifies into

$$\sin \alpha \cos \gamma \sin w = \cos \alpha \sin \gamma \sin w.$$

This is equivalent to

$$\sin w \sin(\alpha - \gamma) = 0$$

where $\sin w \neq 0$, so this equation has the two solutions $\alpha = \gamma$ and $\alpha = \pi + \gamma$. The second is not valid since $\alpha < \pi$ by the angle sum of a triangle. Hence we conclude that AB and DC are parallel. But since $\{E, E'\} \in AD$ and $\{F, F'\} \in BC$ are also possible, we get in the same way that AD and BC are parallel, so $ABCD$ is indeed a parallelogram.

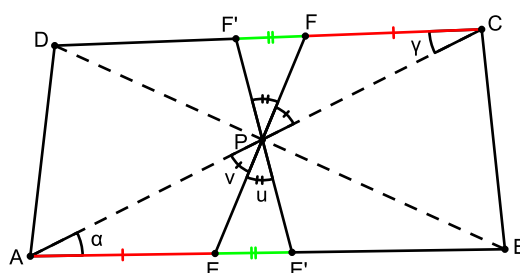


FIGURE 12. $AE = CF$ and $AE' = CF'$

(i) In a parallelogram, let E and F be points on AB and CD respectively such that $P \in EF$. Then $AE = CF$, $BE = DF$ and $BC = AD$, so EF is a perimeter bisector. In the same way we realize that any transversal through the diagonal intersection is a perimeter bisector.

Conversely, suppose transversal EF is a perimeter bisector in a convex quadrilateral that goes through a fixed point O . Let E' and E'' be points on equal distance from E and draw transversals from them through O (see Figure 13). They intersect the opposite side at points F' and F'' such that

$$FF' = FF'' = EE'' = EE' := k$$

since each transversal EF , $E'F'$, $E''F''$ is a perimeter bisector. Let us denote $\alpha := \angle BE'O$, $\beta := \angle DF'O$, $v := \angle E'OE$, $u := \angle EOE''$, and $w := v + u$. Applying the Law of Sines, we get

$$\frac{\sin \alpha}{OE} = \frac{\sin v}{k} = \frac{\sin \beta}{OF}$$

and

$$\frac{\sin(\alpha + w)}{OE} = \frac{\sin u}{k} = \frac{\sin(\beta + w)}{OF}.$$

From these, we deduce that

$$\frac{\sin(\alpha + w)}{\sin(\beta + w)} = \frac{\sin \alpha}{\sin \beta}$$

which by some algebraic manipulations is rewritten as

$$\sin w \sin(\alpha - \beta) = 0.$$

Since $\sin w \neq 0$ and $\alpha < \pi$, the only possible solution is $\alpha = \beta$. This implies that AB and DC are parallel. By choosing E and F on the other pair of opposite sides, we get that AD and BC must also be parallel, confirming that $ABCD$ is a parallelogram. \square

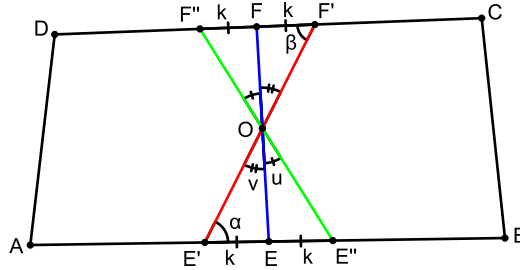


FIGURE 13. Three perimeter bisectors

6. VECTORS AND TRIANGLES

In the penultimate theorem we have ten necessary and sufficient conditions for when a convex quadrilateral is a parallelogram that are about vectors (written in **bold font**) or triangles. Condition (a) is due to Celia Hoyles in the anthology [75, pp. 177–178], the sufficient condition in (d) is from [18], (e), which generalize Theorem 5.1 (d) in Part 1, is from [80] and we cite the published proof of that problem (this theorem can easily be generalized even further), the sufficient condition in (f) is from [53], the necessary condition in (g) is a problem due to Miguel Ochoa Sanchez that was proved in the same way at [12] but the converse was not considered, and (i) was formulated and proved in another way in [26]. The last four are related to Theorem 3.1 (l) in Part 2.

We denote by T_{XYZ} the area of triangle XYZ .

Theorem 6.1. *A convex quadrilateral $ABCD$ satisfies any one of:*

- (a) $\mathbf{EI} = \mathbf{0}$, where E is any point in the plane of $ABCD$, F is the reflection of E in D , G is the reflection of F in C , H is the reflection of G in B , and I is the reflection of H in A
- (b) $\mathbf{AG}_1 + \mathbf{CG}_3 = \mathbf{0}$ or $\mathbf{BG}_2 + \mathbf{DG}_4 = \mathbf{0}$, where G_1, G_2, G_3, G_4 are the centroids of triangles BCD, CDA, DAB, ABC respectively
- (c) $\mathbf{A''A'} = \mathbf{C'C''}$ or $\mathbf{B'B''} = \mathbf{D''D'}$, where the four external squares $ABB'A'', BCC'B'', CDD'C'', DAA'D''$ are erected on the sides

- (d) $\mathbf{AB} + \mathbf{AD} + \mathbf{AP} = \mathbf{BC} + \mathbf{DC} + \mathbf{PC}$, where $P = AC \cap BD$
 (e) all nine quadrilaterals formed by joining appropriate trisection points of the sides (see Figure 16) have equal area
 (f) $T_{ABG_0} = T_{CDG_0}$ and $T_{DAG_0} = T_{BCG_0}$, where G_0 is the vertex centroid of $ABCD$
 (g) triangle NMK is equilateral, where CBJ , DCK , ADL are positively oriented equilateral triangles, and M and N are the centroids of CBJ and ADL respectively
 (h) triangle $N'M'K'$ is equilateral, where CBJ' , DCK' , ADL' are negatively oriented equilateral triangles, and M' and N' are the centroids of CBJ' and ADL' respectively
 (i) triangle SRQ is equilateral, where ADS , BCR , ACQ are positively oriented equilateral triangles
 (j) triangle $S'R'Q'$ is equilateral, where ADS' , BCR' , ACQ' are negatively oriented equilateral triangles

if and only if it's a parallelogram.

Proof. (a) Using notations as in Figure 14, we have $2\mathbf{w} + 2\mathbf{x} + 2\mathbf{y} + 2\mathbf{z} = \mathbf{EI}$, so

$$\mathbf{EI} = 2(\mathbf{y} + \mathbf{z} + \mathbf{w} + \mathbf{x}) = -2(\mathbf{AB} + \mathbf{CD}).$$

Hence $\mathbf{EI} = \mathbf{0}$ is equivalent to $\mathbf{AB} + \mathbf{CD} = \mathbf{0}$, which is a characterization of parallelograms according to Theorem 3.1 (b) in Part 2.

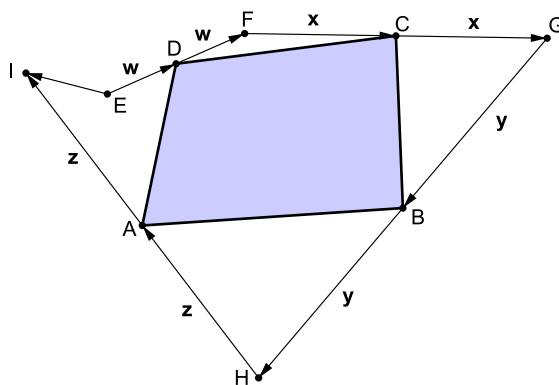


FIGURE 14. The vectors \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z}

(b) If the vertices A , B , C , D are represented by the position vectors \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} respectively, then

$$\mathbf{AG}_1 = \frac{\mathbf{b} + \mathbf{c} + \mathbf{d}}{3} - \mathbf{a} = \frac{\mathbf{b} + \mathbf{c} + \mathbf{d} - 3\mathbf{a}}{3}$$

and since $\mathbf{AG}_1 + \mathbf{CG}_3 = \mathbf{0}$ is equivalent to $3\mathbf{AG}_1 = -3\mathbf{CG}_3$, we get

$$\mathbf{b} + \mathbf{c} + \mathbf{d} - 3\mathbf{a} = -(\mathbf{a} + \mathbf{b} + \mathbf{d} - 3\mathbf{c}) \Leftrightarrow \mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$$

which is a characterization of parallelograms according to Theorem 3.1 (c) in Part 2.

(c) We place the quadrilateral in the Argand plane. Then $A' = A + \mathbf{A}\mathbf{A}' = A + i(D - A)$ and $A'' = A + \mathbf{A}\mathbf{A}'' = A - i(B - A)$, see Figure 15, so

$$\mathbf{A}''\mathbf{A}' = A' - A'' = i(D - A + B - A)$$

and in the same way,

$$\mathbf{C}'\mathbf{C}'' = C'' - C' = -i(D - C + B - C).$$

Hence

$$\mathbf{A}''\mathbf{A}' = \mathbf{C}'\mathbf{C}'' \Leftrightarrow i(D + B - 2A) = -i(D + B - 2C)$$

which is equivalent to $A + C = B + D$, and this is a characterization of parallelograms according to Theorem 3.1 (e) in Part 2.

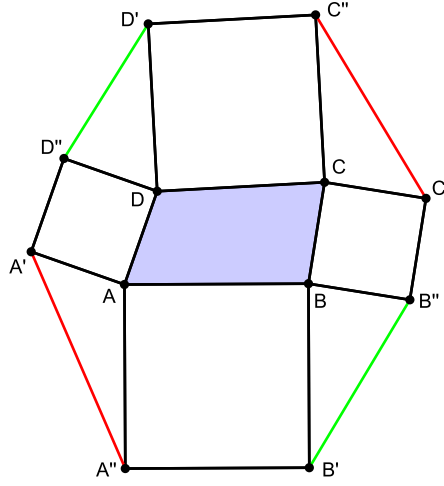


FIGURE 15. Four external squares

(d) Assume the quadrilateral $ABCD$ is placed in a coordinate system such that $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, $D = (d, e)$, so here a, b, c, d does *not* denote the lengths of the quadrilateral sides. The diagonal intersection has coordinates

$$\left(\frac{abe}{ac + be - cd}, \frac{ace}{ac + be - cd} \right) = \mathbf{AP}$$

according to equation (2) in Part 3. Then we have that $\mathbf{AB} + \mathbf{AD} + \mathbf{AP} = \mathbf{BC} + \mathbf{DC} + \mathbf{PC}$ is equivalent to

$$(a, 0) + (d, e) + \mathbf{AP} = (b - a, c) + (b - d, c - e) + (b, c) - \mathbf{AP}.$$

Inserting the expression for \mathbf{AP} and simplifying, we get for the abscissa and the ordinate the two equalities

$$\begin{cases} 2a + 2d + \frac{2abe}{ac + be - cd} = 3b \\ 2e + \frac{2ace}{ac + be - cd} = 3c \end{cases}$$

which we rewrite as

$$\frac{2ae}{ac + be - cd} = \frac{3b - 2a - 2d}{b} = \frac{3c - 2e}{c}.$$

Hence $c(3b - 2a - 2d) = b(3c - 2e)$, from which we get

$$(1) \quad ac = be - cd,$$

and inserting this back into the denominator of the ordinate yields

$$2e + \frac{2ace}{2ac} = 3c,$$

that is, $c = e$. Then (1) simplifies into $b = a + d$, and together with $c = e$, these characterize a parallelogram according to Theorem 1.1 in Part 3.

(e) When $ABCD$ is a parallelogram, then all nine small quadrilaterals are clearly congruent parallelograms. Hence they all have the same area.

Conversely, let A' , B' , C' , D' be intersection points between trisection lines (see Figure 16) in a convex quadrilateral $ABCD$ where all nine small quadrilaterals have the same area. We first prove that these four are also trisection points. With notations as in the figure, we have $\mathbf{TU} = \frac{2}{3}\mathbf{AC}$ and $\mathbf{WV} = \frac{1}{3}\mathbf{AC}$, so $\mathbf{TU} = 2\mathbf{WV}$, making triangles $D'VW$ and $D'TU$ similar (AA) and $\mathbf{D'U} = 2\mathbf{WD'}$. This proves that D' is a trisection point, and by symmetry, so are A' , B' , and C' .

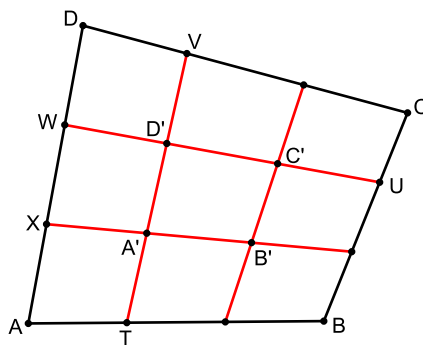


FIGURE 16. All sides are trisected

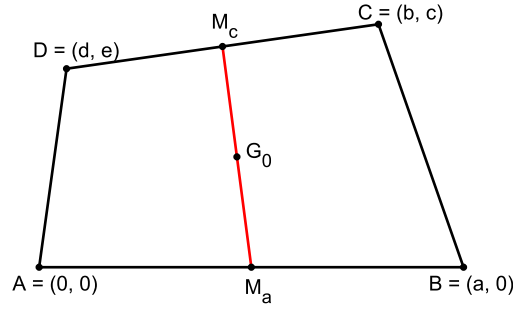
Next, given that quadrilaterals $ATA'X$ and $XA'D'W$ have equal area, and since triangles $TA'X$ and $D'A'X$ have equal area, so do triangles TAX and $D'WX$ by subtraction. This implies that AD and TD' and thus TV are parallel. By symmetry in the configuration, it follows that when all nine small quadrilaterals have equal area, they must all be parallelograms. Then $ABCD$ is also a parallelogram.

(f) Assume the quadrilateral $ABCD$ is placed in a coordinate system such that $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, $D = (d, e)$, so here a , b , c , d does *not* denote the lengths of the quadrilateral sides. If M_a and M_c are the midpoints of AB and CD , then

$$G_0 = \frac{M_a + M_c}{2} = \frac{A + B + C + D}{4} = \left(\frac{a + b + d}{4}, \frac{c + e}{4} \right)$$

since the vertex centroid coincide with the intersection of the bimedians and the bimedians bisect each other in all quadrilaterals (see Figure 17).

Applying the formula for the area of a triangle in terms of the coordinates of its vertices, we get that $2T_{ABG_0} = 2T_{CDG_0}$ is equivalent to (we have

FIGURE 17. The vertex centroid G_0

omitted terms that are zero and rotate through the vertices in each triangle in positive direction so we don't need an absolute value)

$$a \cdot \frac{1}{4}(c + e) = be - dc + (d - b) \cdot \frac{1}{4}(c + e) + (c - e) \cdot \frac{1}{4}(a + b + d)$$

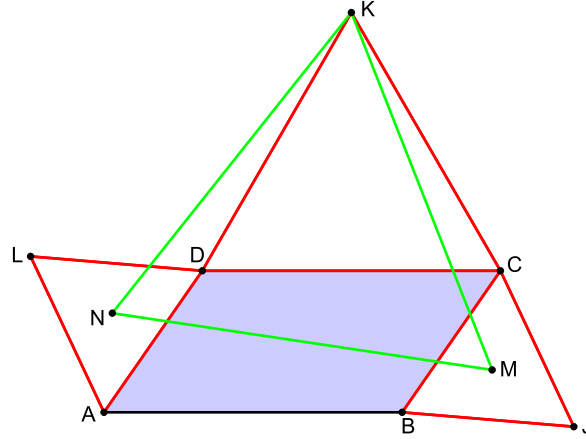
which is expanded and simplified into

$$(2) \quad cd = be - ae.$$

In the same way, $2T_{BCG_0} = 2T_{DAG_0}$ is equivalent to

$$ac + \frac{1}{4}(c + e)(b - a) - \frac{1}{4}(a + b + d)c = \frac{1}{4}(a + b + d)e - d \cdot \frac{1}{4}(c + e)$$

and it simplifies as $2ac = 2ae$, that is, $c = e$ since $a \neq 0$. Substituting $c = e$ in (2) yields $b = a + d$, and these two equalities characterize parallelograms according to Theorem 1.1 in Part 3.

FIGURE 18. Equilateral triangles CBJ , DCK , ADL

(g) When $ABCD$ is a parallelogram, $DN = CM$, and $KD = KC$ since DCK is equilateral (see Figure 18). Let $\angle DAB := \alpha$. Then

$$\angle KDN = 360^\circ - (180^\circ - \alpha) - 60^\circ - 30^\circ = 90^\circ + \alpha$$

and

$$\angle KCM = \alpha + 60^\circ + 30^\circ = 90^\circ + \alpha = \angle KDN$$

so triangles KDN and KCM are congruent (SAS). Hence $KN = KM$ and $\angle DKN = \angle CKM$, yielding

$$\angle NKM = \angle DKC + \angle DKN - \angle CKM = \angle DKC = 60^\circ$$

so triangle KMN is isosceles with an apex angle of 60° , proving that it is equilateral.

Conversely, when triangle KMN as well as CBJ , DCK , ADL are equilateral, we have $KN = KM$, $DK = CK$ and $\angle NKM = \angle DKC = 60^\circ$. This implies that $\angle NKD = \angle MKC$, so triangles NKD and MKC are congruent (SAS), and we get that $ND = MC$. Then $AD = BC$ since triangles ADL and CBJ are equilateral. We further have $\angle MCK = \angle NDK$, implying $\angle BCK = \angle LDK$ and thus

$$\angle ADC = 360^\circ - 2 \cdot 60^\circ - (\angle BCD + 60^\circ) = 180^\circ - \angle BCD.$$

This proves that AD is parallel to BC , and since also $AD = BC$, we conclude that $ABCD$ is a parallelogram according to Theorem 2.1 (b) in Part 1.

(h) This proof is very similar to the previous one, so we leave it as an exercise for the reader.

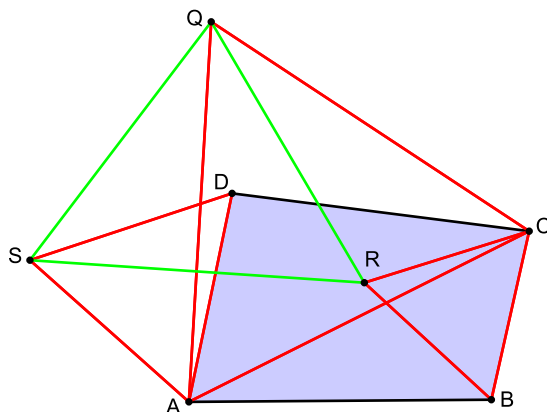


FIGURE 19. Equilateral triangles ADS , BCR , ACQ

(i) We will use the fact that a triangle XYZ in the Argand plane is equilateral and positively oriented if and only if $X + \omega Y + \omega^2 Z = 0$, where $\omega = e^{2i\pi/3}$ (proved in [10, p. 77]). Since triangles ADS , BCR , ACQ are equilateral (see Figure 19), it holds that

$$\begin{cases} S + \omega A + \omega^2 D = 0 \\ C + \omega R + \omega^2 B = 0 \\ A + \omega C + \omega^2 Q = 0. \end{cases}$$

Therefore triangle SRQ is equilateral if and only if

$$\begin{aligned} 0 &= S + \omega R + \omega^2 Q \\ &= -\omega A - \omega^2 D - C - \omega^2 B - A - \omega C \\ &= -\omega^2(B + D) - (1 + \omega)(A + C) \\ &= \omega^2(A + C - B - D) \end{aligned}$$

where we in the last equality applied $\omega + \omega^2 = -1$, that is, $1 + \omega = -\omega^2$, which follows from Euler's formula. Since $\omega^2 \neq 0$, triangle SRQ is equilateral if and only if $A + C = B + D$, which is a characterization of parallelograms according to Theorem 3.1 (e) in Part 2.

(j) This proof is very similar to the previous one except that we use that a triangle $X'Y'Z'$ in the Argand plane is equilateral and *negatively* oriented if and only if

$$X' + \omega^2 Y' + \omega Z' = 0,$$

where $\omega = e^{2i\pi/3}$ (proved in [10, p. 78]). Therefore we omit this proof. \square

7. METRIC RELATIONS

We conclude by proving eight necessary and sufficient conditions that are expressed as metric relations for when a quadrilateral is a parallelogram. Only the first is about a trapezoid, the rest concern a general convex quadrilateral. To prove the necessary condition in (a) was given as a problem at [31]. Condition (b) is from [36, p. 76], (c) was formulated differently (including a second redundant requirement) in [27], (d) was proved in the same way in [73, pp. 187–188]. (e) is from [84] and (f) is a variant of a similar characterization of isosceles trapezoids proved in [10, pp. 206–207]. Condition (g), which is closely related to Theorem 2.1 (h) in Part 2, was Problem 3 in 1967 on the Hungarian Kürschák Competition [1]. The main idea in the proof we present is based on a solution given in [25, pp. 119–120]. (h) is from [52] and we give a translation of the original Hungarian proof by Emil Kiss.

We denote by T_{XYZ} and K_{WXYZ} the area of triangle XYZ and the area of quadrilateral $WXYZ$ respectively.

Theorem 7.1. *A convex quadrilateral $ABCD$ with area K and diagonal intersection P satisfies any one of:*

- (a) *it is a trapezoid ($AD \parallel BC$) with $T_{EFG} = K_{ABCG}$, where G is an interior point such that $E = AB \cap CG$ and $F = BC \cap AG$*
- (b) *it is extangential with infinite exradius*
- (c) *$M_a M_c = \max\{AD, BC\}$ or $M_b M_d = \max\{AB, CD\}$, where M_a, M_b, M_c, M_d are the midpoints of AB, BC, CD, DA respectively*
- (d) *$\max\{T_a, T_c\} \leq \min\{T_b, T_d\}$, where T_a, T_b, T_c, T_d are the area of the triangles ABP, BCP, CDP, DAP respectively*
- (e) *$K \geq \frac{K_L^2}{K_R} + \frac{K_R^2}{K_L}$ and $K \geq \frac{K_D^2}{K_U} + \frac{K_U^2}{K_D}$, where each bimedians divide $ABCD$ in two parts with area K_L, K_R and K_D, K_U*
- (f) *$AG_1 = CG_3$ and $BG_2 = DG_4$, where G_1, G_2, G_3, G_4 are the centroids of the triangles BCD, CDA, DAB, ABC respectively*
- (g) *$h(A) = h(B) = h(C) = h(D)$, where $h(X)$ denotes the sum of the distances from vertex X to the two sides not containing it*
- (h) *RQ bisects the area of the quadrilateral for all $R \in AB$ and $Q \in CD$ satisfying $\frac{AR}{RB} = \frac{CQ}{QD}$*

if and only if it's a parallelogram.

Proof. (a) When $ABCD$ is a parallelogram, triangles EAD and EBF are similar (AA), so

$$\frac{EA}{EB} = \frac{AD}{FB} \Rightarrow \frac{EA}{EB \sin \beta} = \frac{AD}{FB \sin \beta} \Rightarrow \frac{EA}{d(E, BC)} = \frac{AD}{d(F, EA)}$$

where β is a supplementary angle to B , and $d(X, YZ)$ denotes the distance between point X and line YZ . We get

$$EA \cdot d(F, EA) = BC \cdot d(E, BC) \Rightarrow T_{EFA} = T_{BCE}$$

and this implies that $T_{EFG} = K_{ABCG}$ by subtracting the common area of triangle EAG .

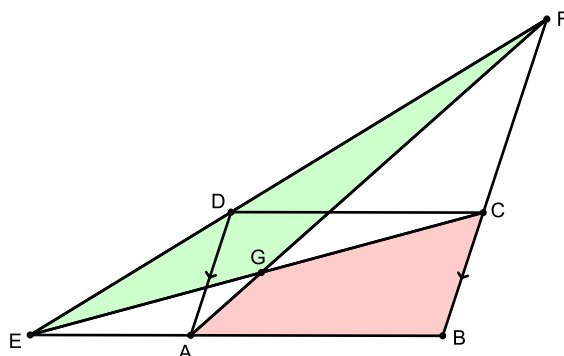


FIGURE 20. EFG and $ABCG$ have equal area

Conversely, given that $T_{EFG} = K_{ABCG}$ holds in a trapezoid where AD and BC are parallel, we get $T_{EFA} = T_{BCE}$ by adding the common area of triangle EAG (see Figure 20). Since the quotient of the area of triangles with the same height is equal to the quotient of their bases,

$$\frac{EA}{EB} = \frac{BC}{FB}.$$

Triangles EAD and EBF are still similar (AA), so

$$\frac{EA}{EB} = \frac{AD}{FB}$$

and equating the last two right hand sides, we get $AD = BC$. Hence opposite sides AD and BC are both parallel and equal, implying that $ABCD$ is a parallelogram according to Theorem 2.1 (b) in Part 1.

(b) Theorem 8 in [36] states that the exradius of an extangential quadrilateral with area K and consecutive sides a, b, c, d is given by

$$\rho = \frac{K}{|a - c|} = \frac{K}{|b - d|}.$$

This is infinite if and only if $a - c = b - d = 0$, that is, only when opposite sides have equal length, which characterize parallelograms.

(c) In a parallelogram, each bimedian is equal to the two sides it doesn't connect, so now we prove the converse.

Let M_p be the midpoint of diagonal AC . Applying the Triangle Inequality in triangle $M_bM_dM_p$ (see Figure 21), we get

$$M_bM_d \leq M_bM_p + M_dM_p = \frac{AB + CD}{2}.$$

We have $M_bM_d = \max\{AB, CD\}$, and assuming without loss of generality that AB is the longest of the two sides, we get $2AB = AB + CD$, so $AB = CD$. This implies that $AB = CD = M_bM_d$. Hence we have the degenerate case of the Triangle Inequality, so the three points M_b, M_p, M_d are collinear and we conclude that AB, CD, M_bM_d are parallel. A pair of opposite equal and parallel sides implies that $ABCD$ is a parallelogram.

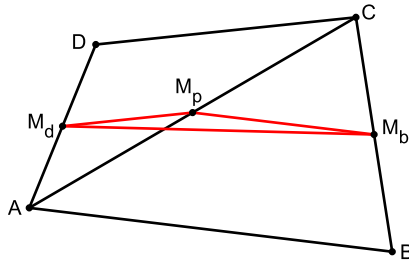


FIGURE 21. Midpoints M_b, M_d and M_p

(d) First we prove that if $T_a \leq T_b$ and $T_c \leq T_d$, then $AP = CP$. Let $\{A', C'\} \in BD$ be points such that $AA' \perp BD$ and $CC' \perp BD$ (see Figure 22). From $T_a \leq T_b$, we get

$$\frac{BP \cdot AA'}{2} \leq \frac{BP \cdot CC'}{2}$$

implying that $AA' \leq CC'$. In the same way $T_c \leq T_d$ implies $AA' \geq CC'$. Thus we must have $AA' = CC'$, and it follows that triangles $AA'P$ and $CC'P$ are congruent (AAS). Hence $AP = CP$.

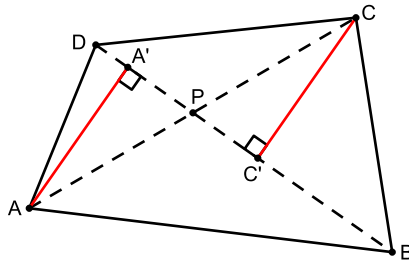


FIGURE 22. Projections on diagonal BD

Returning to the proof that $\max\{T_a, T_c\} \leq \min\{T_b, T_d\}$ characterize parallelograms, we know that $T_a = T_b = T_c = T_d$ in a parallelogram (Theorem 8.1 (g) in Part 1), so let's prove the converse.

The given inequality yields firstly that $T_a \leq T_b$ and $T_c \leq T_d$, implying $AP = CP$, and secondly that $T_c \leq T_b$ and $T_a \leq T_d$, implying in the same

way $BP = DP$. Hence $ABCD$ is a parallelogram according to Theorem 3.1 (a) in Part 1.

(e) The first inequality is equivalent to

$$(K_L + K_R)K_RK_L \geq K_L^3 + K_R^3,$$

which can be factored as

$$(K_L + K_R)K_RK_L \geq (K_L + K_R)(K_L^2 - K_LK_R + K_R^2)$$

and further as

$$0 \geq (K_L - K_R)^2(K_L + K_R).$$

It has only one possible solution $K_L = K_R$. In the same way we get that the second inequality yields only one possible solution $K_D = K_U$. That both bimedians bisect the area is a characterization of parallelograms according to Theorem 5.1 (f) in Part 1.

(f) We will use the real product of complex numbers (also called the complex dot product) to prove this characterization. For two complex numbers X and Y , it is defined as

$$X \circ Y := \frac{1}{2}(\overline{X}Y + X\overline{Y}),$$

where the symbol for the multiplication sign can vary between different sources, and the bar just means the complex conjugate. Three important properties are that $X \circ X = |X|^2$, the real product is commutative, and it is distributive with respect to addition according to [10, p. 89].

Let us denote the vertices of $ABCD$ in the Argand plane simply by A , B , C , D . Then

$$G_1 = \frac{B + C + D}{3} = \frac{S - A}{3} \quad \text{and} \quad G_3 = \frac{S - C}{3},$$

where we defined $S := A + B + C + D$. Now we shall establish a sequence of equivalent equalities. Since $AG_1 = CG_3$ is the same as $|A - G_1| = |C - G_3|$, we get $|4A - S| = |4C - S|$ and using the real product, this is equivalent to

$$(4A - S) \circ (4A - S) = (4C - S) \circ (4C - S).$$

Expanding yields

$$16|A|^2 - 8A \circ S + |S|^2 = 16|C|^2 - 8C \circ S + |S|^2,$$

that is,

$$(3) \quad 2(|A|^2 - |C|^2) = (A - C) \circ S.$$

In the same way, $BG_2 = DG_4$ is equivalent to

$$(4) \quad 2(|B|^2 - |D|^2) = (B - D) \circ S.$$

Subtracting (3) and (4) gives

$$2(|A|^2 - |C|^2 - |B|^2 + |D|^2) = (A - C - B + D) \circ (A + B + C + D),$$

that is,

$$2(|A|^2 - |C|^2 - |B|^2 + |D|^2) = |A + D|^2 - |B + C|^2$$

and applying the property $|X|^2 = X\overline{X}$, we get

$$2(|A|^2 - |C|^2 - |B|^2 + |D|^2) = (A + D)(\overline{A + D}) - (B + C)(\overline{B + C}).$$

Expanding (using the property $\overline{X+Y} = \overline{X} + \overline{Y}$) and simplifying yields

$$A\overline{A} - C\overline{C} - B\overline{B} + D\overline{D} = A\overline{D} + \overline{A}D - B\overline{C} - \overline{B}C$$

which is equivalent to

$$(A - D)(\overline{A - D}) = (B - C)(\overline{B - C}),$$

that is,

$$|A - D|^2 = |B - C|^2.$$

Hence $AD = BC$, so one pair of opposite sides have equal length.

Next, adding (3) and (4) gives

$$2(|A|^2 - |C|^2 + |B|^2 - |D|^2) = (A + B - C - D) \circ (A + B + C + D),$$

and by similar calculations, we obtain $AB = DC$. Another way is using the symmetry and realizing that we only have to make the change $B \leftrightarrow D$ in the first result. Since we have equivalence throughout, we have proved that

$$\begin{cases} AG_1 = CG_3 \\ BG_2 = DG_4 \end{cases} \Leftrightarrow \begin{cases} AD = BC \\ AB = DC \end{cases}$$

where the latter two equalities characterize parallelograms according to Theorem 2.1 (a) in Part 1.

(g) The necessary condition is trivial as $h(X)$ is equal to the sum of the two different heights in a parallelogram for all $X \in \{A, B, C, D\}$.

To prove the sufficient condition, we use the notation $s(Y, \theta)$ for the sum of the distances from a point Y inside an angle θ to the arms of that angle. In the proof of Theorem 2.1 (h) in Part 2, we proved a locus, which for the purpose here can be formulated as follows (see Figure 23):

Suppose Y is a point on the line EF perpendicular to the angle bisector of $\angle EAF$, and Q is another point inside the same angle, then

$$s(Y, \angle EAF) = s(Q, \angle EAF)$$

if and only if Q also lies on EF .

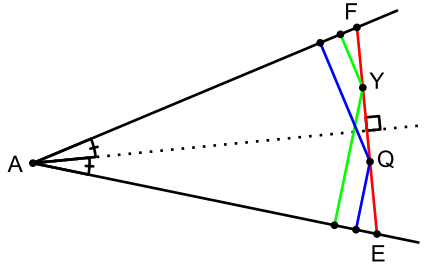


FIGURE 23. $s(Y, \angle EAF) = s(Q, \angle EAF) \Leftrightarrow Q \in EF$

Given a convex quadrilateral $ABCD$, we construct a rhombus $AEKF$ such that $B \in AE$, $D \in AF$, $C \in EF$, and with $G = CD \cap KE$ and $H = BC \cap KF$ (see Figure 24). Then triangles CBE and CHF are similar (AA), as are triangles CGE and CDF (AA). Hence the quadrilaterals $CBEG$ and $CHFD$ are similar, implying that BG is parallel to DH . By assumption,

$$s(B, \angle ADC) = s(C, \angle BAD)$$

and by the reformulated locus property,

$$s(C, \angle BAD) = s(E, \angle BAD).$$

We also have

$$s(E, \angle BAD) = s(G, \angle ADC)$$

since both are equal to the distance between EK and AF . Thus we conclude

$$s(B, \angle ADC) = s(G, \angle ADC)$$

meaning that BG is perpendicular to the bisector of $\angle ADC$. In the same way, DH is perpendicular to the bisector of $\angle ABC$. Since BG is parallel to DH , the two bisectors to $\angle ADC$ and $\angle ABC$ are parallel.

With a similar argument we can prove that the two bisectors to $\angle DAB$ and $\angle BCD$ are parallel. Two pairs of opposite parallel angle bisectors imply that $ABCD$ is a parallelogram according to Theorem 5.1 (a) in Part 1.

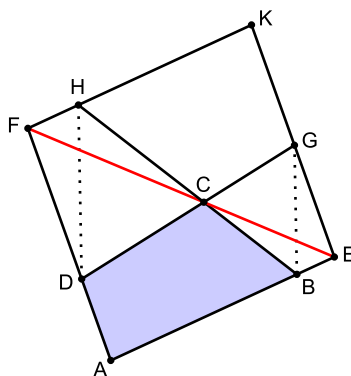


FIGURE 24. The constructed rhombus $AEKF$

(h) We start by proving the converse. The assumption means that for every pair of points R and Q for which

$$(5) \quad \frac{AR}{RB} = \frac{CQ}{QD} := \lambda (> 0)$$

the following is also true:

$$(6) \quad [ARQD] = [BRQC] = \frac{1}{2}[ABCD],$$

where the square brackets denote area of the included polygon. We show that from this, with the special value $\lambda = 1$, we obtain that CD is parallel to AB . Indeed, in the case of $\lambda = 1$, R bisects AB and Q bisects CD (called R_1 and Q_1 respectively in Figure 25, left), so

$$(7) \quad [AR_1Q_1D] = [BR_1Q_1C].$$

On the other hand,

$$(8) \quad [R_1Q_1D] = [R_1Q_1C]$$

also holds, because their bases Q_1D and Q_1C are equal and their altitudes drawn from R_1 are the same. Subtracting (8) from (7) yields

$$[AR_1D] = [BR_1C].$$

But the bases AR_1 and BR_1 of these two triangles are equal, so their heights, i.e. the distances of D and C from the line AB are also equal. Since the quadrilateral $ABCD$ is convex, C and D are on the same side of AB , so CD is parallel to AB . In other words: $ABCD$ is a trapezoid.

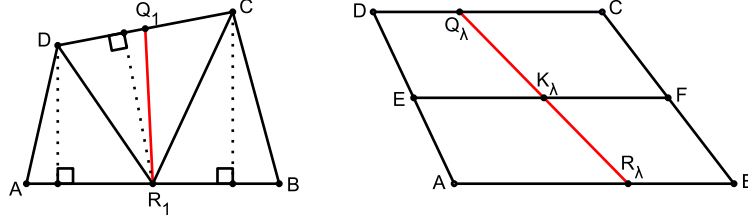


FIGURE 25. Area bisecting line RQ satisfying $\frac{AR}{RB} = \frac{CQ}{QD}$

Next let R_λ and Q_λ be a pair of points satisfying (5) with an arbitrary positive value $\lambda \neq 1$ in a trapezoid $ABCD$ (see Figure 25, right). Thus

$$\begin{aligned} R_\lambda B &= \frac{1}{1+\lambda} AB, & R_\lambda A &= \frac{\lambda}{1+\lambda} AB, \\ Q_\lambda D &= \frac{1}{1+\lambda} CD, & Q_\lambda C &= \frac{\lambda}{1+\lambda} CD. \end{aligned}$$

Quadrilaterals $AR_\lambda Q_\lambda D$ and $BR_\lambda Q_\lambda C$ are trapezoids with a common height and equal area according to (6), so their bimedians EK_λ and $K_\lambda F$ are also equal, where E and F are the midpoints of the legs AD and BC respectively, and K_λ is the intersection of the line segments EF and $R_\lambda Q_\lambda$. This follows from the well-known formula that the area of a trapezoid is equal to the height times the bimedian parallel to the bases. Hence

$$\begin{aligned} EK_\lambda - K_\lambda F &= \frac{1}{2(1+\lambda)}(\lambda AB + CD) - \frac{1}{2(1+\lambda)}(AB + \lambda CD) \\ (9) \quad &= \frac{\lambda - 1}{2(1+\lambda)}(AB - CD) = 0, \end{aligned}$$

and this can only be fulfilled for any ratio value $\lambda \neq 1$ if $AB = CD$, i.e. $ABCD$ is a parallelogram according to Theorem 2.1 (b) in Part 1.

For the direct part, in a parallelogram we have $AB = CD$, so for every λ it holds that

$$EK_\lambda = K_\lambda F$$

according to (9). Since $R_\lambda Q_\lambda$ bisects the bimedian EF , it also bisects the area of the parallelogram. \square

8. CHRONOLOGICAL COMPILATION

Here we summarize all 200 characterizations of parallelograms that have been studied in the four papers [45, 46, 48, 49]. They are given in chronological order with respect to the oldest source for the *sufficient* condition that we know of, but some of them have probably been published earlier.

If it's not specified which type of quadrilateral a characterization is about, it concerns an arbitrary convex quadrilateral $ABCD$ with sides $AB = a$,

$BC = b$, $CD = c$, $DA = d$, their respective midpoints M_a , M_b , M_c , M_d , diagonals $AC = p$, $BD = q$, semidiagonals $AP = a'$, $BP = b'$, $CP = c'$, $DP = d'$ where $P = AC \cap BD$, bimedians m , n , and area K . The area of triangles ABP , BCP , CDP , DAP are denoted by T_a , T_b , T_c , T_d respectively. For the meaning of other notations, see the corresponding theorems. *As superscript to the sources, we indicate in which of the four parts each characterization was proved.*

BriMO, CroMO, DutMO, LenMO, MedMO, MexMO, and RomMO are short for the British, Croatian, Dutch, Leningrad, Mediterranean, Mexican, and Romanian Mathematical Olympiad respectively, and BDQ stands for bisect-diagonal quadrilateral, Q^s for quadrilaterals, OC^s for orthocenters, pt for point, and w for with.

SUCEM is an abbreviation for Stanford University Competitive Examination in Mathematics, ConoSurSL for Cono Sur Olympiad Short List problem, ARMO R4 for the Regional All-Russian Mathematical Olympiad Round 4, RSTMF for Russian Southern Tournament Math Fights, OIAJ for Olimpiáda Ibero-Americana Juvenil, and CGMO for Croatia Girls Mathematical Olympiad.

#	Year	Source	Short description	Ref.
1	-300	Euclid ¹	$AB \parallel DC$ and $BC \parallel AD$ (definition)	[51]
2	-300	Euclid ¹	$AB \parallel DC$ and $AB = DC$	[51]
3	1750	Euler ¹	$a^2 + b^2 + c^2 + d^2 = p^2 + q^2$	[76]
4	1794	Legendre ¹	$AB = DC$ and $BC = AD$	[59]
5	1833	Young ¹	$\triangle ABC \cong \triangle CDA$	[93]
6	1833	Young ¹	$\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$	[93]
7	1833	Young ¹	$\angle A = \angle C$ and $\angle B = \angle D$	[93]
8	1833	Young ¹	$AP = CP$ and $BP = DP$	[93]
9	1833	Young ¹	Any two adjacent angles are supplementary	[93]
10	1833	Young ¹	$\triangle ABP \cong \triangle CDP$ and $\triangle BCP \cong \triangle DAP$	[93]
11	1871	Todhunter ¹	Both diagonals bisect the area of quad	[83]
12	1879	Henrici ²	It has a center of symmetry	[21]
13	1896	Halsted ¹	$AB = DC > AC$ and $\angle ACB = \angle CAD$	[20]
14	1896	Halsted ¹	$AB = DC < AC$ and $\angle B = \angle D$	[20]
15	1952	SUCEM ¹	$T_a = T_b = T_c = T_d$	[81]
16	1954	Gordon ²	$A + C = B + D$ for $\{A, B, C, D\} \in \mathbb{C}$	[29]
17	1967	Keedy ¹	Trapezoid with two pairs of parallel sides	[89]
18	1967	Kürschák ⁴	$h(A) = h(B) = h(C) = h(D)$	[1]
19	1967	Garfunkel ⁴	Triangle SRQ is equilateral	[26]
20	1968	Kung ⁴	All lines thru fixed pt are perim. bisectors	[57]
21	1968	Klamkin ²	$\mathbf{PA} + \mathbf{PB} + \mathbf{PC} + \mathbf{PD} = \mathbf{0}$	[53]
22	1968	Klamkin ²	$G_0 = G_2$	[53]
23	1968	Klamkin ²	$G_0 = G_1$	[53]
24	1968	Klamkin ⁴	$T_{ABG_0} = T_{CDG_0}$ and $T_{DAG_0} = T_{BCG_0}$	[53]
25	1972	Eperson ²	$\mathbf{OA} + \mathbf{OC} = \mathbf{OB} + \mathbf{OD}$ for arbitrary pt O	[22]
26	1972	Kiss ⁴	RQ bisects the area such that $\frac{AR}{RB} = \frac{CQ}{QD}$	[52]
27	1973	Sagar ²	$\mathbf{AB} + \mathbf{CD} = \mathbf{0}$	[90]
28	1977	Posamentier ¹	$AB \parallel DC$ and $\angle B = \angle D$	[74]
29	1977	Milenkovic ²	$T_{ABY} + T_{CDY} = T_{BCY} + T_{DAY} \quad \forall Y \in \text{quad}$	[78]
30	1977	Klamkin ³	Quadrilateral perimeter $L = IB + BJ$	[54]
31	1980	LenMO ¹	Quadrilateral semiperimeter $s = m + n$	[23]
32	1981	LenMO ²	Sum of dist. fr. any int. pt to sides is const.	[23]
33	1985	Wagner ¹	$T_a = T_c$ and $T_b = T_d$	[55]
34	1987	LenMO ³	Bimedians divide Q into 4 Q^s of eq. perim.	[23]

35	1988	Okumura ¹	$\frac{\angle CAD}{\angle CAB} = \frac{\angle ACB}{\angle ACD}$ and $\frac{\angle DBA}{\angle DBC} = \frac{\angle BDC}{\angle BDA}$	[63]
36	1989	LenMO ⁴	P is the midpoint of X_1X_3 and X_2X_4	[23]
37	1990	Schmidt ⁴	9 Q^s from trisection points have equal area	[80]
38	1991	SavinComp ³	$AB \parallel CD, B'C' \parallel D'A'$, corresp. sides equal	[65]
39	1993	Hoehn ¹	$da \cos A + ab \cos B + bc \cos C + cd \cos D = 0$	[34]
40	1993	Seimiya ¹	$PA \sin A = PC \sin C$ & $PB \sin B = PD \sin D$	[79]
41	1993	ConoSurSL ³	P is the centroid of triangle AM_bM_c	[66]
42	1994	Toumasis ¹	Trapezoid where one diag. bisects the other	[85]
43	1994	Toumasis ¹	$\angle B = \angle D$ and $AP = CP$	[85]
44	1994	Toumasis ¹	$\angle B = \angle D$ and $BP = DP$ and $BD \not\perp AC$	[85]
45	1996	MexMO ¹	E and F are the midpoints of AB and BC	[92]
46	1996	CroMO ¹	$T_b^2 = T_aT_c$ and $2T_d = T_a + T_c$	[50]
47	1996	Alkan ³	$K = 8\sqrt[4]{T_A T_B T_C T_D}$	[6]
48	1997	Pompe ²	$h_a = h_c$ and $h_b = h_d$	[72]
49	1997	RomMO ¹	$AD \parallel BC$ with $AO = 4OM_c$	[2]
50	2000	MedMO ¹	$a^2 + b^2 + c^2 + d^2 = \frac{4}{5}(m_A^2 + m_B^2 + m_C^2 + m_D^2)$	[9]
51	2001	DeTemple ³	$w^2 + x^2 + y^2 + z^2 = 2(a^2 + b^2 + c^2 + d^2)$	[17]
52	2002	Sauerberg ⁴	Centered to non-triv. quad w same angles	[77]
53	2003	Contreras ¹	All 4 interior angle bisectors form rectangle	[14]
54	2003	Contreras ³	All 4 exterior angle bisectors form rectangle	[14]
55	2003	Hoyles ⁴	EI = 0	[75]
56	2004	ARMO R4 ³	$AB \parallel DC$ with M_bM_d bisecting O_1O_2	[68]
57	2005	Tydd ¹	2 bimedians and 2 diagonals are concurrent	[87]
58	2005	Graumann ⁴	Both bimedians are axes of sloping symmetry	[30]
59	2005	Graumann ⁴	Both diagonals are axes of sloping symmetry	[30]
60	2005	Graumann ⁴	1 diag. & 1 bimedian are axes of sloping sym.	[30]
61	2006	BriMO ²	$\exists X$ s.t. any line thru X bisects area of quad	[13]
62	2006	Iurea ²	$BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$	[28]
63	2006	ConoSur ²	$DR = RQ = QC$	[3]
64	2006	Andrescu ³	$(a + c)^2 + (b + d)^2 = 4(m^2 + n^2)$	[11]
65	2006	RSTMF ³	$AM_b \parallel CM_d$ and $BM_c \parallel DM_a$	[67]
66	2007	MacMO ³	$AB \parallel DC$ with AK, CL, BO concurrent	[61]
67	2007	Nicula ³	$\frac{JA}{JD} = \frac{MC}{MN}$ and $\frac{LB}{LC} = \frac{ND}{MN}$	[60]
68	2008	Usiskin ²	Rotational symmetry of order 2	[89]
69	2008	KöMaL ¹	Both bimedians bisect the area of quad	[56]
70	2009	Al-Sharif ¹	Both diagonals bisect perimeter of quad	[5]
71	2009	Al-Sharif ²	$G_0 = P$	[5]
72	2009	Al-Sharif ²	$G_2 = P$	[5]
73	2009	Al-Sharif ²	$G_1 = P$	[5]
74	2009	Webb ¹	Bimedians have constant length	[16]
75	2009	Zhao ⁴	No spiral similarity sends a to c or d to b	[94]
76	2010	Pillay ²	BDQ with equipartitioning point G_2	[71]
77	2012	Josefsson ⁴	Extangential quad w infinite exradius	[36]
78	2012	[leader] ⁴	$\angle B = \angle D$ & $AM = CM, M = \text{midp. of } BD$	[58]
79	2013	Pop ¹	$T_{ABC} \leq T_{CDA} \leq T_{BCD} \leq T_{DAB}$	[73]
80	2013	Pop ¹	Trapezoid w 1 diag. bisecting the perimeter	[73]
81	2013	Pop ³	$AB \parallel DC$ with P, Q, R collinear	[73]
82	2013	Pop ³	$AB \parallel DC$ with $QR = \frac{ad+bc}{a+c}$	[73]
83	2013	Pop ⁴	P is the midpoint of X_1X_3 and X_5X_6	[73]
84	2013	Pop ⁴	$\max\{T_a, T_c\} \leq \min\{T_b, T_d\}$	[73]
85	2013	Fritsch ²	External van Aubel quad is a square	[24]
86	2013	Josefsson ³	$K = \frac{1}{2}\sqrt{(a^2 + c^2)(b^2 + d^2)} \sin \phi$	[37]
87	2014	OIAJ ¹	Bimedians quadrisect the area of quad	[62]
88	2014	Shapovalov ³	Bisect. to A & C paral., $BR = DQ, R \neq Q$	[7]
89	2015	DutMO ³	$AB \parallel DC$ with $F_4G_4 \parallel E_4H_4$	[19]
90	2016	Volchkevich ⁴	BDQ where bisected diag. bisects perimeter	[91]
91	2016	Ufnarovski ¹	Two pairs of opp. parallel angle bisectors	[88]

92	2016	Ufnarovski ¹	1 angle bisect. is perp. to 2 adj. angle bisect.	[88]
93	2016	Ufnarovski ¹	Eq. dist. from opp. vertices to corresp. diag.	[88]
94	2016	DutMO ²	$PX = PY$ w X & Y OC ^s of ABP & CDP	[64]
95	2016	Tadeev ³	$AB \parallel DC$ with $\angle DAC = \angle ACB$	[82]
96	2016	Tadeev ³	$AB \parallel DC$ with $\angle C + \angle D = \pi$	[82]
97	2017	Josefsson ⁴	Extantential and quasi-isosceles quad	[40]
98	2017	Josefsson ¹	BDQ ($AP = CP$) with $a^2 + b^2 = c^2 + d^2$	[41]
99	2017	Tuchilus ¹	$a' + b' = c' + d'$ and $d' + a' = b' + c'$	[86]
100	2017	Tuchilus ³	$S_1P = S_3P$ and $S_2P = S_4P$	[86]
101	2017	Tuchilus ³	$S_1S_2S_3S_4$ is a rhombus	[86]
102	2018	Aassila ¹	$(a + c)^2 + (b + d)^2 = 2(p^2 + q^2)$	[4]
103	2018	Totoi ⁴	$K \geq \frac{K_L^2}{K_R} + \frac{K_R^2}{K_L}$ and $K \geq \frac{K_U^2}{K_D} + \frac{K_D^2}{K_U}$	[84]
104	2020	Alsina ¹	$AB \parallel DC$ with $T_a + T_c = T_b + T_d$	[8]
105	2021	Ghita ⁴	$M_aM_c = \max\{AD, BC\}$	[27]
106	2022	JMAP ²	$AH_B = CH_D$ and $\angle ADH_D = \angle CBH_B$	[35]
107	2022	CGMO ³	AC trisected and $T_{ADN} = T_{CDO} = \frac{1}{4}K$	[69]
108	2023	Hajja ¹	Both bimedians bisect perimeter of quad	[32]
109	?	? ¹	$T_b = T_d = \frac{1}{4}K$	[32]
110	2023	Hajja ²	$UVWX$ is a square	[33]
111	2023	Hajja ²	Triangle CIJ is equilateral	[33]
112	2023	Dolcan ⁴	$\mathbf{AB} + \mathbf{AD} + \mathbf{AP} = \mathbf{BC} + \mathbf{DC} + \mathbf{PC}$	[18]
113	2024	Josefsson ¹	$T_{DAX} = T_{ABX}$ & $T_{BCX} = T_{CDX}$, $X \in AC$	[45]
114	2024	Josefsson ¹	Sum of any 2 adjacent angles is a constant	[45]
115	2024	Josefsson ¹	Angles btw. extensions of opp. sides are 0	[45]
116	2024	Josefsson ¹	Bimedians quadrisect perimeter of quad	[45]
117	2024	Josefsson ¹	Bimedians intersect at a diagonal midpoint	[45]
118	2024	Josefsson ¹	Trapezoid w 1 diagonal bisecting the area	[45]
119	2024	Josefsson ¹	$AB \parallel DC$ with $QI = RI$	[45]
120	2024	Josefsson ¹	$AB \parallel DC$ with $BJ = \frac{1}{3}BD$	[45]
121	2024	Josefsson ¹	$AB \parallel DC$ with $AE = \sqrt{EF \cdot EG}$	[45]
122	2024	Josefsson ¹	$AB \parallel DC$ with $T_a = T_c$	[45]
123	2024	Josefsson ¹	BDQ ($AP = PC$) with $IP = PJ$	[45]
124	2024	Josefsson ¹	BDQ ($AP = PC$) with $BK = DL$	[45]
125	2024	Josefsson ¹	BDQ ($AP = PC$) with $\angle MAP = \angle NCP$	[45]
126	2024	Josefsson ¹	$ab = cd$ and $da = bc$	[45]
127	2024	Josefsson ¹	$(a - c)^2 + (b - d)^2 = 0$	[45]
128	2024	Josefsson ¹	$a^2 + b^2 + c^2 + d^2 = 2(m^2 + n^2)$	[45]
129	2024	Josefsson ¹	$K = \frac{1}{2}\sqrt{(s^2 - p^2)(s^2 - q^2)}$	[45]
130	2024	Josefsson ²	Eq. dist. fr. any 2 adj. vertices to opp. side	[46]
131	2024	Josefsson ²	$AM_c = CM_a$ and $AM_c \parallel CM_a$	[46]
132	2024	Josefsson ²	$AH = CF$ and $BG = DE$	[46]
133	2024	Josefsson ²	Internal tang. to opp. incircles bisect diag.	[46]
134	2024	Josefsson ²	$R_1 = R_3$ and $R_2 = R_4$	[46]
135	2024	Josefsson ²	$r_A = r_C$ and $r_B = r_D$	[46]
136	2024	Josefsson ²	$m_A = m_C$ and $m_B = m_D$	[46]
137	2024	Josefsson ²	$R_a = R_c$ and $R_b = R_d$	[46]
138	2024	Josefsson ²	$t_a = t_c$ and $t_b = t_d$	[46]
139	2024	Josefsson ²	$m_a = m_c$ and $m_b = m_d$	[46]
140	2024	Josefsson ²	P bisects MN and UV	[46]
141	2024	Josefsson ²	Any line segment thru P is bisected by P	[46]
142	2024	Josefsson ²	$T_{AYC} \cot AYC - T_{BYD} \cot BYD$ indep. of Y	[46]
143	2024	Josefsson ²	The coordinates $c = e$ and $b = a + d$	[46]
144	2024	Josefsson ²	$\mathbf{PA} = -\mathbf{PC}$ and $\mathbf{PB} = -\mathbf{PD}$	[46]
145	2024	Josefsson ²	$BE = \frac{1}{3}BD$ and $EN = \frac{1}{3}AN$	[46]
146	2024	Josefsson ²	Internal van Aubel quad is a square	[46]
147	2024	Josefsson ²	$U'V'W'X'$ is a square	[46]
148	2024	Josefsson ²	Triangle $CI'J'$ is equilateral	[46]

149	2025	Guerra ⁴	$a + b = c + d$ and $a' + b' = c' + d'$	[15]
150	2026	Josefsson ³	Two pairs of equal opposite exterior angles	[48]
151	2026	Josefsson ³	Triangles DEF & BGH are directly similar	[48]
152	2026	Josefsson ³	$\triangle BIJ$ is similar to $\triangle CDJ$ and $\triangle AID$	[48]
153	2026	Josefsson ³	Const. \triangle area w base = a , 3 rd vert. on CD	[48]
154	2026	Josefsson ³	Centroids of AM_bM_c & BM_cM_d coincide	[48]
155	2026	Josefsson ³	2 pairs of opp. parallel ext. angle bisectors	[48]
156	2026	Josefsson ³	1 angle bis. is \parallel to 2 adj. ext. angle bis.	[48]
157	2026	Josefsson ³	2 adj. angle bis. perp. to opp. ext. angle bis.	[48]
158	2026	Josefsson ³	$M_aM_c = \frac{1}{2}(b + d)$ and $M_bM_d = \frac{1}{2}(a + c)$	[48]
159	2026	Josefsson ³	G_0 is the centroid of triangle AM_bM_c	[48]
160	2026	Josefsson ³	$AB \parallel DC$ w E_1G_1, F_1H_1, BD concurrent	[48]
161	2026	Josefsson ³	$AB \parallel DC$ with BE_2, DG_2, CI concurrent	[48]
162	2026	Josefsson ³	$AB \parallel CD$ & $\frac{AM}{MB} = \frac{CN}{ND}$, MN bisects area	[48]
163	2026	Josefsson ³	$AB \parallel CD \parallel F_3H_3$ and $K = 2K_{E_3F_3G_3H_3}$	[48]
164	2026	Josefsson ³	$a^2 + b^2 = c^2 + d^2$ and $d^2 + a^2 = b^2 + c^2$	[48]
165	2026	Josefsson ³	$\frac{AF}{FB} = \frac{CG}{GD}$ and $\frac{BH}{HC} = \frac{DI}{IA}$	[48]
166	2026	Josefsson ³	$T_{ABM_c} = T_{CDM_a}$ and $T_{BCM_d} = T_{DAM_b}$	[48]
167	2026	Josefsson ³	$K = 2\sqrt{a'b'c'd'} \sin \theta$	[48]
168	2026	Josefsson ³	$K = 4\sqrt{T_aT_c}$	[48]
169	2026	Josefsson ³	$K = 4\sqrt[4]{T_aT_bT_cT_d}$	[48]
170	2026	Josefsson ³	$K = \frac{1}{4}(\sqrt{T_a} + \sqrt{T_b} + \sqrt{T_c} + \sqrt{T_d})^2$	[48]
171	2026	Josefsson ³	$w = y$ and $x = z$	[48]
172	2026	Josefsson ³	$AG_3 = CG_1$ and $BG_4 = DG_2$	[48]
173	2026	Josefsson ³	$GC = CH$ and $FB = BH$ (rev. Varignon)	[48]
174	2026	Josefsson ³	Quad perimeter $L = E_aE_c + E_bE_d$	[48]
175	2026	Josefsson ³	$r_a = r_c$ and $r_b = r_d$	[48]
176	2026	Josefsson ³	$\rho_a = \rho_c$ and $\rho_b = \rho_d$	[48]
177	2026	Josefsson ³	$I_aI_bI_cI_d$ is a rhombus	[48]
178	2026	Josefsson ³	$E_aE_bE_cE_d$ is a rhombus	[48]
179	2026	Josefsson ³	$AC \perp EM$ and $BD \perp FG$	[48]
180	2026	Josefsson ³	$G_1 \vee G_3 \in AC$ and $G_2 \vee G_4 \in BD$	[48]
181	2026	Josefsson ³	Midpoints of AC and $A'C'$ coincide	[48]
182	2026	Josefsson ³	Midpoints of $A'C'$ and $B'D'$ coincide	[48]
183	2026	Dalcín ⁴	Two pairs of opp. parallel perp. bisectors	[49]
184	2026	Dalcín ⁴	Any bimedians divide Q into 2 congr. Q^s	[49]
185	2026	Dalcín ⁴	1 diag. bis. perim. & 1 bimedians bis. area	[49]
186	2026	Dalcín ⁴	1 diagonal & 1 bimedians bisect perimeter	[49]
187	2026	Dalcín ⁴	$FH \parallel AB$ and $EG \parallel BC$	[49]
188	2026	Dalcín ⁴	Pythagorean and quasi-isosceles quad	[49]
189	2026	Josefsson ⁴	$a + d = b + c$ and $a^2 + b^2 = c^2 + d^2$	[49]
190	2026	Josefsson ⁴	$\angle A = \angle C$ and $h_B = h_D$	[49]
191	2026	Dalcín ⁴	$\angle A = \angle C$ and $a^2 + b^2 = c^2 + d^2$	[49]
192	2026	Josefsson ⁴	$\angle A = \angle C$ and $m_B = m_D$	[49]
193	2026	Josefsson ⁴	$AE = CF$, $E \in \{a, d\}$, $F \in \{c, b\}$, $P \in EF$	[49]
194	2026	Josefsson ⁴	$AG_1 = CG_3$ and $BG_2 = DG_4$	[49]
195	2026	Josefsson ⁴	$\mathbf{AG}_1 + \mathbf{CG}_3 = \mathbf{0}$	[49]
196	2026	Josefsson ⁴	$\mathbf{A}''\mathbf{A}' = \mathbf{C}'\mathbf{C}''$	[49]
197	2026	Dalcín ⁴	Triangle NMK is equilateral	[49]
198	2026	Josefsson ⁴	Triangle $N'M'K'$ is equilateral	[49]
199	2026	Josefsson ⁴	Triangle $S'R'Q'$ is equilateral	[49]
200	2026	Dalcín ⁴	$AD \parallel BC$ with $T_{EFG} = K_{ABCG}$	[49]

TABLE 1. Characterizations of parallelograms

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