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THE BEST MINORATION AND MAJORATION FOR THE SUM OF MEDIANS IN A TRIANGLE

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Abstract. The purpose of this article is to find the best bounds $f(R,r)$ and $g(R,r)$ such that $f(R,r) \leq m_a + m_b + m_c \leq g(R,r)$ which is true in every triangle or in every non-obtuse triangle. Using the above result we find the best constant α_1 for which $m_a + m_b + m_c \geq \alpha_1 R + (288 - 64\alpha_1)\dfrac{r^6}{R^5}$ is true in non-obtuse triangle and taking in account of inequality $\alpha_1 R + (288 - 64\alpha_1)\dfrac{r^6}{R^5} \geq 4R + \dfrac{32r^6}{R^5}$ which is true in every triangle, we give a proof for the conjecture of J. Liu from [7]. Also, we find the best constant α_2 for which $m_a + m_b + m_c \geq \alpha_2 R + (9 - 2\alpha_2)r$ is true in every non-obtuse triangle.

1. Introduction

In this section we will recall some known results, which we will use in the following.

In a given triangle ABC, we denote the lengths of the sides with $AB = c$, $BC = a$, $CA = b$, F the area, r, R the radius of the inscribed circle with the center I, respectively of the circumscribed circle with the center O of the triangle, $s = \dfrac{a+b+c}{2}$ the semiperimeter, the distance between O and I by $d = \sqrt{R^2 - 2Rr}$, with A, B, C the measures of the angles and m_a, m_b, m_c the lengths of the medians in A, B and C respectively.

W.J. Blundon in [2] has proved in 1965 the following inequalities

$$(1) \qquad 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq$$
$$\leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}.$$

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The inequalities from (1) represent necessary and sufficient conditions for the existence of a triangle with given elements R, r and s.

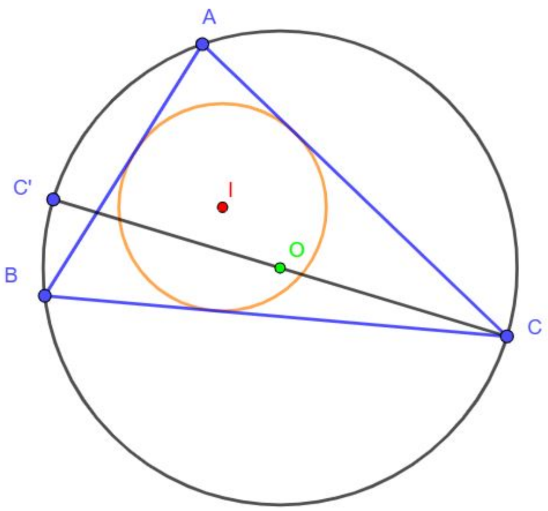

FIGURE 1

Lemma 1. *Let ABC be a given triangle.*
(i) If $d < r$, then ABC is an acute triangle.
(ii) If $d \geq r$, then ABC can be an obtuse, acute or right triangle.

Proof. (i) Because $d < r$ it turns out that O is located inside the triangle ABC (see Figure 1). We denote by C' the point where the line CO intersect the circle $\mathcal{C}(O, R)$ for the second time and then it results that C' is located inside the arc $\overset{\frown}{AB}$. Then we have that
$A = \frac{1}{2} measure(\overset{\frown}{BC}) < \frac{1}{2} measure(\overset{\frown}{CBC'}) = \frac{1}{2} \cdot \pi$, from which it follows that A is an acute angle. The same is true for angle B and C.
(ii) If $d \geq r$ then in Figure 2 we have that triangle ABC is acute, triangle $A'B'C'$ is right and $A"B"C"$ is obtuse $\left(A" > \frac{\pi}{2}\right)$.

Let's recall some results found in the paper [5].
In the following we consider given the triangle ABC, $\mathcal{C}(O, R)$ the circumscribed circle and $\mathcal{C}(I, r)$ the inscribed circle. The half-lines $(OI$, $(IO$ intersect $\mathcal{C}(O, R)$ in A_1, respectively A_2.
According to Poncelet's Theorem, are obtained the triangles $A_1B_1C_1$ and $A_2B_2C_2$, tangent to the circle $\mathcal{C}(O, I)$ (see Figure 3).

Lemma 2. (i) *The lengths of the sides of the triangle $A_1B_1C_1$ are given by*

$$(2) \qquad a_1 = 2\sqrt{R^2 - (r-d)^2}, \quad b_1 = c_1 = \sqrt{2R(R + r - d)},$$

while those of the sides of the triangle $A_2B_2C_2$ are given by

$$(3) \qquad a_2 = 2\sqrt{R^2 - (r+d)^2}, \quad b_2 = c_2 = \sqrt{2R(R + r + d)},$$

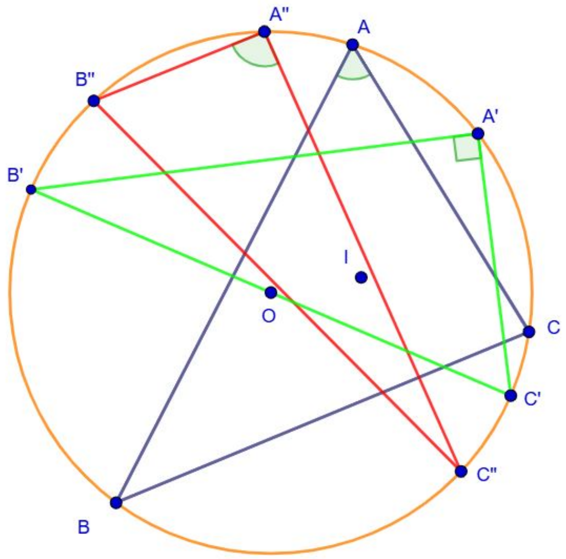

FIGURE 2

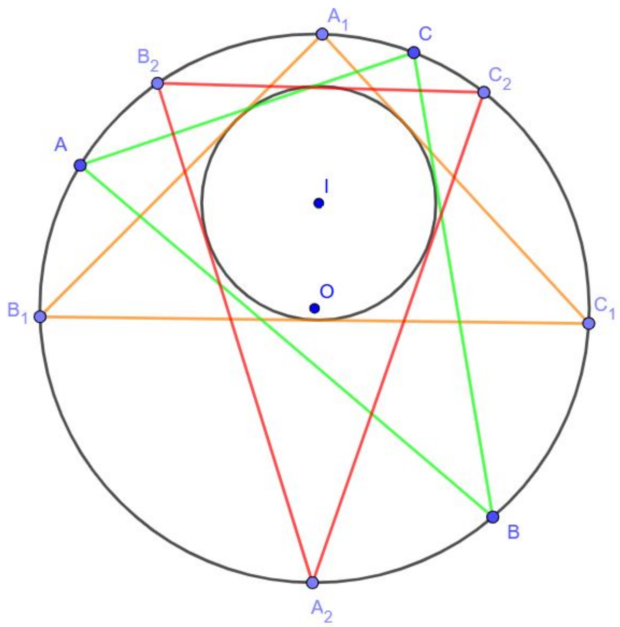

FIGURE 3

(ii) The semiperimeter of the triangle $A_1B_1C_1$ is

$$s_1 = \sqrt{\frac{(R+r-d)^3}{R-r-d}} = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}}, \tag{4}$$

while that of the triangle $A_2B_2C_2$ is

$$s_2 = \sqrt{\frac{(R+r+d)^3}{R-r+d}} = \sqrt{2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}}. \tag{5}$$

Remark 1. It is immediately verified that $a_1 > b_1$ and $a_2 < b_2$.

Theorem 1 (Fundamental triangle inequalities of Blundon)**.** *The following inequalities*

$$s_1 \le s \le s_2 \tag{6}$$

hold. The equality occurs on the left-side and respectively on right-side of inequality if and only if the triangle ABC becomes triangle $A_1B_1C_1$, respectively triangle $A_2B_2C_2$, with the sides from the Lemma 2.

In [3], C. Ciamberlini proved the identity

$$s^2-(2R+r)^2 = 4R^2\cos A\cos B\cos C, \tag{7}$$

where A, B, C are the angles of the triangle ABC. From (7), the following results emerge.

Lemma 3 (C. Ciamberlini, see [3])**.** *In a triangle ABC we have*
(i) $s < 2R+r$ if and only if the triangle is obtuse;
(ii) $s = 2R+r$ if and only if the triangle is right;
(iii) $s > 2R+r$ if and only if the triangle is acute.

In the following we will study the case $d \ge r$, equivalently $R \ge \left(\sqrt{2}+1\right) r$, that is the case when the point O is exterior to the circle $\mathcal{C}(I,r)$. The tangents through O to the circle $\mathcal{C}(I,r)$ intersect the circle $\mathcal{C}(O,R)$ in two pairs of points B_3, C_3 and B_4, C_4. Next we construct the right triangles $A_3B_3C_3$ and $A_4B_4C_4$ inscribed in $\mathcal{C}(O,R)$ and circumscribed to $\mathcal{C}(I,r)$ (see the Figure 4).

Remark 2. Immediately check that triangles $A_3B_3C_3$ and $A_4B_4C_4$ are congruent. If $d = r$, equivalent to $R = (\sqrt{2}+1)r$, then triangle $A_4B_4C_4$ coincides with triangle $A_3B_3C_3$.

Lemma 4. *If $\dfrac{R}{r} \ge \sqrt{2}+1$, then the right triangle $A_3B_3C_3$ has the sides*

$$a_3 = 2R,\ b_3 = R+r-\sqrt{R^2-2Rr-r^2},\ c_3 = R+r+\sqrt{R^2-2Rr-r^2} \tag{8}$$

and semiperimeter

$$s_3 = 2R+r. \tag{9}$$

Lemma 5. *If $\dfrac{R}{r} \ge \sqrt{2}+1$, then the inequalities*

$$s_1 \le s_3 \le s_2 \tag{10}$$

hold. The equality occurs if and only if $\dfrac{R}{r} = 1+\sqrt{2}$.

Proof. The first inequality from (10) is equivalent with
$2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3} \le 4R^2+4Rr+r^2$, equivalent with
$-R^2+3Rr-r^2 \le \sqrt{R(R-2r)^3}$. If $\dfrac{R}{r} > \dfrac{3+\sqrt{5}}{2}$, then $-R^2+3Rr-r^2 < 0$,

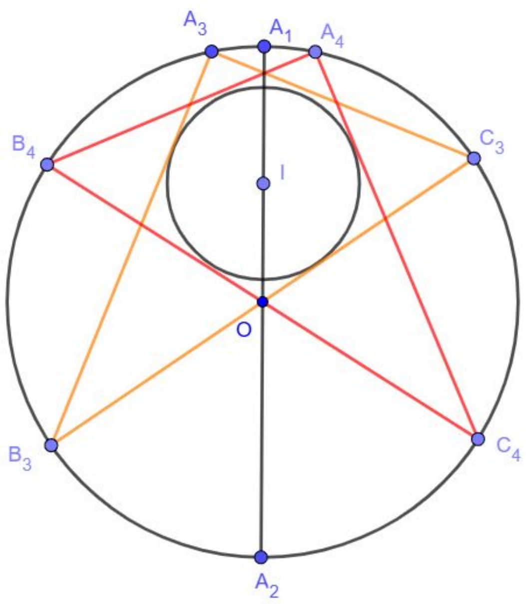

FIGURE 4

so inequality holds. If $2 \leq \frac{R}{r} \leq \frac{3+\sqrt{5}}{2}$, then $-R^2+3Rr-r^2 \geq 0$ and by squaring we have $(-R^2+3Rr-r^2)^2 \leq R(R-2r)^3$, equivalent after performing some calculation with $0 \leq R^2-2Rr-r^2$, or $\frac{R}{r} \geq 1+\sqrt{2}$.
The inequality $s_2 \geq s_3$ is equivalent to

$$2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3} \geq 4R^2+4Rr+r^2,$$

equivalent to $\sqrt{R(R-2r)^3} \geq R^2-3Rr+r^2$. If $\frac{R}{r} \in \left[2, \frac{3+\sqrt{5}}{2}\right)$, the inequality occurs because the right-hand side is negative.

If $\frac{R}{r} \geq \frac{3+\sqrt{5}}{2}$ then $R^2-3Rr+r^2 \geq 0$ and by squaring we have $R(R-2r)^3 \geq (R^2-3Rr+r^2)^2$, equivalent after performing some calculation to $R^2-2Rr-r^2 \geq 0$, or $\frac{R}{r} \geq \sqrt{2}+1$.

Lemma 6. *Triangles $A_1B_1C_1$ and $A_2B_2C_2$ can be equilateral triangles, but triangle $A_3B_3C_3$ cannot.*

Proof. From (2), $A_1B_1C_1$ can be equilateral, equivalent to $a_1 = b_1$, equivalent to $R-2r+2\sqrt{R(R-2r)} = 0$, equivalent to $R = 2r$, which is true equality in an equilateral triangle. Similar for triangle $A_2B_2C_2$. If triangle $A_3B_3C_3$ was equilateral, then from (8) we have $a_3 = b_3 = c_3$, equivalent to $R^2-2Rr-r^2 = 0$ and $2R = R+r$ from where $R = r$ which is a false equality.

Theorem 2. *Let ABC be a triangle with the property that* $\dfrac{R}{r} \geq \sqrt{2}+1$.
(*i*) *If ABC is acute or right triangle, then*

$$s_1 \leq s_3 \leq s, \tag{11}$$

with equality on the right-side if and only if the triangle ABC becomes triangle $A_3B_3C_3$.
(*ii*) *If ABC is obtuse or right triangle, then*

$$s \leq s_3 \leq s_2, \tag{12}$$

with equality on the left-side if and only if the triangle ABC becomes triangle $A_3B_3C_3$.

Proof. Inequalities (11) and (12) are obtained from Theorem 1, Lemma 3 and Lemma 5.

Remark 3. In the above we showed the existence of triangles $A_1B_1C_1$, $A_2B_2C_2$ and $A_3B_3C_3$ for which the equalities in (10)-(12) hold.

Using the above results we obtain the following theorems.

Theorem 3. *Let ABC an acute or right triangle. If* $2 \leq \dfrac{R}{r} \leq \sqrt{2}+1$, *then*

$$s_1 \leq s \leq s_2, \tag{13}$$

the equality on the left-side and respectively on the right-side of inequality hold if and only if the triangle ABC becomes triangle $A_1B_1C_1$, respectively triangle $A_2B_2C_2$, with the sides from Lemma 2.
If $\dfrac{R}{r} \geq \sqrt{2}+1$, *then*

$$s_3 \leq s \leq s_2 \tag{14}$$

the equality on the left-side and respectively on the right-side of inequality hold if and only if the triangle ABC becomes triangle $A_3B_3C_3$, respectively $A_2B_2C_2$, with the sides from Lemma 2 *and Lemma* 4.

Theorem 4. *If ABC is obtuse or right triangle and* $\dfrac{R}{r} \geq \sqrt{2}+1$, *then*

$$s_1 \leq s \leq s_3, \tag{15}$$

with equality in (1.15) *on the left-side and respectively on the right-side if and only if the triangle ABC becomes triangle $A_1B_1C_1$, respectively $A_3B_3C_3$.*

Remark 4. If the triangle ABC is acute or obtuse triangle then the left-side of inequality from (14) and the right-side of the inequality from (15) are strict.

2. Main results

In the following, we will denote by $m_{a_i}, m_{b_i}, m_{c_i}$ the lengths of the medians in A_i, B_i and C_i respectively, from the triangle $A_iB_iC_i$, where $i \in \{1,2,3\}$. The triangles $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ are defined in Introduction.

Lemma 7. *The following equalities*

$$m_{a_1} = R + r - d, \quad m_{b_1} = m_{c_1} = \frac{1}{2}\sqrt{2(R+r-d)(5R-4r+4d)} \tag{16}$$

and

$$m_{a_2} = R + r + d, \quad m_{b_2} = m_{c_2} = \frac{1}{2}\sqrt{2(R+r+d)(5R-4r-4d)} \tag{17}$$

hold.

Proof. Taking (2) into account, we have

$$m_{a_1} = \sqrt{\frac{2(b_1^2+c_1^2)-a_1^2}{4}} = \sqrt{\frac{4b_1^2-a_1^2}{4}}$$
$$= \frac{1}{2}\sqrt{8R(R+r-d) - 4\big(R^2-(r-d)^2\big)} = \sqrt{(R+r-d)^2} = R+r-d$$

and

$$m_{b_1} = m_{c_1} = \frac{1}{2}\sqrt{2a_1^2+b_1^2} = \frac{1}{2}\sqrt{8\big(R^2-(r-d)^2\big)+2R(R+r-d)}$$
$$= \frac{1}{2}\sqrt{2(R+r-d)\big(4(R-r+d)+R\big)} = \frac{1}{2}\sqrt{2(R+r-d)(5R-4r+4d)}.$$

The equalities in (17) are proven similarly.

Lemma 8. *If* $\frac{R}{r} \geq \sqrt{2}+1$ *then the equalities*

$$m_{a_3} = R \tag{18}$$

and

$$m_{b_3} = \frac{1}{2}\sqrt{10R^2 + 6(R+r)\sqrt{R^2-2Rr-r^2}},$$
$$m_{c_3} = \frac{1}{2}\sqrt{10R^2 - 6(R+r)\sqrt{R^2-2Rr-r^2}} \tag{19}$$

Proof. The relations from (8) are used.

Theorem 5. *In every triangle ABC are true the following inequalities.*

$$R + r - d + \sqrt{2(R+r-d)(5R-4r+4d)} \leq m_a + m_b + m_c \leq$$
$$\leq R + r + d + \sqrt{2(R+r+d)(5R-4r-4d)}. \tag{20}$$

The equality occurs on the left-side and respectively on the right-side of the inequality if and only if the triangle ABC *becomes triangle* $A_1B_1C_1$*, respectively triangle* $A_2B_2C_2$*, with the sides from Lemma* 2.

Proof. We denote $w = m_a + m_b + m_c = \sum \sqrt{m_a^2}$ and after squaring we obtain

$$w^2 = \sum m_a^2 + 2\sum m_a m_b = \sum m_a^2 + 2\sqrt{\left(\sum m_a m_b\right)^2} =$$
$$\sum m_a^2 + 2\sqrt{\sum m_a^2 m_b^2 + 2m_a m_b m_c \cdot w}.$$

The following identities are very well known

$$\sum m_a^2 = \frac{3}{4}(a^2+b^2+c^2) = \frac{3}{2}(s^2 - -r^2 - 4Rr)$$
$$\sum m_a^2 m_b^2 = \frac{9}{16}\left[(s^2+r^2+4Rr)^2 - 16Rrs^2\right]$$

and $16m_a^2m_b^2m_c^2 = s^6 + (33r^2-12Rr)s^4 - (60R^2r^2+120Rr^3+33r^4)s^2 - (4Rr+r^2)^3$.
We define the functions $f,g,h : [s_1,s_2] \to \mathbb{R}$, $f(s) = \sum m_a^2$, $g(s) = \sum m_a^2 m_b^2$, $h(s) = m_a m_b m_c$. According to the identities above, this functions depend on s, where $s \in [s_1,s_2]$. We have $w^2 = f(s) + 2\sqrt{g(s)+2h(s)\cdot w}$, equivalent to $w^2 - f(s) = 2\sqrt{g(s)+2h(s)\cdot w}$. If we consider the variable u, the equality above becomes

$$u^2 - f(s) = 2\sqrt{g(s)+2h(s)\cdot u}, \tag{21}$$

where $u \geq \sqrt{f(s)}$, and for $u(s) = w$ in (21) the equality holds.
Let $F : [\sqrt{f(s)}, +\infty) \to \mathbb{R}$ be a function defined by

$$F(u) = \big(u^2 - f(s)\big)^2 - 4\big(g(s)+2h(s)u\big) = u^4 - 2f(s)u^2 - 8h(s)u + f^2(s) - 4g(s),$$

where $u \in [\sqrt{f(s)}, +\infty)$. We have $F(w) = 0$ and $F'(u) = 4u^3 - 4f(s)u - 8h(s) = 4u\big(u^2 - f(s)\big) - 8h(s)$, $u \in [\sqrt{f(s)}, +\infty)$.
It follows that F' is increasing on $[\sqrt{f(s)}, +\infty)$.
But $F'\big(\sqrt{f(s)}\big) = -8h(s) < 0$ and $\lim\limits_{u\to\infty} F'(u) = +\infty$, it follows that F' has a single root in $[\sqrt{f(s)}, +\infty)$.
Since $F\big(\sqrt{f(s)}\big) = -8h(s)\sqrt{f(s)} - 4g(s) < 0$, $\lim\limits_{u\to\infty} F(u) = +\infty$ and $F(w) = 0$, it results that the equation $F(u) = 0$ has w as it's only root on $[\sqrt{f(s)}, +\infty)$.
So, the equation $F(u) = 0$ is equivalent with

$$u^4 - 2f(s)u^2 - 8h(s)u + f^2(s) - 4g(s) = 0, \tag{22}$$

equivalent, taking identities above, with

$$\begin{aligned} &u^4 - 3(s^2 - r^2 - 4Rr)u^2 \\ &-2\sqrt{s^6 + (33r^2-12Rr)s^4 - (60R^2r^2+120Rr^3+33r^4)s^2 - (4Rr+r^2)^3}\cdot u \\ &+\frac{9}{4}(s^2 - r^2 - 4Rr)^2 - \frac{9}{4}[(s^2+r^2+4Rr)^2 - 16Rrs^2] = 0. \end{aligned} \tag{23}$$

Taking into account [8], it follows that the positive root of the equation from (23) is expressed as operations with differentiable functions, so the root of (23) is a differentiable function, $u : [s_1,s_2] \to \mathbb{R}$, the variable of the function u being s. Differentiating relation (22), we can write it

$$\begin{aligned} &2\Big(u(s)\big(u^2(s) - f(s)\big) - 2f(s)\Big)u'(s) \\ &= f'(s)\big(u^2(s) - f(s)\big) + 2g'(s) + 4h'(s)u(s), \end{aligned} \tag{24}$$

where $s \in [s_1, s_2]$.
Applying the inequality of means, we have

$$\begin{aligned} g(s) + 2h(s)u(s) &= \sum m_a^2 m_b^2 + 2m_a m_b m_c (m_a + m_b + m_c) \\ &= \sum m_a^2 m_b^2 + \sum 2m_a^2 m_b m_c \geq 6\sqrt[6]{8m_a^8 m_b^8 m_c^8} \\ &= 6\sqrt{2}\sqrt[3]{(m_a m_b m_c)^4} = 6\sqrt{2}\sqrt[3]{h^4(s)}] \end{aligned}$$

and taking (2.6) into account we have $u^2(s) - f(s) = 2\sqrt{g(s) + 2h(s)u(s)} \geq 2\sqrt{6\sqrt{2}\sqrt[3]{h^4(s)}} = 2\sqrt{6\sqrt{2}}\sqrt[3]{h^2(s)}$. From this inequality it follows that $u^2(s) > 2\sqrt{6\sqrt{2}}\sqrt[3]{h^2(s)}$, from where $u(s) > \sqrt{2\sqrt{6\sqrt{2}}}\sqrt[3]{h(s)}$. From the inequalities above we have $u(s)\big(u^2(s) - f(s)\big) - 2h(s) \geq \big(2\sqrt{2}\sqrt[4]{(6\sqrt{2})^3} - 2\big)h(s)$, from where

$$u(s)\big(u^2(s) - f(s)\big) - 2h(s) > 0, \tag{25}$$

for every $s \in [s_1, s_2]$.
Because $f(s) = \sum m_a^2 = \dfrac{3}{2}(s^2 - r^2 - 4Rr)$, then $f'(s) = 3s$, so

$$f'(s) > 0, \tag{26}$$

for any $s \in [s_1, s_2]$.
We have $g(s) = \sum m_a^2 m_b^2 = \dfrac{9}{16}\big((s^2 + r^2 + 4Rr)^2 - 16Rrs^2\big)$, from where $g'(s) = \dfrac{9}{4}s(s^2 + r^2 - 4Rr)$ and taking Gerretsen's Inequality $s^2 \geq 16Rr - 5r^2$ into account, we have

$$g'(s) \geq 0, \tag{27}$$

for any $s \in [s_1, s_2]$.
We have $h(s) = m_a m_b m_c = \dfrac{1}{4}\sqrt{z(s)}$, where $z(s) = s^6 + (33r^2 - 12Rr)s^4 - (60R^2r^2 + 120Rr^3 + 33r^4)s^2 - (4Rr + r^2)^3$ and if noting $s^2 = t$, we obtain the function $p : [s_1^2, s_2^2] \to \mathbb{R}$, $p(t) = t^2 + (33r^2 - 12Rr)t - (60R^2r^2 + 120Rr^3 + 33r^4)$, $t \in [s_1^2, s_2^2]$ and $z(s) = tp(t) - (4Rr + r^2)^3$, $s \in [s_1, s_2]$. Then $p(t) = t\big(t - (12Rr - 33r^2)\big) - (60R^2r^2 + 120Rr^3 + 33r^4)$ and according Gerretsen's Inequality we have $t = s^2 \geq 16Rr - 5r^2 > 12Rr - 33r^2$, so p is an increasing function on $[s_1^2, s_2^2]$. After calculus, we have that $p(t) \geq p(16Rr - 5r^2) = r^2(4R^2 + 308Rr - 173r^2) > 0$, so p is a positive function on $[s_1^2, s_2^2]$.
From above it results that $z(s)$ is increasing on $[s_1, s_2]$, from where $z'(s) \geq 0$, for any $s \in [s_1, s_2]$.
From these remarks, follows that the function $h(s) = \dfrac{1}{4}\sqrt{z(s)}$ is increasing on $[s_1, s_2]$, so

$$h'(s) \geq 0, \tag{28}$$

for any $s \in [s_1, s_2]$. Taking (21), (25)-(28) into account, from (24) we obtain $u'(s) \geq 0$ for all $s \in [s_1, s_2]$, so function u is increasing on $[s_1, s_2]$, from where $u(s_1) \leq u(s) \leq u(s_2)$, for any $s \in [s_1, s_2]$, equivalent to $m_{a_1} + m_{b_1} + m_{c_1} \leq m_a + m_b + m_c \leq m_{a_2} + m_{b_2} + m_{c_2}$, for any $s \in [s_1, s_2]$.

Theorem 6. *Let ABC be a triangle with the property that* $\dfrac{R}{r} \geq \sqrt{2}+1$.
(i) If ABC is acute or right triangle, then

$$R+r-d+\sqrt{2(R+r-d)(5R-4r+4d)} \leq$$

$$\leq R+\frac{1}{2}\left(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\right) \leq \tag{29}$$

$$\leq m_a+m_b+m_c,$$

with equality on the right-side if and only if the triangle ABC becomes triangle $A_3B_3C_3$.
(ii) If ABC is obtuse or right triangle, then

$$m_a+m_b+m_c \leq$$

$$\leq R+\frac{1}{2}\left(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\right) \leq \tag{30}$$

$$\leq R+r+d+\sqrt{2(R+r+d)(5R-4r-4d)},$$

with equality on the left-side if and only if the triangle ABC becomes triangle $A_3B_3C_3$.

Proof. The results of this theorem are obtained taking into account Theorem 2, Lemmas 7 and 8 and that from the proof of Theorem 5 it follows that the function u is increasing.

According Theorem 5 and Theorem 6 we obtained the following theorems.

Theorem 7. *Let ABC an acute or right triangle.*
(i) If $2 \leq \frac{R}{r} \leq \sqrt{2}+1$, *then*

$$R+r-d+\sqrt{2(R+r-d)(5R-4r+4d)} \leq m_a+m_b+m_c \leq$$

$$\leq R+r+d+\sqrt{2(R+r+d)(5R-4r-4d)}, \tag{31}$$

with equality on the left-side and respectively on the right-side of inequality if and only if the triangle ABC becomes $A_1B_1C_1$, *respectively triangle* $A_2B_2C_2$.
(ii) If $\dfrac{R}{r} \geq \sqrt{2}+1$, *then*

$$R+\frac{1}{2}\left(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\right) \tag{32}$$

$$\leq m_a+m_b+m_c \leq R+r+d+\sqrt{2(R+r+d)(5R-4r-4d)},$$

with equality on the left-side and respectively on the right-side of inequality if and only if the triangle ABC becomes $A_3B_3C_3$, *respectively triangle* $A_2B_2C_2$.

Theorem 8. *If ABC is obtuse or right triangle and* $\dfrac{R}{r} \leq \sqrt{2}+1$, *then*

$$m_a+m_b+m_c$$

$$\leq R+\frac{1}{2}\left(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\right), \tag{33}$$

with equality if and only if the triangle ABC becomes $A_3B_3C_3$.

Lemma 9. *The following inequality*

$$R+r+d+\sqrt{2(R+r+d)(5R-4r-4d)} \leq 4R+r, \tag{34}$$

holds, with equality if and only if ABC is an equilateral triangle.

Proof. The inequality from (34) is equivalent to $\sqrt{2R^2+18Rr-8r^2+2d(R-8r)} \le 3R-d$. Since $3R-d>0$, squaring it we have $2R^2+18Rr-8r^2+2d(R-8r) \le 10R^2-6Rd-2Rr$, equivalent to $0 \le (R-2r)(2R-r)-2d(R-2r)$, equivalent to $0 \le (R-2r)(2R-r-2d)$. We demonstrate that $2R-r>2d$. Because $R \ge 2r$, it results that $2R-r>0$ and squaring we have $4R^2-4Rr+r^2>4(R^2-2Rr)$, which is a true inequality. So, $2R-r-2d>0$ and $R>2r$, it results that $(R-2r)(2R-r-2d) \ge 0$. Equality holds if and only if $R=2r$, equivalent to ABC is an equilateral triangle.

From Theorem 7 and Lemma 8 we obtain a well known inequality contained in Corollary 1.

Corollary 1. *In every triangle ABC is true the inequality*

$$m_a+m_b+m_c \le 4R+r, \tag{35}$$

with equality if and only if the triangle ABC is an equilateral triangle.

In the following, we find the best constants $\alpha,\beta \in \mathbb{R}$ such that $\alpha R+\beta r \le m_a+m_b+m_c$ is true in every triangle ABC, with the condition that the equality holds for the equilateral triangle ABC, so $R=2r$. Then, we have that $m_a=m_b=m_c=\dfrac{l\sqrt{3}}{2}$, $R=\dfrac{l\sqrt{3}}{3}$, $r=\dfrac{l\sqrt{3}}{6}$, where l is the length of the side of the triangle ABC, so $\alpha\cdot\dfrac{l\sqrt{3}}{3}+\beta\cdot\dfrac{l\sqrt{3}}{6}=3\cdot\dfrac{l\sqrt{3}}{2}$, from where $2\alpha+\beta=9$, so $\beta=9-2\alpha$.

Remark 5. From above it follows that if $\alpha \in \mathbb{R}$ verify the inequality $\alpha R+(9-2\alpha)r \le m_a+m_b+m_c$, then for $R=2r$ the equality in the inequality above holds.

Lemma 10. *Let ABC be a triangle and $\alpha \le 3\sqrt{10}+4\sqrt{5}-7\sqrt{6}-6 \approx 2,53161$. If*

(i) $2 \le \dfrac{R}{r} \le \sqrt{2}+1$, *then*

$$\alpha R+(9-2\alpha)r \le R+r-d+\sqrt{2(R+r-d)(5R-4r+4d)}; \tag{36}$$

(ii) $\dfrac{R}{r} \ge \sqrt{2}+1$, *then*

$$\alpha R+(9-2\alpha)r \le R+\frac{1}{2}\Big(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+ +\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\Big); \tag{37}$$

(iii) $\alpha_0=3\sqrt{10}+4\sqrt{5}-7\sqrt{6}-6$, *then the equalities in* (36) *and* (37) *hold if and only if* $\dfrac{R}{r} \in \{2,\sqrt{2}+1\}$.

Proof. (i) If we note $\dfrac{R}{r}=x$, $d(x)=\sqrt{x^2-2x}$, dividing in (36) by r^2, we obtain $\alpha x+(9-2\alpha) \le x+1-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}$, from where $\alpha \le \dfrac{x-8-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}}{x-2}$, for

$x \in (2, \sqrt{2}+1]$. We have that

$$\lim_{\substack{x\to 2\\ x>2}} \frac{x-8-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}}{x-2}$$
$$= \lim_{\substack{x\to 2\\ x>2}} \left(1+\frac{\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}-\big(d(x)+6\big)}{x-2}\right)$$
$$= \lim_{\substack{x\to 2\\ x>2}} \left(1+\frac{(x-2)(x+22)-2d(x)(x-2)}{\left(\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}+\big(d(x)+6\big)\right)(x-2)}\right)$$
$$= 3$$

and let $u_1 : [2, \sqrt{2}+1] \to \mathbb{R}$ be a function defined by

$$u_1(x) = \begin{cases} \dfrac{x-8-d(x)+\sqrt{2(x+1-d(x))(5x-4+4d(x))}}{x-2}, & x \in (2, \sqrt{2}+1] \\ 3, \quad x = 2. \end{cases}$$

Using the program Wolfram|Alpha we have

$$\alpha \le \inf_{2\le x\le \sqrt{2}+1} u_1(x) = 3\sqrt{10}+4\sqrt{5}-7\sqrt{2}-6.$$

From the above it follows that inequality (36) holds.
(ii) According to the idea from (i) if we note $t(x) = \sqrt{x^2-2x-1}$, we have $\alpha x+(9-2\alpha) \le x+\frac{1}{2}\left(\sqrt{10x^2+6(x+1)t(x)}+\sqrt{10x^2-6(x+1)t(x)}\right)$ and then let $v_1 : [\sqrt{2}+1, +\infty) \to \mathbb{R}$ be a function defined by $v_1(x) = \dfrac{x-9+\frac{1}{2}\left(\sqrt{10x^2+6(x+1)t(x)}+\sqrt{10x^2-6(x+1)t(x)}\right)}{x-2}$, $x \in [\sqrt{2}+1, +\infty)$.
Using the program Wolfram|Alpha we have

$$\alpha \le \inf_{x\ge \sqrt{2}+1} v_1(x) = 3\sqrt{10}+4\sqrt{5}-7\sqrt{2}-6.$$

From (i), (ii), Theorem 7, Remark 5 and program Wolfram|Alpha, follows (iii).

Theorem 9. *In every acute or right triangle is true the inequality*

(38) $(3\sqrt{10}+4\sqrt{5}-7\sqrt{2}-6)R+(21+14\sqrt{2}-6\sqrt{10}-8\sqrt{5})r \le m_a+m_b+m_c,$

with equality if and only if $\dfrac{R}{r} \in \{2, \sqrt{2}+1\}$.

Proof. It follows from Lemma 10.

Corollary 2. *In every acute or right triangle, the following inequalities*

$$m_a+m_b+m_c \ge \alpha_0 R+(9-2\alpha_0)r \ge \frac{5}{2}R+4r \tag{39}$$

and

$$m_a+m_b+m_c \ge \frac{5}{2}R+4r \tag{40}$$

hold.

Proof. We consider the function $w : (0,+\infty) \to \mathbb{R}$ defined by $w(\alpha) = \alpha R+(9-2\alpha)r = \alpha(R-2r)+9r$, $\alpha \in (0,+\infty)$. Because $R \geq 2r$, the function w is increasing on $(0,+\infty)$ and $\alpha_0 > \frac{5}{2}$ it results that $w(\alpha_0) \geq w\left(\frac{5}{2}\right)$, so (39) is obtained. The equality holds if and only if $R = 2r$, equivalent to ABC is an equilateral triangle. From (39) follows (40).

Remark 6. Inequality (40) is proven in [7] and then the first inequality in (39) is a refinement of the inequality (40).

Lemma 11. *Let ABC be a triangle and $\alpha \leq 4$. If*
(i) $2 \leq \frac{R}{r} \leq \sqrt{2}+1$, *then*

$$(41)\quad \alpha R + (288-64\alpha)\frac{r^6}{R^5} \leq R + r - d + \sqrt{2(R+r-d)(5R-4r+4d)};$$

(ii) $\frac{R}{r} \geq \sqrt{2}+1$, *then*

$$(42)\quad \alpha R+ (288-64\alpha)\frac{r^6}{R^5} \leq R+ \frac{1}{2}\Bigg(\sqrt{10R^2+6(R+r)\sqrt{R^2-2Rr-r^2}}+ +\sqrt{10R^2-6(R+r)\sqrt{R^2-2Rr-r^2}}\Bigg);$$

(iii) $\alpha_1 = 4$ *then the equalities in* (41) *and* (42) *hold if and only if* $\frac{R}{r} = 2$.

Proof. Because $\frac{R}{r} \geq 2$ we have that $m_a + m_b + m_c \geq \alpha R + (9-2\alpha)r \geq \alpha R + (9-2\alpha)r \cdot 32\left(\frac{r}{R}\right)^5 = \alpha R + (288-64\alpha)\frac{r^6}{R^5}$ and taking Remark 5 into account, then for $R = 2r$ the inequalities above the equalities become equal. Using the ideas from Lemma 10, from (41) we have
$\alpha \leq \frac{x^5}{x^6-64}\left(x+1-\frac{288}{x^5}-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}\right)$, for $x \in (2, \sqrt{2}+1]$. Because
$\lim\limits_{\substack{x\to 2\\ x>2}} \frac{x^5}{x^6-64}\left(x+1-\frac{288}{x^5}-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}\right)$
is equal to $\frac{17}{4}$, let $u_2 : [2, \sqrt{2}+1] \to \mathbb{R}$ be a function defined by

$$u_2(x) = \begin{cases} \frac{x^5}{x^6-64}\left(x+1-\frac{288}{x^5}-d(x)+\sqrt{2\big(x+1-d(x)\big)\big(5x-4+4d(x)\big)}\right), & x \in (2, \sqrt{2}+1] \\ \frac{17}{4}, & x = 2. \end{cases}$$

Using the program Wolfram|Alpha we have

$$\alpha \leq \inf_{2\leq x\leq\sqrt{2}+1} u_2(x) = \frac{1}{245}(181\sqrt{10}+256\sqrt{5}-448\sqrt{2}+469) \approx 4,00097$$

and is touched for $x = \sqrt{2}+1$. From the above it follows that inequality (41) holds. From (42) we obtain that

$$\alpha \leq v_2(x) = \frac{x^5}{x^6-64}\left(x-\frac{288}{x^5}+\frac{1}{2}\left(\sqrt{10x^2+6(x+1)t(x)}+\sqrt{10x^2-6(x+1)t(x)}\right)\right)$$

and using the program Wolfram|Alpha we have $\alpha \leq \inf_{x \geq \sqrt{2}+1} v_2(x) = 4$, at the limit when x tends to infinity. From the above the best constant is 4. From (i), (ii), Theorem 7, Remark 5 and program Wolfram|Alpha follows (iii).

Corollary 3. *In every acute or right triangle ABC, the inequality*

$$m_a + m_b + m_c \geq 4R + \frac{32r^6}{R^5} \tag{43}$$

holds, with equality if and only if ABC is an equilateral triangle.

Proof. Is obtained immediately from Lemma 11.

Remark 7. The inequality from (43) represent the Conjecture 5.1 from [7].

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