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SEQUENCE OF FINSLER–HADWIGER REFINEMENTS

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Abstract. We demonstrated that, given an improvement of the Finsler–Hadwiger inequality, it is possible to generate a sequence of enhancements that converge exponentially to the left-hand side of the Finsler–Hadwiger inequality.

1. Introduction

Geometric inequalities have attracted the attention of many generations of mathematicians, shedding profound understanding not only on matters of symmetry and optimality but also on the basic structure of geometric figures. A close relationship exists, particularly in triangle geometry, between side lengths and angles, radius, and area. The Finsler–Hadwiger inequality represents a significant contribution to this highly fruitful field. This elegant inequality gives a sharper form of the classical Weitzenböck inequality and relates to broader themes in Euclidean geometry as well as in the theory of algebraic inequalities.

In triangle ABC, we use the standard notations: α, β, and γ represent the measures of the angles, while a, b, and c denote the lengths of the sides opposite angles A, B, and C, respectively. Additionally, R, r, s, and T represent the circumradius, inradius, semiperimeter, and area of triangle ABC, respectively. Throughout the text, we also use the notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Weitzenböck inequality states that:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}T,$$

with equality if and only if the triangle is equilateral. This result appeared in the early 20th century with Julius Weitzenböck and acted as a geometric analogue to inequalities involving means. It shows how equilateral triangles are optimal in terms of minimizing the total squared side length for a given area.

In 1937, Paul Finsler and Hans Hadwiger published a note in Commentarii Mathematici Helvetici [2], refining Weitzenböck's inequality by adding a correction term that measures the asymmetry of the triangle:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}T + (a-b)^2 + (b-c)^2 + (c-a)^2, \tag{1}$$

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with equality if and only if the triangle is equilateral. The extra squared terms provide something like a "distance" from the equilateral shape. This refinement came during a time of very active research in geometric inequalities, framed by other impressive results such as the Erdős–Mordell inequality (1935) and the "fundamental inequality of a triangle" [1] (also known as "Blundon's inequality"). The latter was originally proved by Rouché as a solution to a problem posed by C. Ramus in 1851 [11]. All of these results aimed to capture extremal properties of triangles via algebraic expressions.

For many years, the Finsler–Hadwiger inequality was not frequently used, despite its elegance. Because of its potential applications in inequality theory and its openness to improvement, a number of authors have expanded or refined the inequality including D. S. Mitrinović, J. E. Pečarić, V. Volenec [9], D. S. Mitrinović, J. E. Pečarić, V. Volenec, J. Chen. [10], S. Wu [13], S. Wu and L. Debnat [14], C. Lupu and C. Pohoaţă [8], C. Lupu and V. Nicula [7], M. Lukarevski [4, 5], M. Lukarevski and D. S. Marinescu [6], W. D. Jiang [3], and Q. H. Tran [12].

An example of geometric interpretation. The quantitative indicator of deviation from the equilateral shape is the squared differences $(a-b)^2 + (b-c)^2 + (c-a)^2$. Because (1) takes into account both global and local geometric properties, it is especially powerful.

Consider a right triangle with side lengths $a = 3$, $b = 4$, and $c = 5$ as a specific example. Since $T = 6$ is its area, the right-hand side of (1) is as follows:

$$4\sqrt{3}\cdot 6 + (3-4)^2 + (4-5)^2 + (5-3)^2 = 24\sqrt{3} + 6 = 47.5692\ldots$$

The left-hand side is:

$$3^2 + 4^2 + 5^2 = 50,$$

which demonstrates that, in this instance, the inequality is strictly satisfied.

In this study, we develop an infinite series of modifications of the Finsler–Hadwiger inequality that become increasingly sharp.

Theorem 1.1. *Let ABC be a triangle with sides a, b, and c, and let $\varphi_0 > 0$ be a real number such that $a^2+b^2+c^2 \geq \varphi_0 + (b-a)^2 + (c-a)^2 + (c-b)^2$. If $\varphi_n = 16Rr + \frac{r}{4R+r}\varphi_{(n-1)}$ for $n \in \{1, 2, 3, \cdots\}$, then $(\varphi_n)_n$ is increasing and satisfies $(b-a)^2+(c-a)^2+(c-b)^2+\lim\limits_{n\to+\infty}\varphi_n = a^2+b^2+c^2$, and the convergence is exponential. Moreover, the following conditions are equivalent:*

(i) $a^2+b^2+c^2 = \varphi_0 + (b-a)^2 + (c-a)^2 + (c-b)^2$;
(ii) $a^2+b^2+c^2 = \varphi_n + (b-a)^2 + (c-a)^2 + (c-b)^2$, *for some* $n \in \mathbb{N}_0$;
(iii) $a^2+b^2+c^2 = \varphi_n + (b-a)^2 + (c-a)^2 + (c-b)^2$, *for all* $n \in \mathbb{N}_0$.

Recently, Q. H. Tran [12] showed the following improvement of Finsler–Hadwiger inequality.

(2)
$$a^2+b^2+c^2 \geq 4T\sqrt{4 - \frac{2r}{R} + \frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}} + (b-a)^2 + (c-a)^2 + (c-b)^2$$

In light of (2), consider $\varphi_0 = 4T\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}}$ and apply Theorem 1.1. Then $\varphi_1 = 16Rr + \frac{r}{4R+r}\varphi_0$ and
$\varphi_1 = 16Rr + \frac{r}{4R+r}4T\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}}$. This yields the following result.

Corollary 1.1. *If ABC is a triangle with sides a, b, and c, then the inequalities*

$$a^2+b^2+c^2 \geq 16Rr + \frac{r}{4R+r}4T\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}} + (b-a)^2+(c-a)^2+(c-b)^2$$

and

$$(3)\quad \varphi_1 + (b-a)^2+(c-a)^2+(c-b)^2 \geq \varphi_0 + (b-a)^2+(c-a)^2+(c-b)^2$$

hold.

To illustrate the improvement provided by Corollary 1.1, consider the triangle with side lengths $a = 3$, $b = 4$, and $c = 5$. In this case, we have:

$$a^2+b^2+c^2 = 50, \quad (a-b)^2+(a-c)^2+(b-c)^2 = 6, \quad T = 6,$$

$$R = 2.5, \quad \text{and } r = 1.$$

Tran's sharp inequality yields:

$$4T\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}}+(b-a)^2+(c-a)^2+(c-b)^2 = 49.0296\ldots$$

whereas the improved bound from Corollary 1.1 gives:

$$16Rr + (b-a)^2+(c-a)^2+(c-b)^2+$$

$$\frac{r}{4R+r}4T\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}} = 49.9117\ldots$$

Similarly, considering the Finsler–Hadwiger's inequality (1), the Theorem 1.1 is applied with $\varphi_0 = 4\sqrt{3}T$, and then $\varphi_1 = 16Rr + \frac{r}{4R+r}\varphi_0 = 16Rr + \frac{r}{4R+r}4\sqrt{3}T$. We obtain the following.

Corollary 1.2. *If ABC is a triangle with sides a, b, and c, then the inequalities*

$$(4)\quad a^2+b^2+c^2 \geq 16Rr + \frac{r}{4R+r}4\sqrt{3}T + (b-a)^2+(c-a)^2+(c-b)^2$$

and

$$(5)\quad \varphi_1 + (b-a)^2+(c-a)^2+(c-b)^2 \geq \varphi_0 + (b-a)^2+(c-a)^2+(c-b)^2$$

hold, with equality if and only if the triangle is equilateral.

In Corollary 1.2, to verify that equality holds if and only if the triangle is equilateral, note from equation (1) that the identity

$$a^2+b^2+c^2=\varphi_0+(b-a)^2+(c-a)^2+(c-b)^2$$

holds if and only if the triangle is equilateral. This observation, together with Theorem 1.1, completes the argument.

As an example demonstrating the refinement achieved in Corollary 1.2, consider the triangle with side lengths $a=3$, $b=4$, and $c=5$. For this triangle, we compute:

$$a^2+b^2+c^2=50,\quad (a-b)^2+(a-c)^2+(b-c)^2=6,\quad T=6,$$

$$R=2.5,\quad \text{and } r=1.$$

Using the sharp form of the Finsler–Hadwiger inequality, we obtain:

$$4\sqrt{3}T+(b-a)^2+(c-a)^2+(c-b)^2=47.5692\ldots$$

On the other hand, the bound provided by Corollary 1.2 results in:

$$16Rr+\frac{r}{4R+r}\cdot 4\sqrt{3}T=49.7790\ldots$$

New refinements can be developed, as outlined in Corollaries 1.1 and 1.2, by utilizing the bounds established in [3, 4, 5, 6, 7, 8, 13, 14].

2. Auxiliar result

Theorem 1.1 is a consequence of the following general result. Suppose that $\xi=u+v\xi$ such that $1>v>0$ and $u\neq 0$.

Theorem 2.1. *Let φ_0 be a real number such that $\xi\geq\varphi_0$. If $\varphi_n=u+v\varphi_{(n-1)}$ for $n\in\{1,2,3,\cdots\}$, then $(\varphi_n)_n$ is increasing and satisfies $\xi\geq\cdots\geq\varphi_n\geq\cdots\geq\varphi_2\geq\varphi_1\geq\varphi_0$, with $\lim_{n\to+\infty}\varphi_n=\xi$, and the convergence is exponential. Moreover, the following conditions are equivalent:*

(i) $\xi=\varphi_0$;
(ii) $\xi=\varphi_n$, *for some* $n\in\mathbb{N}_0$;
(iii) $\xi=\varphi_n$, *for all* $n\in\mathbb{N}_0$.

2.1. Proof of Theorem 2.1.

Lemma 2.1. *Let φ_0 be a number such that $\xi\geq\varphi_0$. If $\varphi_1:=u+v\varphi_0$, then $\xi\geq\varphi_1\geq\varphi_0$.*

Proof. Suppose that $\xi\geq\varphi_0$. Then $\xi\geq\varphi_1$ since $\xi=u+v\xi\geq u+v\varphi_0=\varphi_1$.

Recall that $\xi=u+v\xi$, and then $(1-v)\xi=u$, so

$$\varphi_1\geq\varphi_0\Leftrightarrow u+v\varphi_0\geq\varphi_0\Leftrightarrow u\geq\Big(1-v\Big)\varphi_0\Leftrightarrow$$
$$(1-v)\xi\geq\Big(1-v\Big)\varphi_0\Leftrightarrow\xi\geq\varphi_0.$$

Given $\varphi_0>0$ such that $\xi\geq\varphi_0$, define recursively

$$\varphi_n=u+v\varphi_{n-1},\quad \text{for } n\in\mathbb{N}.$$

By Lemma 2.1, $(\varphi_n)_n$ is an increasing sequence of real numbers such that

$$\xi\geq\cdots\geq\varphi_n\geq\cdots\geq\varphi_2\geq\varphi_1\geq\varphi_0.$$

Finally, we note that $(\varphi_n)_n$ converges to ξ.

Lemma 2.2. *Let* φ_0 *be a number such that* $\xi \geq \varphi_0$*. If* $\varphi_n = u + v\varphi_{n-1}$ *for* $n \in \mathbb{N}$*, then* $\lim_{n\to+\infty} \varphi_n = \xi$*, and the convergence is exponential.*

Proof. Note that

$$|\varphi_n - \xi| = |\varphi_n - (u + v\xi)| = |u + v\varphi_{(n-1)} - u - v\xi| = \\ |v\varphi_{(n-1)} - v\xi| = v|\varphi_{(n-1)} - \xi| = \cdots = v^n|\varphi_0 - \xi|.$$

Since $0 < v < 1$ and $\lim_{n\to+\infty} v^n = 0$, we have that $\lim_{n\to+\infty} \varphi_n = \xi$.

In order to conclude the proof of Theorem 2.1, we need to prove that the following conditions are equivalent:

(i) $\xi = \varphi_0$.
(ii) $\xi = \varphi_n$, for some $n \in \mathbb{N}_0$.
(iii) $\xi = \varphi_n$, for all $n \in \mathbb{N}_0$.

Proof. Indeed, let's see first that (i) implies (iii). We proceed by induction. The equality holds for $n = 0$. Assuming that, for some fixed $n \in \mathbb{N}_0$, we have that $\xi = \varphi_n$ it follows that $\varphi_{n+1} = u + v\varphi_n = u + v\xi = \xi$.

Clearly, (iii) implies (ii), so it remains to prove that (ii) implies to (i). Indeed, if $n = 0$, there is nothing to prove. Otherwise, $\xi = \varphi_n$ implies $u + v\xi = u + v\varphi_{n-1}$ and we have $\xi = \varphi_{n-1}$. Repeating this process n times, we get that $\xi = \varphi_0$.

3. Proof of Theorem 1.1

Using that $ab + bc + ca = s^2 + 4Rr + r^2$ and $T = rs$, one has

$$z = a^2 + b^2 + c^2 - [(a - b)^2 + (a - c)^2 + (b - c)^2] = 4r(4R + r). \tag{6}$$

So,

$$z = 16Rr + 4r^2 = 16Rr + \frac{r}{(4R + r)} 4r(4R + r) \tag{7}$$

Then

$$z = 16Rr + \frac{r}{4R + r} z \tag{8}$$

Suppose that $\varphi_0 > 0$ be a number such that $z \geq \varphi_0$. By equation (8), consider $u = 16Rr$ and $v = \frac{r}{4R+r}$, and then $z = u + vz$. Given that $u \neq 0$ and $0 < v < 1$, considering $z = \xi$, Theorem 2.1 implies that if $\varphi_n = u + v\varphi_{(n-1)}$ for $n \in \{1, 2, 3, \cdots\}$, then $(\varphi_n)_n$ is increasing and satisfies $\lim_{n\to+\infty} \varphi_n = \xi$. Moreover, the following conditions are equivalent:

(i) $\xi = \varphi_0$;
(ii) $\xi = \varphi_n$, for some $n \in \mathbb{N}_0$;
(iii) $\xi = \varphi_n$, for all $n \in \mathbb{N}_0$.

Since $\xi = a^2 + b^2 + c^2 - [(a - b)^2 + (a - c)^2 + (b - c)^2]$ we have $(a - b)^2 + (a - c)^2 + (b - c)^2 + \lim_{n\to+\infty} \varphi_n = a^2 + b^2 + c^2$ and the following conditions are equivalent:

(i) $a^2 + b^2 + c^2 = \varphi_0 + (b - a)^2 + (c - a)^2 + (c - b)^2$;

(ii) $a^2 + b^2 + c^2 = \varphi_n + (b-a)^2 + (c-a)^2 + (c-b)^2$, for some $n \in \mathbb{N}_0$;
(iii) $a^2 + b^2 + c^2 = \varphi_n + (b-a)^2 + (c-a)^2 + (c-b)^2$, for all $n \in \mathbb{N}_0$.

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