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CHARACTERIZATIONS OF PARALLELOGRAMS PART 3

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Abstract. We collect, categorize and prove an additional 56 necessary and sufficient conditions for when a convex quadrilateral is a parallelogram.

1. Introduction

In the paper [39] from 1940, Yates wrote about the parallelogram that it is a "rather unpretentious figure" that "has been laid aside to rust". This is an accurate description of how I felt about the parallelogram a few years ago until I read [13] and realized there is more to this figure than first meets the eye. The goal with Yates' paper was "to apply a bit of polish to see if its lustre cannot be brought fourth in true brilliance". That was also one of my intentions with the two part papers [21, 22] about characterizations of parallelograms, which will be referred to as Part 1 and Part 2 respectively. I worked almost a year on them and when they were finished I thought for sure I had collected most of the existing necessary and sufficient conditions for when a convex quadrilateral is a parallelogram. Surpassing one hundred characterizations was a goal I had thought was unattainable, especially since I knew of only a quarter of these a year before that work began.

When writing Part 1 and Part 2 I realized there is nothing unpretentious about the parallelogram, and it became one of my favorite quadrilaterals. Hence it's not so surprising it was hard to stop thinking about new possible sufficient conditions that might characterize parallelograms. After another year has passed, I have now collected an additional 56 characterizations that are the subject of this third part. About half of these were found online as former Olympiad problems or in less known papers or books, but the other half are, as far as I know, new *sufficient* conditions for a convex quadrilateral to be a parallelogram, although some of these are known properties of parallelograms. The total number of characterizations of parallelograms in the three parts is 159.

Keywords and phrases: Convex quadrilateral, parallelogram, angle bisector, bimedian, trapezoid, area, diagonal, converse

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The parallelogram is studied early in geometry courses since its basic necessary and sufficient conditions can be proved with just congruence theorems. However, several other mathematical tools are required when proving many of the more advanced characterizations as was shown in Part 1 and Part 2, like trigonometry, vectors, coordinates, and proof by contradiction to mention a few. We shall see more of this in the present paper, but before we get started, let us formulate one characterization that was used in the proof of Theorem 2.1 *(g)* in Part 2 without being stated as a theorem there. We will apply this necessary and sufficient condition in six proofs throughout this paper.

Theorem 1.1. *A convex quadrilateral $ABCD$ placed in a coordinate system such that $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, and $D = (d, e)$ satisfies $c = e$ and $b = a + d$ if and only if it's a parallelogram.*

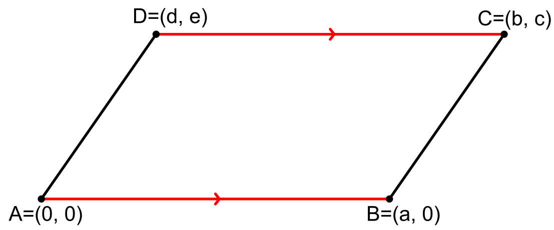

Figure 1. Coordinates of the vertices

Proof. These equalities are equivalent to that a pair of opposite sides are parallel ($c = e$) and have equal length ($b = a + d$), which characterize parallelograms according to Theorem 2.1 *(b)* in Part 1 (see Figure 1). □

2. Angles and triangles

Here we shall prove six characterizations of parallelograms that are about angles or triangles. The first four as well as the last one were discovered by the author. To prove the penultimate necessary and sufficient condition was a shortlisted problem on the 1993 Cono Sur Mathematical Olympiad [30].

Theorem 2.1. *A convex quadrilateral $ABCD$, where M_b, M_c, M_d are the midpoints of the sides BC, CD, DA respectively, satisfies any one of:*

- *(a) it has two pairs of equal opposite exterior angles*
- *(b) triangles DEF and BGH are directly similar, where $F \in DC$ and $G \in BC$, and the line FG intersect the extensions of AD and AB at E and H respectively*
- *(c) triangle BIJ is directly similar to CDJ and AID, where $I \in AB$ and the extension of DI intersects the extension of BC at J*
- *(d) any triangle with its base equal to one of the quadrilateral sides and its third vertex on the opposite side has a constant area*
- *(e) the diagonal intersection is the centroid of triangle AM_bM_c*
- *(f) the centroids of triangles AM_bM_c and BM_cM_d coincide*

if and only if it's a parallelogram.

Proof. *(a)* A convex quadrilateral is a parallelogram if and only if $\angle A = \angle C$ and $\angle B = \angle D$ according to Theorem 4.1 *(a)* in Part 1 [21]. Denoting the exterior angles at A, B, C, D by α, β, γ, δ respectively, we have $\angle A = \pi - \alpha$ and similar relations for the other three vertices. We get that $ABCD$ is a parallelogram if and only if $\pi - \alpha = \pi - \gamma$ and $\pi - \beta = \pi - \delta$, which are equivalent to $\alpha = \gamma$ and $\beta = \delta$.

(b) In a parallelogram $ABCD$, $\angle B = \angle D$ so $\angle GBH = \angle EDF$, and since $DC \parallel AB$, we get $\angle DFE = \angle BHG$ making triangles DEF and BGH directly similar (see Figure 2).

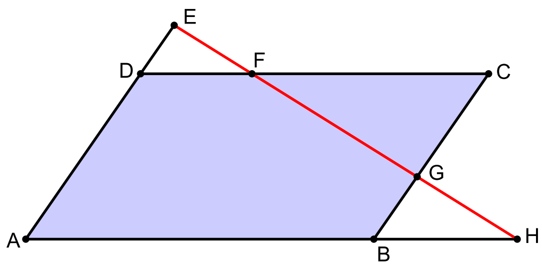

FIGURE 2. Transversal FG

Conversely, when triangles DEF and BGH are directly similar, $\angle BGH = \angle DEF$ implies $BC \parallel AD$ and $\angle BHG = \angle DFE = \angle CFG$ implies $AB \parallel DC$. This proves that $ABCD$ is a parallelogram.

(c) In a parallelogram, the three triangles BIJ, CDJ and AID are similar (AA) due to opposite sides of the quadrilateral being parallel. Conversely, if triangles BIJ and CDJ are similar, then $\angle IBJ = \angle DCJ$, implying that $AB \parallel DC$ (see Figure 3). In the same way, in similar triangles BIJ and AID we have $\angle IBJ = \angle IAD$, so $AD \parallel BC$. Hence $ABCD$ is a parallelogram.

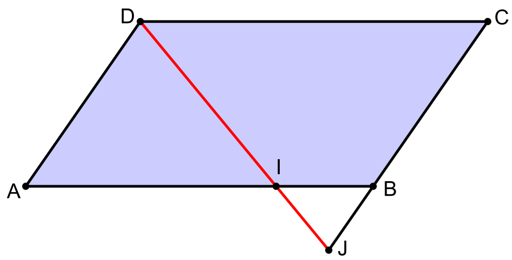

FIGURE 3. Transversal DI

(d) In a parallelogram, any such triangle always has an area equal to one half of the area of the parallelogram, which is a constant.

Conversely, if two such triangles with base AB have the same area, then AB and CD are parallel, since two lines are parallel if and only if they are everywhere equidistant. In the same way BC and DA must be parallel, making $ABCD$ a parallelogram.

(e) Assume the quadrilateral $ABCD$ is placed in a coordinate system such that $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, $D = (d,e)$. Then $M_b = \left(\frac{a+b}{2}, \frac{c}{2}\right)$ and

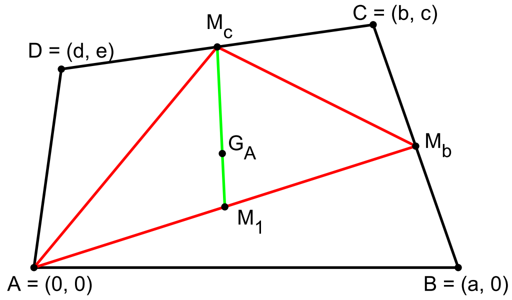

FIGURE 4. The centroid of triangle AM_bM_c

$M_c = \left(\frac{b+d}{2}, \frac{c+e}{2}\right)$. The midpoint M_1 of AM_b has coordinates $M_1 = \left(\frac{a+b}{4}, \frac{c}{4}\right)$, see Figure 4, so the centroid G_A of AM_bM_c lies on M_1M_c such that its coordinates satisfy

$$(1)\qquad G_A = \frac{2}{3}\left(\frac{a+b}{4}, \frac{c}{4}\right) + \frac{1}{3}\left(\frac{b+d}{2}, \frac{c+e}{2}\right) = \left(\frac{a+2b+d}{6}, \frac{2c+e}{6}\right).$$

The line through $(a, 0)$ and (d, e) has the equation

$$y = \frac{e}{d-a}x - \frac{ae}{d-a}$$

and the line through $(0, 0)$ and (b, c) has the equation $y = \frac{c}{b}x$. Equating these lines, we have

$$\frac{e}{d-a}x - \frac{ae}{d-a} = \frac{c}{b}x \quad\Rightarrow\quad x = \frac{abe}{ac+be-cd} \quad\Rightarrow\quad y = \frac{ace}{ac+be-cd}$$

so the diagonals in $ABCD$ intersect at

$$(2)\qquad \left(\frac{abe}{ac+be-cd}, \frac{ace}{ac+be-cd}\right).$$

If this point shall coincide with the centroid of AM_bM_c, it is required that

$$\begin{cases} \dfrac{abe}{ac+be-cd} = \dfrac{a+2b+d}{6} \\ \dfrac{ace}{ac+be-cd} = \dfrac{2c+e}{6} \end{cases}$$

which we rewrite as

$$\frac{a+2b+d}{b} = \frac{6ae}{ac+be-cd} = \frac{2c+e}{c}.$$

Equating the first and third fractions yields $be - cd = ac$ after simplification. From the second and third fractions, we have

$$6ace = (2c+e)(ac+be-cd) = (2c+e)\cdot 2ac \quad\Rightarrow\quad 4ac(e-c) = 0$$

and since $ac \neq 0$, this yields $e = c$ and thus $b = a + d$. These equalities are equivalent to that $ABCD$ is a parallelogram according to Theorem 1.1.

(f) Using the same method as in the previous proof, we get that the centroid of triangle BM_cM_d has coordinates

$$\frac{2}{3}\left(\frac{2a+d}{4}, \frac{e}{4}\right) + \frac{1}{3}\left(\frac{b+d}{2}, \frac{c+e}{2}\right) = \left(\frac{2a+b+2d}{6}, \frac{c+2e}{6}\right).$$

Equating this with (1), we obtain that the two centroids coincide if and only if the following two equalities hold:

$$\begin{cases} 2a+b+2d = a+2b+d \\ c+2e = 2c+e \end{cases} \quad \Leftrightarrow \quad \begin{cases} b = a+d \\ c = e, \end{cases}$$

where the latter two formulas characterize a parallelogram according to Theorem 1.1. □

3. Bisectors and bimedians

Next we have nine necessary and sufficient conditions for when a convex quadrilateral is a parallelogram that concern bisectors or bimedians. A *bimedian* is a line segment that connects the midpoints of opposite sides. The first three characterizations are my own discoveries, while *(d)* was stated in [5, p. 275] but not proved there. Those four conditions are closely related to Theorem 5.1 *(a)*, *(b)* and *(c)* in Part 1. *(e)* and *(f)* are closely related to Theorem 5.1 *(j)* in Part 1 and Theorem 2.1 *(e)* in this paper respectively; they were discovered by the author. The last three conditions are Russian Olympiad problems. Characterization *(g)* was Problem 1 on the 1987 Leningrad Mathematical Olympiad for Grade 7 [11, p. 5] and *(h)* is from a 2014 Mathematical Reggata proposed by A. Shapovalov [2] (we cite a translation of this Russian proof). *(i)* is from the 2006 Southern Tournament Math Fights [31] and was also used as a problem on Round 4 of the 2010 German Mathematical Olympiad [34].

Theorem 3.1. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d and their respective midpoints M_a, M_b, M_c, M_d satisfies any one of:*

- *(a) it has two pairs of opposite parallel exterior angle bisectors*
- *(b) it has one interior angle bisector parallel to two exterior angle bisectors that are adjacent to it*
- *(c) it has two adjacent interior angle bisectors perpendicular to opposite exterior angle bisectors*
- *(d) all four exterior angle bisectors form a rectangle*
- *(e) $M_aM_c = \frac{1}{2}(b+d)$ and $M_bM_d = \frac{1}{2}(a+c)$*
- *(f) the bimedians intersect at the centroid of triangle AM_bM_c*
- *(g) the bimedians divide $ABCD$ into four quadrilaterals of equal perimeter*
- *(h) the interior angle bisectors at A and C are parallel and intersect diagonal BD in two distinct points Q and R respectively such that $BR = DQ$*
- *(i) $AM_b \parallel CM_d$ and $BM_c \parallel DM_a$*

if and only if it's a parallelogram.

Proof. *(a)* A convex quadrilateral is a parallelogram if and only if it has two pairs of parallel opposite interior angle bisectors according to Theorem 5.1 *(a)* in Part 1. An exterior angle bisector is always perpendicular to an interior angle bisector at the same vertex in a convex quadrilateral, so the quadrilateral is a parallelogram if and only if it has two pairs of opposite parallel exterior angle bisectors (since two lines that are perpendicular to two parallel lines are themselves parallel).

(b) A convex quadrilateral is a parallelogram if and only if one interior angle bisector is perpendicular to two adjacent interior angle bisectors according to Theorem 5.1 *(b)* in Part 1. Hence one interior angle bisector parallel to two exterior angle bisectors adjacent to it characterize a parallelogram since an exterior angle bisector is always perpendicular to an interior angle bisector at the same vertex in a convex quadrilateral, and two lines that are both perpendicular to a third line are themselves parallel.

(c) This is a direct consequence of *(a)* and the fact that an exterior angle bisector is always perpendicular to an interior angle bisector at the same vertex in a convex quadrilateral (see Figure 5).

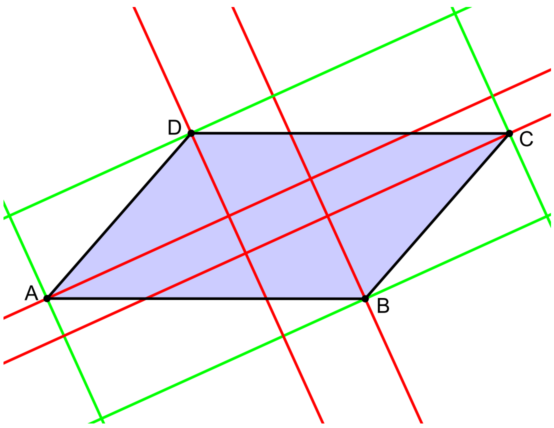

FIGURE 5. The interior and exterior angle bisectors

(d) The angle between the exterior angle bisectors at A and B is given by

$$\pi - \frac{\pi - \angle A}{2} - \frac{\pi - \angle B}{2} = \frac{\angle A + \angle B}{2}$$

and in the same way, the angle between the exterior angle bisectors at B and C is given by $\frac{\angle B+\angle C}{2}$ (see Figure 5). Similar expressions hold at the other two intersections of exterior angle bisectors. A rectangle is characterized by four right vertex angles, so the proof concludes by applying Theorem 4.1 *(b)* in Part 1. (Hence we don't need all four angles between the exterior angle bisectors to be right, but just two adjacent such angles.)

(e) In the proof of Theorem 5.1 *(j)* in Part 1, we proved that $M_aM_c \leq \frac{1}{2}(b+d)$ where equality holds if and only if the angle between the extensions of b and d is zero, and that $M_bM_d \leq \frac{1}{2}(a+c)$ where equality holds if and only if the angle between the extensions of a and c is zero. Hence equality in both hold if and only if the quadrilateral is a parallelogram according to Theorem 4.1 *(d)* in Part 1.

(f) We consider a convex quadrilateral $ABCD$ placed in a coordinate system such that $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, and $D = (d,e)$, so here b, c, d does *not* denote side lengths. Then the bimedians intersect at a point

with coordinates

$$(3)\qquad \frac{1}{2}\left(\frac{a}{2}+\frac{b+d}{2},\frac{c+e}{2}\right)=\left(\frac{a+b+d}{4},\frac{c+e}{4}\right)$$

and the centroid of AM_bM_c has coordinates

$$\left(\frac{a+2b+d}{6},\frac{2c+e}{6}\right)$$

according to (1). Equating these, we get that the bimedians intersect at the centroid of AM_bM_c if and only if

$$\begin{cases}4(a+2b+d)=6(a+b+d)\\4(2c+e)=6(c+e)\end{cases}\quad\Leftrightarrow\quad\begin{cases}b=a+d\\c=e\end{cases}$$

which is equivalent to that $ABCD$ is a parallelogram according to Theorem 1.1.

(g) The bimedians are the diagonals in Varignon's parallelogram, so they bisect each other in all quadrilaterals. The four quadrilaterals they create have equal perimeter if and only if

$$\frac{d}{2}+\frac{a}{2}+\frac{m}{2}+\frac{n}{2}=\frac{a}{2}+\frac{b}{2}+\frac{m}{2}+\frac{n}{2}=\frac{b}{2}+\frac{c}{2}+\frac{m}{2}+\frac{n}{2}=\frac{c}{2}+\frac{d}{2}+\frac{m}{2}+\frac{n}{2}$$

where $m=M_aM_c$ and $n=M_bM_d$, which is equivalent to $a=c$ and $b=d$. The latter equalities characterize parallelograms according to Theorem 2.1 *(a)* in Part 1.

(h) In a parallelogram $ABCD$, triangles ABQ and CDR are congruent (ASA), so $BQ=DR$. Then $\angle AQD=\angle CRB$, proving that AQ and CR are parallel.

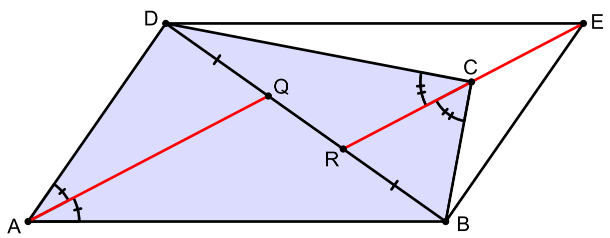

Figure 6. Two interior parallel angle bisectors

Conversely, when AQ and CR are parallel angle bisectors and $BR=DQ$, assume for the sake of contradiction that $AQ>CR$. Extend the segment RC beyond C so that the resulting segment RE is equal to AQ (see Figure 6). Then triangles AQD and ERB are congruent (SAS), implying that $\angle QAD=\angle REB$ and $AD=BE$. In the same way we get that $\angle QAB=\angle RED$ and $AB=DE$. From this we conclude that $ABED$ is a parallelogram. If points C and E do not coincide, then triangles BEC and CED are congruent (ASA). We get that triangle BCD is isosceles and R is the midpoint of side BD. But then point Q must coincide with point R, which contradicts the assumption. Hence $AQ>CR$ is not possible, and in a similar way $AQ<CR$ is not possible either, so we conclude that

$AQ = CR$. This means that points C and E coincide, proving that $ABCD$ is a parallelogram.

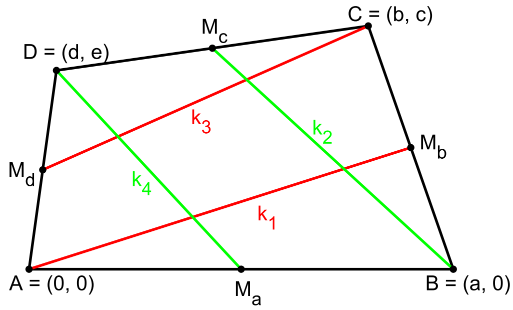

FIGURE 7. Slopes of four medians

(i) We place quadrilateral $ABCD$ in a coordinate system in such a way that $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, and $D = (d, e)$. Then the side midpoints are $M_a = (\frac{a}{2}, 0)$, $M_b = (\frac{a+b}{2}, \frac{c}{2})$, $M_c = (\frac{b+d}{2}, \frac{c+e}{2})$, $M_d = (\frac{d}{2}, \frac{e}{2})$, see Figure 7. The slopes of the lines AM_b and CM_d are given by

$$k_1 = \frac{c}{a+b} \quad \text{and} \quad k_3 = \frac{2c-e}{2b-d}$$

respectively. They are parallel if and only if

$$\frac{c}{a+b} = \frac{2c-e}{2b-d}$$

which is equivalent to $cd = -2ac + ae + be$. The other two lines BM_c and DM_a have slopes

$$k_2 = \frac{c+e}{b+d-2a} \quad \text{and} \quad k_4 = \frac{-2e}{a-2d}$$

respectively, and they are parallel if and only if

$$\frac{c+e}{b+d-2a} = \frac{-2e}{a-2d}$$

which is equivalent to $2be - 3ae + ac = 2cd$. Now solving the system of equations

$$\begin{cases} cd = -2ac + ae + be \\ 2be - 3ae + ac = 2cd \end{cases}$$

we obtain, by inserting the first equation into the second and simplifying, that $5a(-c + e) = 0$. Thus $c = e$ since $a \neq 0$. Inserting this into the first equation yields $c(d + a - b) = 0$, and with $c \neq 0$ we have $b = a + d$. The equalities $c = e$ and $b = a + d$ characterize a parallelogram according to Theorem 1.1. □

Theorem 5.1 *(i)* in Part 1 states that the diagonals and the bimedians in a convex quadrilateral are concurrent if and only if the quadrilateral is a parallelogram, which we proved with geometrical methods. We can now get a coordinate proof of this characterization by using (2) and (3). Equating the abscissa and the ordinate for the intersection of the two diagonals and the two bimedians yields

$$\begin{cases}(a+b+d)(ac+be-cd)=4abe\\(c+e)(ac+be-cd)=4ace.\end{cases}$$

We let the interested reader solve this system of equations and thus complete this alternative proof.

4. Trapezoids

In the following theorem, we study twelve necessary and sufficient conditions for when a trapezoid is a parallelogram. This is possible when using inclusive definitions so that a parallelogram is a special case of a trapezoid, as advocated in [16, p. 23], which is the preferred way in higher mathematics. The first two characterizations are from [37, p. 46]. The necessary condition in *(d)* was a longlisted problem proposed by USA to the International Mathematical Olympiad in 1977 [8, p. 120], the sufficient condition in *(e)* is from the 2007 Macedonian Mathematical Olympiad [27], *(h)* is a reformulation of Problem 3 (version for junior students) from the Final round of the 2015 Dutch Mathematical Olympiad [9, p. 330], the sufficient condition in *(k)* is from the 1991 Savin Competition [29] and our proof is based on a translation from the original Russian at [25], and the sufficient condition in *(l)* is from the 2004 Regional All-Russian Mathematical Olympiad Round 4, proposed by N. Agakhanov [32]. The characterizations *(i)* and *(j)* are from the book [35, p. 184]. The rest were discovered by the author.

Theorem 4.1. *A convex trapezoid $ABCD$ with consecutive sides a, b, c, d, their respective midpoints M_a, M_b, M_c, M_d, and diagonal intersection P satisfies any one of:*

- *(a)* $AB \parallel CD$ *and* $\angle DAC = \angle ACB$
- *(b)* $AB \parallel CD$ *and* $\angle C + \angle D = \pi$
- *(c)* $AB \parallel CD$ *and* E_1G_1, F_1H_1, BD *are concurrent, where* $E_1 \in DA$, $F_1 \in BC$, $G_1 \in AB$, $H_1 \in CD$, $E_1F_1 \parallel AB$ *and* $G_1H_1 \parallel BC$
- *(d)* $AB \parallel CD$ *and* BE_2, DG_2, CI *are concurrent, where* $E_2 \in DA$, $F_2 \in BC$, $G_2 \in AB$, $H_2 \in CD$, $E_2F_2 \parallel AB$, $GH_2 \parallel BC$ *and* $I = E_2F_2 \cap G_2H_2$
- *(e)* $AB \parallel CD$ *and* AK, CL, *and the line* BO *are concurrent, where* $K \in CD$ *and* $L \in AD$ *such that* $AK \perp CD$ *and* $CL \perp AD$, *and* O *is the circumcenter of triangle* ABC
- *(f)* $AB \parallel CD$ *and* $\frac{AM}{MB} = \frac{CN}{ND}$, *where* $M \in AB$ *and* $N \in CD$ *are points such that* MN *bisects the area of the trapezoid and* $MN \neq M_aM_c$
- *(g)* $AB \parallel CD \parallel F_3H_3$ *and the area of* $E_3F_3G_3H_3$ *is half the area of* $ABCD$, *where* $E_3 \in AB$, $F_3 \in BC$, $G_3 \in CD$, $H_3 \in DA$ *and* $F_3H_3 \neq M_bM_d$
- *(h)* $AB \parallel CD$ *and* $F_4G_4 \parallel E_4H_4$, *where* E_4, F_4, G_4, H_4 *are the intersections of the angle bisectors at* A *and* B, B *and* C, C *and* D, D *and* A *respectively*
- *(i)* $AB \parallel CD$ *and* P, Q, R *are collinear, where* $Q \in AB$ *and* $R \in CD$ *such that* $PR \parallel AD$ *and* $PQ \parallel BC$

(j) *$AB \parallel CD$ and $QR = \frac{ad+bc}{a+c}$, where $Q \in AB$ and $R \in CD$ such that $PR \parallel AD$ and $PQ \parallel BC$*

(k) *$AB \parallel CD$, $B'C' \parallel D'A'$ and $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, where $ABCD$ and $A'B'C'D'$ are two different trapezoids*

(l) *$AB \parallel CD$ and M_bM_d bisects O_1O_2, where O_1 and O_2 are the circumcenters of triangles ABD and BCD respectively, and they do not lie on the bimedian M_bM_d*

if and only if it's a parallelogram.

Proof. *(a)* $\angle DAC = \angle ACB$ is equivalent to $AD \parallel BC$.

(b) $\angle C + \angle D = \pi$ is equivalent to $AD \parallel BC$.

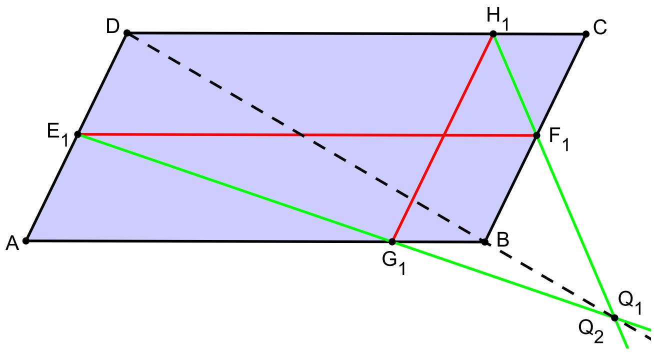

FIGURE 8. Transversals E_1G_1 and H_1F_1

(c) In a parallelogram $ABCD$, we apply Menelaus' theorem (with non-directed distances) in triangle ABD with transversal E_1G_1 (see Figure 8) to get

$$\frac{AG_1}{G_1B} \cdot \frac{BQ_1}{Q_1D} \cdot \frac{DE_1}{E_1A} = 1 \tag{4}$$

where Q_1 is the intersection between E_1G_1 and DB. In the same way in triangle CBD with transversal H_1F_1, we get

$$\frac{CF_1}{F_1B} \cdot \frac{BQ_2}{Q_2D} \cdot \frac{DH_1}{H_1C} = 1 \tag{5}$$

where Q_2 is the intersection between H_1F_1 and DB. It holds that $AG_1 = DH_1$ and $G_1B = H_1C$ and also $CF_1 = DE_1$ and $E_1A = F_1B$, so from (4) and (5), we obtain

$$\frac{BQ_1}{Q_1D} = \frac{BQ_2}{Q_2D}.$$

This proves that $Q_1 \equiv Q_2$ and we conclude that E_1G_1, H_1F_1, and DB are concurrent in a parallelogram.

Conversely, if E_1G_1, H_1F_1, and DB are concurrent in a trapezoid $ABCD$ where $AB \parallel CD$, from the two applications of Menelaus' theorem, we have

$$\frac{AG_1}{G_1B} \cdot \frac{BQ_1}{Q_1D} \cdot \frac{DE_1}{E_1A} = \frac{CF_1}{F_1B} \cdot \frac{BQ_2}{Q_2D} \cdot \frac{DH_1}{H_1C}$$

which, with the help of $\frac{DE_1}{E_1A} = \frac{CF_1}{F_1B}$ that holds in a trapezoid, simplifies into $AG_1 = DH_1$. This in turn implies $AB = DC$, and since also $AB \parallel DC$, we get that $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in Part 1.

(d) In a parallelogram $ABCD$, where $J = BE_2 \cap DG_2$, we apply Menelaus' theorem in triangle DAG_2 with transversal BE_2 (see Figure 9) to get

$$1 = \frac{DE_2}{E_2A} \cdot \frac{AB}{BG_2} \cdot \frac{G_2J}{JD} = \frac{H_2I}{IG_2} \cdot \frac{DC}{CH_2} \cdot \frac{G_2J}{JD}$$

where the second equality follows from properties due to $E_2F_2 \parallel AB$, $G_2H_2 \parallel BC$, and $I = E_2F_2 \cap G_2H_2$. According to the converse to Menelaus' theorem applied in triangle DG_2H_2 with transversal CJ, we conclude that points C, I, J are collinear, proving that BE_2, DG_2, and CI are concurrent at J.

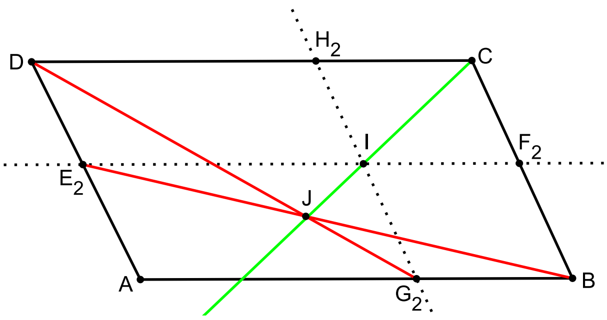

FIGURE 9. Five transversals

Conversely, suppose that BE_2, DG_2, and CI are concurrent at a point J in a trapezoid $ABCD$ with $AB \parallel CD$. This implies that C, I, J are collinear, so by Menelaus' theorem, we get

$$\frac{H_2I}{IG_2} \cdot \frac{G_2J}{JD} \cdot \frac{DC}{CH_2} = 1.$$

Applying Menelaus' theorem in triangle DAG_2 with transversal BE_2 yields

$$\frac{DE_2}{E_2A} \cdot \frac{AB}{BG_2} \cdot \frac{G_2J}{JD} = 1$$

and we get, by equating the previous two left-hand sides, that

$$\frac{H_2I}{IG_2} \cdot \frac{DC}{CH_2} = \frac{DE_2}{E_2A} \cdot \frac{AB}{BG_2}$$

which via $CH_2 = BG_2$ and $\frac{H_2I}{IG_2} = \frac{DE_2}{E_2A}$ (the latter is the basic proportionality theorem in a trapezoid) reduces to $AB = DC$. Together with $AB \parallel CD$ this concludes the proof that $ABCD$ is a parallelogram.

(e) In a parallelogram $ABCD$ where these assumptions hold and $M = AK \cap CL$, we directly get that $ABCM$ is a cyclic quadrilateral with two opposite right angles at A and C, so M, O, B are collinear points (see Figure 10), proving that AK, CL, and the line BO are concurrent at M.

Conversely, if AK, CL, and the line BO are concurrent at a point M in a trapezoid with $AB \parallel CD$, we have that $ABCM$ is a cyclic quadrilateral with two opposite right angles at A and C, since $\angle MAB = 90°$, $O \in MB$, and $OB = OA$, so $OB = OA = OM$. Now $\angle MCB = 90°$, so $BC \perp CL$, and

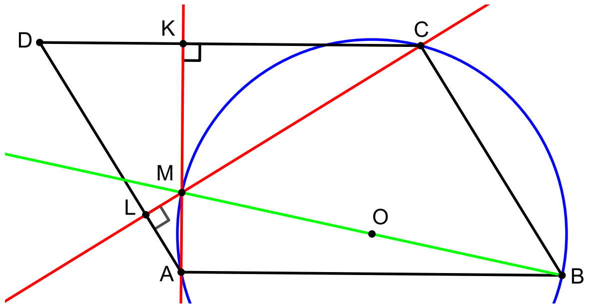

Figure 10. Circumcircle to triangle ABC

we get that $AD \parallel BC$ since $CL \perp AD$ by assumption. Hence $ABCD$ is a parallelogram due to two pairs of opposite parallel sides.

(f) In a parallelogram $ABCD$, any line that bisects the area goes through the diagonal intersection (Theorem 2.1 *(d)* in Part 2), and then it is trivial that $\frac{AM}{MB} = \frac{CN}{ND}$ holds since $AM = CN$ and $MB = ND$.

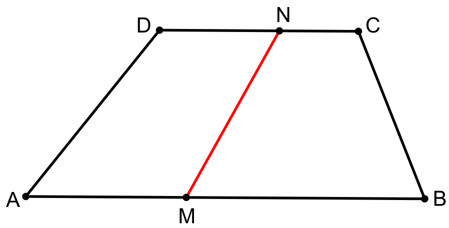

Figure 11. An area bisecting transversal

In a trapezoid where $\frac{AM}{MB} = \frac{CN}{ND}$ holds, the area bisection yields (see Figure 11)

$$\frac{AM \cdot h}{2} + \frac{DN \cdot h}{2} = \frac{BM \cdot h}{2} + \frac{CN \cdot h}{2},$$

where h is the distance between AB and CD, so we get $AM + DN = BM + CN$. Inserting $AM = \frac{BM \cdot CN}{DN}$ and simplifying yields

$$(CN - DN)(BM - DN) = 0.$$

Here we cannot have $CN = DN$, since that would imply $AM = BM$ and we assumed that $MN \neq M_a M_c$. Hence we get $BM = DN$, which implies $AM = CN$, and so

$$AB = AM + MB = CN + DN = CD$$

which together with $AB \parallel CD$ proves that $ABCD$ is a parallelogram.

(g) This is such a simple conclusion that a PWW will suffice (see Figure 12). We let the reader write the words if they are needed.

(h) In a parallelogram, both pairs of opposite angle bisectors are parallel by Theorem 5.1 *(a)* in Part 1. Conversely, let J be the intersection of the line CG_4 and AB (see Figure 13). Then

$$\tfrac{1}{2}\angle BAD = \angle E_4AB = \angle CJB = \angle JCD = \tfrac{1}{2}\angle DCB$$

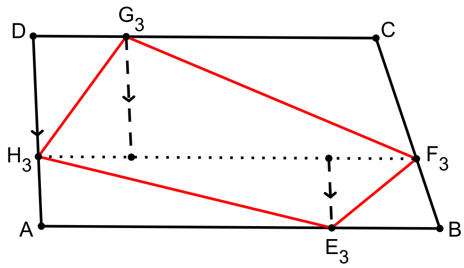

FIGURE 12. A proof without words

by corresponding angles and alternate interior angles. Hence the trapezoid $ABCD$ has a pair of opposite equal angles, so it is a parallelogram according to Theorem 6.1 *(b)* in Part 1.

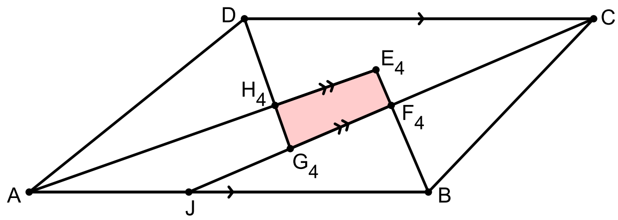

FIGURE 13. Two pairs of opposite parallel line segments

(i) In triangle PQR it holds that P, Q, R are collinear if and only if $PQ \parallel PR$ (see Figure 14), which is equivalent to $AD \parallel BC$ since it was given that $PR \parallel AD$ and $PQ \parallel BC$. Trapezoid $ABCD$ was defined by $AB \parallel CD$, so it is a parallelogram if and only if $AD \parallel BC$, which is thus equivalent to that P, Q, R are collinear.

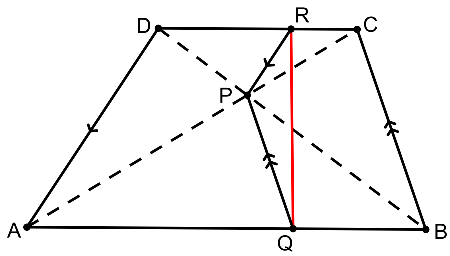

FIGURE 14. Two pairs of parallel line segments

(j) In triangles ABC and ADC (see Figure 14), we get

$$\frac{PQ}{BC} = \frac{AP}{AC} \quad \text{and} \quad \frac{PR}{AD} = \frac{PC}{AC}$$

due to $PQ \parallel BC$ and $PR \parallel AD$. Triangles ABP and CDP are similar (AA), so

$$\frac{AP}{PC} = \frac{AB}{CD} \quad \Leftrightarrow \quad \frac{AP+PC}{PC} = \frac{AB+CD}{CD} \quad \Leftrightarrow \quad \frac{AC \cdot CD}{PC} = AB + CD.$$

Combining these equalities, we get that

$$PQ = \frac{BC \cdot AP}{AC} = \frac{BC}{AC} \cdot \frac{PC \cdot AB}{CD} = \frac{BC \cdot AB}{AB + CD}$$

and in a similar way,

$$PR = \frac{CD \cdot DA}{AB + CD}.$$

Applying the triangle inequality in triangle PQR yields

$$QR \le PQ + PR = \frac{AB \cdot BC + CD \cdot DA}{AB + CD}$$

where equality holds if and only if P, Q, R are collinear, that is, if and only if $ABCD$ is a parallelogram according to *(i)*.

(k) It is trivial that these assumptions are satisfied in a parallelogram $ABCD$, so let us consider the converse. In [16, p. 34] we proved (as part of a larger discussion on trapezoids) that in a trapezoid, *the sum of the longest base and any of the legs is greater than the sum of the shorter base and the other leg.* Thus there are two different inequalities here.

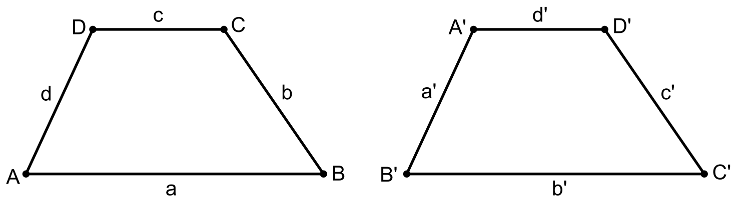

FIGURE 15. The two trapezoids

The assumptions in the sufficient condition are $a \parallel c$, $b' \parallel d'$, $a = a'$, $b = b'$, $c = c'$, and $d = d'$. We also assume first that $a > c$ and $b' > d'$ (see Figure 15). From the quoted property of trapezoids, we get in trapezoid $ABCD$:

$$a + b > c + d \quad \text{and} \quad a + d > b + c$$

and in trapezoid $A'B'C'D'$:

$$b' + a' > c' + d' \quad \text{and} \quad b' + c' > a' + d'$$

which, since $a = a'$, $b = b'$, $c = c'$, and $d = d'$, implies

$$b + a > c + d \quad \text{and} \quad b + c > a + d.$$

Here we have reached a contradiction, that we have both $a + d > b + c$ and $a + d < b + c$, so the assumption that $a > c$ must be wrong. A similar study of the assumption $a < c$ will also lead to a contradiction, so we conclude that we must have $a = c$. Then we have both $a \parallel c$ and $a = c$, proving that $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in Part 1.

Finally we note that there is another case where we assume $b' < d'$ instead, but it leads to the similar contradiction $a + b > c + d$ and $a + b < c + d$.

(l) In a parallelogram $ABCD$, triangles ABD and CDB are congruent, so $O_1P = O_2P$ where P is the midpoint of diagonal BD. The diagonals and the bimedians are concurrent at P according to Theorem 5.1 *(i)* in Part 1,

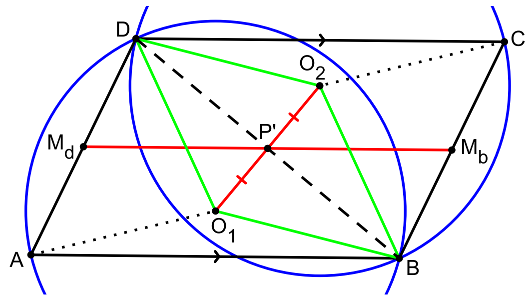

FIGURE 16. Two circumcircles

and since the diagonals bisect each other at P, we get that M_bM_d bisects O_1O_2 at P.

Conversely, suppose M_bM_d bisects O_1O_2 in a trapezoid $ABCD$. It is well-known that the bimedian M_bM_d passes through the midpoint P' of diagonal BD when $AB \parallel CD$ (see Figure 16). Since the center of the circumcircle to a triangle lies on the perpendicular bisectors of its sides, we have that $O_1P' \perp BD$ and $O_2P' \perp BD$. Then segment O_1O_2 passes through $P' \in M_bM_d$, so P' bisects segment O_1O_2. Therefore the diagonals of the quadrilateral BO_1DO_2 are bisected by point P' and are perpendicular. This proves that BO_1DO_2 is a rhombus. Hence $O_1B \parallel O_2D$ and so $\angle O_1BA = \angle O_2DC$. We get that triangles O_1BA and O_2DC are congruent (they are isosceles with equal lateral sides and base angles), and therefore $AB = CD$. Since AB and CD are both parallel and have equal length, $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in Part 1. □

5. Two-dimensional metric relations

In this section we shall prove six necessary and sufficient conditions for when a convex quadrilateral is a parallelogram that are expressed as two-dimensional metric relations or equalities between quotas of lengths. Condition *(b)* is a special case of an inequality for n-sided polygons that was proved as Problem 3.2.10 in [4, pp. 97, 208–210], *(c)* is stated as a special case of an inequality proved in [7, p. 102], and *(e)* was proposed by Virgil Nicula at [28]. The remaining three were discovered by the author

We denote by T_{XYZ} the area of triangle XYZ.

Theorem 5.1. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d and their respective midpoints M_a, M_b, M_c, M_d satisfies any one of:*

(a) $a^2 + b^2 = c^2 + d^2$ *and* $d^2 + a^2 = b^2 + c^2$
(b) $(a + c)^2 + (b + d)^2 = 4(m^2 + n^2)$, *where m, n are the bimedians*
(c) $w^2 + x^2 + y^2 + z^2 = 2(a^2 + b^2 + c^2 + d^2)$, *where w, x, y, z are the distances between neighboring free vertices of external squares erected on the sides*
(d) $\frac{AF}{FB} = \frac{CG}{GD}$ *and* $\frac{BH}{HC} = \frac{DI}{IA}$, *where $F \in AB$, $G \in CD$, $H \in BC$, $I \in DA$ such that FG, HI, AC, BD are concurrent*
(e) $\frac{JA}{JD} = \frac{MC}{MN}$ *and* $\frac{LB}{LC} = \frac{ND}{MN}$, *where $J \in AD$, $L \in BC$, $O \in CJ \cap DL$, $M \in AO \cap CD$ and $N \in BO \cap CD$*

(f) $T_{ABM_c} = T_{CDM_a}$ *and* $T_{BCM_d} = T_{DAM_b}$
if and only if it's a parallelogram.

Proof. *(a)* We solve the system of equations

$$\begin{cases} a^2+b^2=c^2+d^2 \\ d^2+a^2=b^2+c^2 \end{cases} \Leftrightarrow \begin{cases} 2a^2+b^2+d^2=b^2+2c^2+d^2 \\ b^2-d^2=d^2-b^2 \end{cases} \Leftrightarrow \begin{cases} a=c \\ b=d \end{cases}$$

which proves that the quadrilateral is a parallelogram if and only if these two equations are satisfied according to Theorem 2.1 *(a)* in Part 1.

(b) In the proof of Theorem 8.1 *(c)* in Part 1 we noted that in a convex quadrilateral, it holds that

$$2(p^2+q^2) = a^2+c^2+b^2+d^2+2ac\cos\xi+2bd\cos\psi$$

where p and q are the diagonal lengths, and ξ and ψ are the angles between the extensions of a, c and b, d respectively. Applying the Parallelogram Law in Varignon's parallelogram shows that $p^2+q^2 = 2(m^2+n^2)$ also holds in all convex quadrilaterals. By combining these two equalities, we get

$$\begin{aligned} 4(m^2+n^2) &= a^2+c^2+b^2+d^2+2ac\cos\xi+2bd\cos\psi \\ &= (a+c)^2+(b+d)^2-4ac\cdot\sin^2\left(\frac{\xi}{2}\right)-4bd\cdot\sin^2\left(\frac{\psi}{2}\right) \\ &\le (a+c)^2+(b+d)^2 \end{aligned}$$

where equality holds if and only if $\xi=\psi=0$, which according to Theorem 4.1 *(d)* in Part 1 is equivalent to that the quadrilateral is a parallelogram.

(c) Applying the Law of Cosines in the two triangles with sides a, b, p and a, b, w yields $p^2=a^2+b^2-2ab\cos B$ and $w^2=a^2+b^2-2ab\cos(\pi-B) = a^2+b^2+2ab\cos B$ (see Figure 17). Adding these, we get

$$p^2+w^2 = 2(a^2+b^2). \tag{6}$$

In the same way we obtain from other pairs of neighboring triangles that

$$p^2+y^2=2(c^2+d^2), \quad q^2+x^2=2(b^2+c^2), \quad q^2+z^2=2(a^2+d^2). \tag{7}$$

Adding the four equalities (6) and (7) yields

$$2(p^2+q^2)+(w^2+x^2+y^2+z^2) = 4(a^2+b^2+c^2+d^2).$$

Next we apply Euler's quadrilateral theorem (for a proof, see [3, pp. 9–10]), which states that

$$a^2+b^2+c^2+d^2 = p^2+q^2+4v^2$$

where v is the distance between the diagonal midpoints, to eliminate the sum of the squared diagonals. We get

$$w^2+x^2+y^2+z^2 = 2(a^2+b^2+c^2+d^2)+8v^2 \ge 2(a^2+b^2+c^2+d^2)$$

where equality holds if and only if $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in Part 1. This proves that the area of the four outer squares is at least twice the area of the four inner squares and that equality holds if and only if the original quadrilateral is a parallelogram.

(d) In a parallelogram $ABCD$, we have $AP=CP$, $\angle APF = \angle CPG$, and $FP=PG$ (see Theorem 2.1 *(d)* in Part 2), so triangles APF and CPG

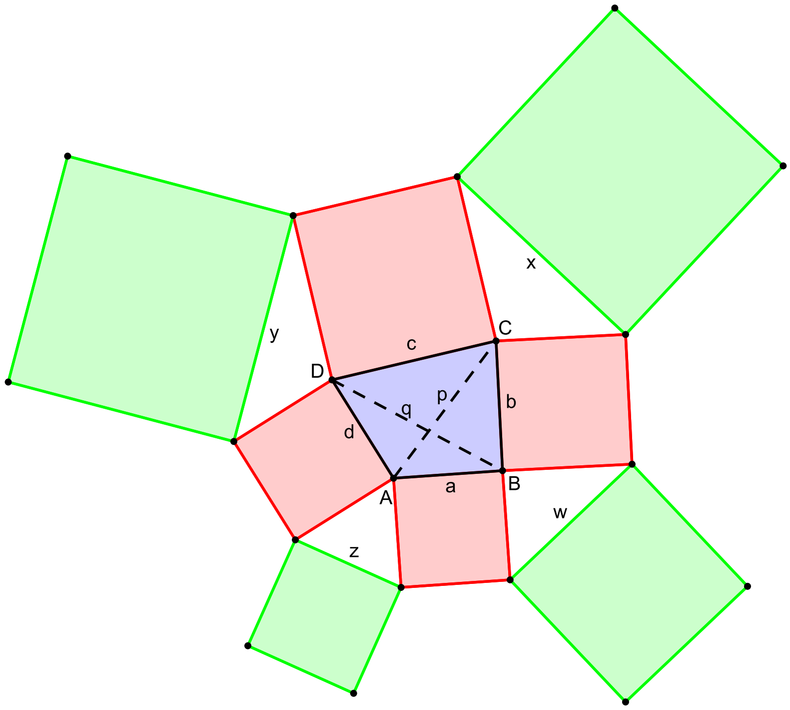

Figure 17. Eight external squares

are congruent (SAS), implying that $AF = CG$ and hence also $BF = DG$. Then $\frac{AF}{FB} = \frac{CG}{GD}$ and in the same way the second equality is satisfied.

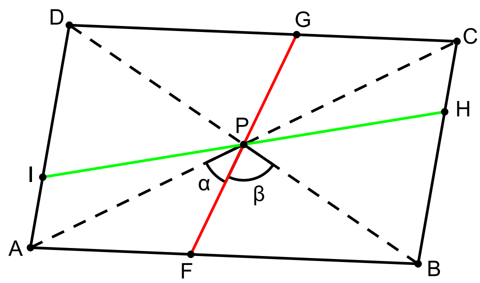

Figure 18. The transversals FG and HI

Conversely, in a convex quadrilateral, we get that $\frac{AF}{FB} = \frac{CG}{GD}$ implies

$$\frac{T_{AFP}}{T_{FBP}} = \frac{T_{CGP}}{T_{GDP}} \quad \Rightarrow \quad \frac{\frac{1}{2}AP \cdot FP \sin\alpha}{\frac{1}{2}FP \cdot BP \sin\beta} = \frac{\frac{1}{2}CP \cdot GP \sin\alpha}{\frac{1}{2}GP \cdot DP \sin\beta},$$

where α is the angle between AC and FG, and β is the angle between BD and FG (see Figure 18). This simplifies into

$$\frac{AP}{CP} = \frac{BP}{DP}$$

and together with $\angle APB = \angle CPD$, we get that triangles ABP and CDP are similar (SAS), so $\angle PAB = \angle PCD$. Hence $AB \parallel CD$. In the same way we prove that $\frac{BH}{HC} = \frac{DI}{IA}$ implies $BC \parallel DA$, confirming that $ABCD$ is a parallelogram.

(e) In a parallelogram $ABCD$, we apply Menelaus' theorem (with non-directed distances) in triangle ADM with transversal CJ to get

$$\frac{AJ}{JD} \cdot \frac{DC}{CM} \cdot \frac{MO}{OA} = 1. \tag{8}$$

Triangles ABO and MNO are similar (AA), so

$$\frac{OA}{MO} = \frac{AB}{MN}$$

and we can rewrite (8) as

$$\frac{JA}{JD} = MC \cdot \frac{AB}{MN} \cdot \frac{1}{DC} = \frac{MC}{MN}$$

since $AB = DC$ in a parallelogram. In the same way we prove that

$$\frac{LB}{LC} = \frac{ND}{NM}.$$

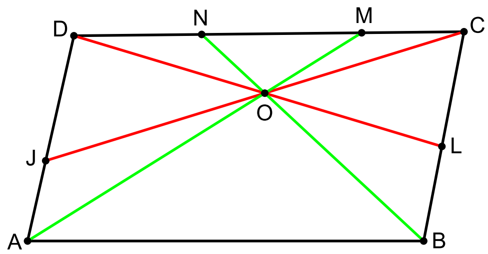

Figure 19. Four concurrent transversals

To prove the converse, we rewrite (8) as

$$\frac{AJ}{JD} = \frac{CM}{DC} \cdot \frac{OA}{MO} = \frac{MC}{MN}$$

where the last equality is due to one of the assumptions. We get

$$\frac{OA}{MO} = \frac{DC}{MN}. \tag{9}$$

A second application of Menelaus' theorem, this time in triangle BCN with transversal DL (see Figure 19), yields

$$\frac{BL}{LC} \cdot \frac{CD}{DN} \cdot \frac{NO}{OB} = 1 \quad \Rightarrow \quad \frac{BL}{LC} = \frac{DN}{CD} \cdot \frac{OB}{NO}$$

and with the help of the second assumption, this rewrites as

$$\frac{ND}{NM} = \frac{DN}{CD} \cdot \frac{OB}{NO} \quad \Rightarrow \quad \frac{OB}{NO} = \frac{CD}{NM},$$

and by (9), as

$$\frac{OA}{MO} = \frac{OB}{NO}.$$

Together with $\angle AOB = \angle MON$, this implies that triangles AOB and MON are similar, so $AB \parallel MN \parallel CD$, and

$$\frac{AO}{MO} = \frac{AB}{MN}.$$

Comparing with (9), we conclude that $AB = DC$ also holds, proving that $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in Part 1.

(f) Assume the quadrilateral is placed in a coordinate system such that $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, $D = (d,e)$, see Figure 7, so here b, c, d does *not* denote side lengths. Applying the formula for the area of a triangle in terms of the coordinates of its vertices, we get that $2T_{ABM_c} = 2T_{CDM_a}$ is equivalent to

$$\begin{aligned} &0\cdot 0 + a\cdot\frac{c+e}{2} + \frac{b+d}{2}\cdot 0 - 0\cdot\frac{c+e}{2} - a\cdot 0 - \frac{b+d}{2}\cdot 0 \\ =&\frac{a}{2}\cdot c + b\cdot e + d\cdot 0 - \frac{a}{2}\cdot e - b\cdot 0 - d\cdot c \end{aligned}$$

which simplifies into $2ae = 2be - 2cd$, that is,

$$cd = e(b-a) \quad\Leftrightarrow\quad \frac{c}{b-a} = \frac{e}{d} \quad\Leftrightarrow\quad k_1 = k_2$$

where k_1 and k_2 are the slopes of the lines BC and AD respectively. This proves that $T_{ABM_c} = T_{CDM_a}$ is equivalent to $BC \parallel AD$.

In the same way we prove that $T_{BCM_d} = T_{DAM_b}$ is equivalent to $AB \parallel CD$. Hence the two area equalities together characterize parallelograms. □

6. Area

Next we prove six different formulas for the area of a quadrilateral that are characterizations of parallelograms. Condition *(a)* was proved earlier in [15] and *(f)* is the equality case of an inequality that was proposed as Problem 10560 in [1] (we reproduce this proof). The remaining four were discovered by the author.

Theorem 6.1. *In a convex quadrilateral $ABCD$ with consecutive sides a, b, c, d, diagonal intersection P, and semidiagonals $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$, let T_a, T_b, T_c, T_d be the areas of the quarter-triangles ABP, BCP, CDP, DAP respectively, ϕ be one of the angles between the bimedians, and θ be one of the angles between the diagonals. The area K of $ABCD$ satisfies any one of:*

- *(a)* $K = \frac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}\sin\phi$
- *(b)* $K = 2\sqrt{a'b'c'd'}\sin\theta$
- *(c)* $K = 4\sqrt{T_aT_c}$ *or* $K = 4\sqrt{T_bT_d}$
- *(d)* $K = 4\sqrt[4]{T_aT_bT_cT_d}$
- *(e)* $K = \frac{1}{4}\left(\sqrt{T_a}+\sqrt{T_b}+\sqrt{T_c}+\sqrt{T_d}\right)^2$
- *(f)* $K = 8\sqrt[4]{T_AT_BT_CT_D}$, *where T_A, T_B, T_C, T_D are the areas of triangles SAU, UBQ, QCR, RDS respectively and $U \in AB$, $Q \in BC$, $R \in CD$, $S \in DA$*

if and only if it's a parallelogram.

Proof. *(a)* It was proved in [15, p. 20] that the area of a convex quadrilateral is equal to

$$K = \tfrac{1}{4}\sqrt{(2(a^2+c^2)-4v^2)(2(b^2+d^2)-4v^2)}\sin\phi$$

where v is the distance between the diagonal midpoints. We get that

$$K \le \tfrac{1}{4}\sqrt{2(a^2+c^2)\cdot 2(b^2+d^2)}\sin\phi$$

where equality holds if and only if the diagonals bisect each other ($v = 0$), which characterize parallelograms according to Theorem 3.1 *(a)* in Part 1. Simplification leads to the desired formula.

(b) The area of a convex quadrilateral with diagonals p and q is given by

$$\begin{aligned} K &= \tfrac{1}{2}pq\sin\theta = \tfrac{1}{2}(a'+c')(b'+d')\sin\theta \\ &\ge \tfrac{1}{2}\cdot 2\sqrt{a'c'}\cdot 2\sqrt{b'd'}\sin\theta = 2\sqrt{a'b'c'd'}\sin\theta \end{aligned}$$

where equality holds if and only if $a' = c'$ and $b' = d'$ according to the AM-GM inequality, that is, only when the diagonals bisect each other, which is a well-known characterization of parallelograms.

(c) The area of a convex quadrilateral is given by

$$K = T_a + T_c + T_b + T_d \ge 2\sqrt{T_aT_c} + 2\sqrt{T_bT_d}$$

where equality holds if and only if $T_a = T_c$ and $T_b = T_d$ according to the AM-GM inequality. These two equalities characterize a parallelogram according to Theorem 8.1 *(h)* in Part 1. The proof concludes by applying the quite well-known fact that the quarter-triangle areas satisfy $T_aT_c = T_bT_d$ (proved for instance in [16, pp. 27–28]), which holds in all convex quadrilaterals.

(d) Using the inequalities $K \ge 4\sqrt{T_aT_c}$ and $K \ge 4\sqrt{T_bT_d}$ from the previous characterization, we get that the area of a convex quadrilateral satisfies

$$K^2 \ge 16\sqrt{T_aT_bT_cT_d}$$

where equality holds if and only if $T_a = T_c$ and $T_b = T_d$, that is, only when it is a parallelogram according to Theorem 8.1 *(h)* in Part 1.

(e) For the area of a convex quadrilateral, we have

$$K = T_a + T_b + T_c + T_d = T_a + T_c + 2\sqrt{T_aT_c} - 2\sqrt{T_bT_d} + T_b + T_d$$

since $T_aT_c = T_bT_d$ holds in all convex quadrilaterals, so we get

$$K = (\sqrt{T_a} + \sqrt{T_c})^2 + (\sqrt{T_b} - \sqrt{T_d})^2.$$

In the same way we derive

$$K = (\sqrt{T_a} - \sqrt{T_c})^2 + (\sqrt{T_b} + \sqrt{T_d})^2.$$

Taking the square root of the last two formulas and adding them yields

$$\begin{aligned} 2\sqrt{K} &= \sqrt{(\sqrt{T_a} + \sqrt{T_c})^2 + (\sqrt{T_b} - \sqrt{T_d})^2} \\ &\quad + \sqrt{(\sqrt{T_a} - \sqrt{T_c})^2 + (\sqrt{T_b} + \sqrt{T_d})^2} \\ &\ge (\sqrt{T_a} + \sqrt{T_c}) + (\sqrt{T_b} + \sqrt{T_d}) \end{aligned}$$

where equality holds if and only if $T_b = T_d$ and $T_a = T_c$, which is equivalent to the quadrilateral being a parallelogram according to Theorem 8.1 *(h)* in Part 1. Now the desired formula follows by solving for K.

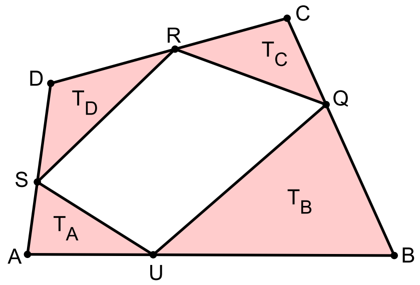

FIGURE 20. The areas T_A, T_B, T_C, T_D

(f) Let U, Q, R, S divide their respective sides in the ratios $\kappa : (1-\kappa)$, $\lambda : (1-\lambda)$, $\mu : (1-\mu)$, $\nu : (1-\nu)$, where $0 < \kappa, \lambda, \mu, \nu < 1$. Also, let F_A, F_B, F_C, F_D denote the areas of triangles DAB, ABC, BCD, CDA respectively. Then we trivially have $F_A + F_C = K$ and $F_B + F_D = K$. Triangles SAU and DAU have collinear bases and the same height from U, so the ratio of their areas is $AS/AD = 1 - \nu$ (see Figure 20). Similarly, the ratio of the areas of triangles DAU and DAB is $AU/AB = \kappa$. It follows that $T_A = F_A(1-\nu)\kappa$, and by symmetry, we also have $T_B = F_B(1-\kappa)\lambda$, $T_C = F_C(1-\lambda)\mu$, and $T_D = F_D(1-\mu)\nu$. Hence we get

$$T_A T_B T_C T_D = [(1-\kappa)\kappa][(1-\lambda)\lambda][(1-\mu)\mu][(1-\nu)\nu][F_A F_C][F_B F_D].$$

For each of the six products in the square brackets, the sum of its two factors is constant, either equal to 1 or K. Applying the AM-GM inequality in the form $xy \leq ((x+y)/2)^2$ yields

$$T_A T_B T_C T_D \leq \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \left(\frac{K}{2}\right)^2 \left(\frac{K}{2}\right)^2 = \frac{K^4}{2^{12}}$$

where equality holds if and only if there is equality in all six applications of the AM-GM inequality. Hence equality holds if and only if $\kappa = \lambda = \mu = \nu = \frac{1}{2}$ and $F_A = F_B = F_C = F_D = \frac{1}{2}K$. The equalities between the Greek letters hold if and only if U, Q, R, S bisect their respective sides. Triangles ABC and ABD, having the same base, have the same area ($F_A = F_B$) if and only if they have the same height. Thus $F_A = F_B = F_C = F_D = \frac{1}{2}K$ hold if and only if $ABCD$ is a parallelogram. □

7. One-dimensional metric relations

Here we will prove eight characterizations of parallelograms that are relations between different lengths. Conditions *(a)*, *(b)*, *(c)*, *(g)* and *(h)* are due to the author, where the third can be considered a reverse version of the Varignon parallelogram theorem. *(d)* was expressed in [24] as a by-product when deriving another inequality, *(e)* was inspired by Problem 4 for Grade 9 on the 1994 Ukrainian Mathematical Olympiad (this was about proving an inequality in a general convex quadrilateral, but the equality case was not mentioned, so the sufficient condition is due to the author), and *(f)* was used in a proof of another characterization of parallelograms in [38, p. 19].

Theorem 7.1. *A convex quadrilateral $ABCD$ with perimeter L and diagonal intersection P satisfies any one of:*

(a) $w = y$ *and* $x = z$, *where* w, x, y, z *are the distances between neighboring free vertices of external squares erected on the sides*

(b) $AG_3 = CG_1$ *and* $BG_4 = DG_2$, *where* G_1, G_2, G_3, G_4 *are the centroids of triangles* BCD, CDA, DAB, ABC *respectively*

(c) $GC = CH$ *and* $FB = BH$, *where* E *is a random point outside* AD *such that* $E - A - F$ *and* $E - D - G$ *are collinear points with* $EA = AF$ *and* $ED = DG$, *and rays* GC *and* FB *intersect at* H

(d) $L = IB + BJ$, *where* I *and* J *are points such that* IB *and* BJ *are both parallel to and have equal length as diagonal* AC

(e) $L = E_aE_c + E_bE_d$, *where* E_a, E_b, E_c, E_d *are the intersections of the exterior angle bisectors at consecutive vertices*

(f) $S_1P = S_3P$ *and* $S_2P = S_4P$, *where* S_1, S_2, S_3, S_4 *form exterior isosceles triangles with the sides of* $ABCD$ *as bases such that* AS_1BP, BS_2CP, CS_3DP, *and* DS_4AP *are cyclic quadrilaterals*

(g) $r_a = r_c$ *and* $r_b = r_d$, *where* r_a, r_b, r_c, r_d *are the inradii of triangles* ABP, BCP, CDP, DAP *respectively*

(h) $\rho_a = \rho_c$ *and* $\rho_b = \rho_d$, *where* ρ_a, ρ_b, ρ_c, ρ_d *are the exradii to triangles* ABP, BCP, CDP, DAP *respectively that are tangent to the sides* AB, BC, CD, DA

if and only if it's a parallelogram.

Proof. *(a)* In a parallelogram with consecutive sides a, b, c, d, opposite sides satisfy $a = c$ and $b = d$. It directly follows from (6) and (7) that

$$p^2 + w^2 = 2(a^2 + b^2) = 2(c^2 + d^2) = p^2 + y^2$$

which implies $w = y$ (see Figure 17). In the same way we have $x = z$.

Conversely, when $w = y$ and $x = z$ hold in a convex quadrilateral, we obtain from (6) and (7) that

$$2(a^2 + b^2) = p^2 + w^2 = p^2 + y^2 = 2(c^2 + d^2),$$

so $a^2 + b^2 = c^2 + d^2$, and in the same way that $d^2 + a^2 = b^2 + c^2$. Then the quadrilateral is a parallelogram according to Theorem 5.1 *(a)*.

(b) In a convex quadrilateral with consecutive sides a, b, c, d and diagonal $q = BD$, it holds that

$$AG_3 = \tfrac{2}{3} \cdot \tfrac{1}{2}\sqrt{2(a^2 + d^2) - q^2} \quad \text{and} \quad CG_1 = \tfrac{1}{3}\sqrt{2(b^2 + c^2) - q^2}$$

where we used a well-known property of the medians in a triangle and Apollonius' theorem. Hence

$$AG_3 = CG_1 \quad \Leftrightarrow \quad a^2 + d^2 = b^2 + c^2$$

and in the same way we get

$$BG_4 = DG_2 \quad \Leftrightarrow \quad a^2 + b^2 = c^2 + d^2.$$

The two equalities $a^2 + d^2 = b^2 + c^2$ and $a^2 + b^2 = c^2 + d^2$ characterize parallelograms according to Theorem 5.1 *(a)*.

(c) When $ABCD$ is a parallelogram, $EA = AF$ and $ED = DG$ imply that $AD \parallel FG$ (see Figure 21). Since we know that $AD \parallel BC$, we get $BC \parallel FG$. But $AD = BC$ and $AD = \frac{1}{2}FG$, so $BC = \frac{1}{2}FG$, which proves

that B and C are midpoints on FH and GH respectively. Hence $FB = BH$ and $GC = CH$.

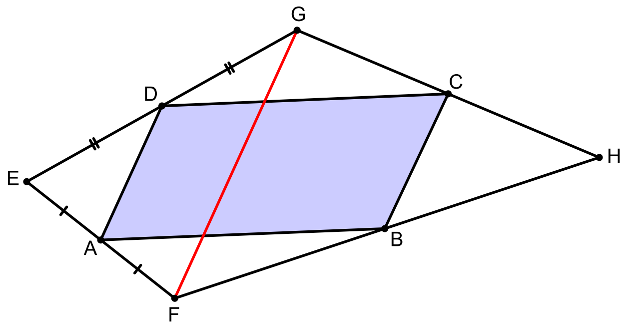

FIGURE 21. A Varignon-related characterization

Conversely, when $FB = BH$ and $GC = CH$ in a convex quadrilateral $ABCD$, then it follows directly that $ABCD$ is a parallelogram since A, B, C, D are the midpoints of the sides of $EFGH$ ($ABCD$ is the Varignon parallelogram of $EFGH$).

(d) From the construction of the points I and J, quadrilaterals $ACBI$ and $ACJB$ are parallelograms with $AI = CB$ and $CJ = AB$ (see Figure 22). Applying the triangle inequality to triangles ADI and DCJ yields $AD + BC \geq ID$ and $DC + AB \geq DJ$, so

$$L = AB + BC + CD + DA \geq ID + DJ.$$

Equality holds if and only if each pair of segments IA, AD and CD, CJ are collinear, which is equivalent to that opposite sides in $ABCD$ are parallel, so it is a parallelogram.

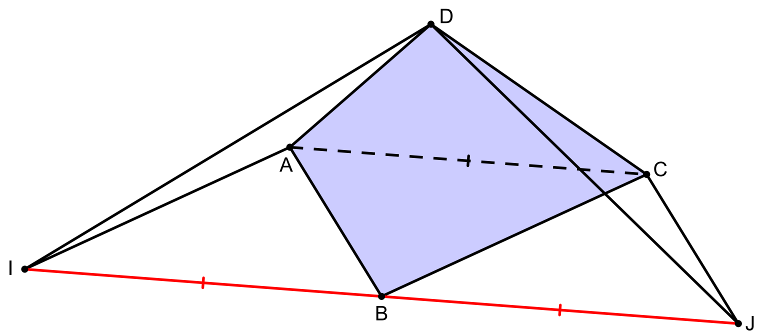

FIGURE 22. IB and BJ are both parallel to AC

(e) If the extensions of AB and DC intersect at O in a convex quadrilateral and the angle at O is 2ω, then by the angle sum in triangle ADO, we have

$$\pi - \angle A + \pi - \angle D + 2\omega = \pi \quad \Leftrightarrow \quad 2\omega = \angle A + \angle D - \pi.$$

With notations as in Figure 23, we get

$$X_1X_2 = E_dX_b \leq E_dE_b$$

with equality if and only if $\omega = 0$, which is equivalent to $\angle A + \angle D = \pi$, that is, $AB \parallel DC$. In the same way for the other two escribed circles, there holds

$$X_3X_4 \leq E_aE_c$$

with equality if and only if $\chi = 0$, which is equivalent to $\angle C + \angle D = \pi$, that is, $AD \parallel BC$ (here 2χ is the angle between the extensions of AD and BC).

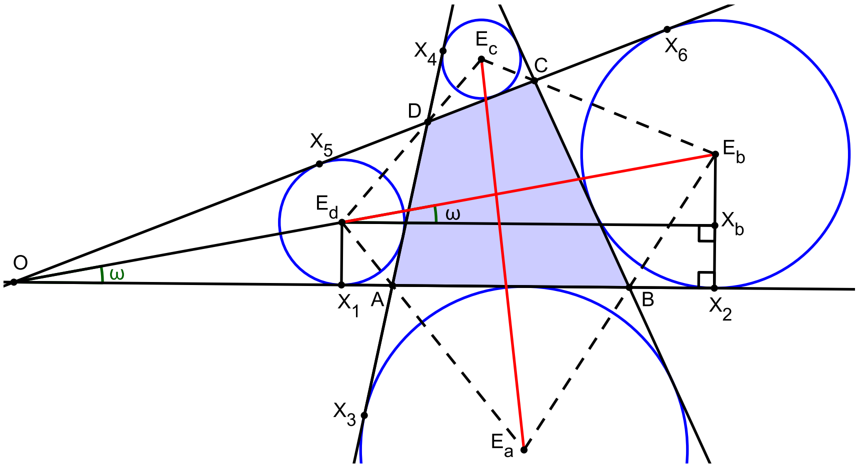

FIGURE 23. Distances between centers in opposite escribed circles

Next we note that $L = X_1X_2 + X_5X_6 = 2X_1X_2$ and $X_1X_2 = X_3X_4$ (the reader can supply the details via equal tangent lengths), so

$$L = X_1X_2 + X_3X_4 \leq E_aE_c + E_bE_d$$

where equality holds if and only if $AD \parallel BC$ and $AB \parallel CD$, that is, only when $ABCD$ is a parallelogram.

(f) Applying Ptolemy's theorem in the cyclic quadrilateral AS_1BP yields

$$S_1P \cdot AB = S_1B \cdot PA + S_1A \cdot PB$$

and since $S_1A = S_1B$ (see Figure 24), this simplifies into

$$S_1P \cdot AB = S_1A(PA + PB). \tag{10}$$

Next we apply the Law of Cosines in triangle AS_1B to get

$$|AB|^2 = |S_1A|^2 + |S_1B|^2 - 2S_1A \cdot S_1B \cos(\pi - \alpha) = |S_1A|^2(2 + 2\cos\alpha)$$

where $\alpha := \angle APB$. Hence

$$AB = S_1A\sqrt{2(1 + \cos\alpha)}$$

and inserting this into (10), we get

$$S_1P = \frac{PA + PB}{\sqrt{2(1 + \cos\alpha)}}. \tag{11}$$

By symmetry

$$S_3P = \frac{PC + PD}{\sqrt{2(1 + \cos\alpha)}} \tag{12}$$

and in the same way,

$$S_2P = \frac{PB+PC}{\sqrt{2(1+\cos\beta)}} \quad \text{and} \quad S_4P = \frac{PD+PA}{\sqrt{2(1+\cos\beta)}} \tag{13}$$

where $\beta := \angle BPC$.

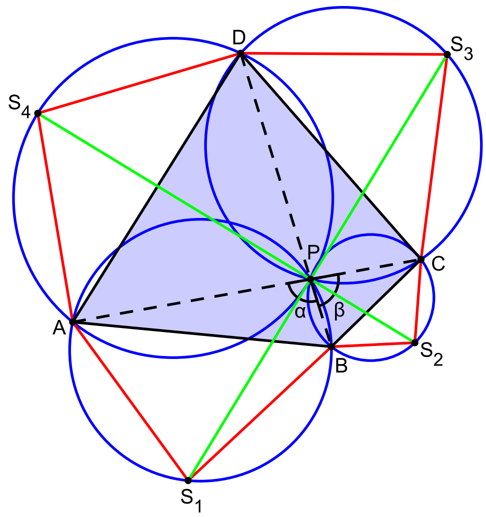

FIGURE 24. Four cyclic quadrilaterals

According to Theorem 3.1 *(f)* in Part 1, $ABCD$ is a parallelogram if and only if

$$\begin{cases} PA+PB = PC+PD \\ PB+PC = PD+PA \end{cases}$$

which is equivalent to

$$\begin{cases} \dfrac{PA+PB}{\sqrt{2(1+\cos\alpha)}} = \dfrac{PC+PD}{\sqrt{2(1+\cos\alpha)}} \\ \dfrac{PB+PC}{\sqrt{2(1+\cos\beta)}} = \dfrac{PD+PA}{\sqrt{2(1+\cos\beta)}} \end{cases}$$

and, in turn, is equivalent to

$$\begin{cases} S_1P = S_3P \\ S_2P = S_4P \end{cases}$$

according to (11), (12) and (13), completing this proof.

(g) That $r_a = r_c$ and $r_b = r_d$ hold in a parallelogram is a direct consequence of the fact that each pair of triangles ABP, CDP and BCP, DAP are congruent (via SSS or SAS or ASA).

We prove the converse with contradiction. Assume that $r_a = r_c$ and $r_b = r_d$ hold in a convex quadrilateral that is not a parallelogram. Then at least one of the diagonals is not bisected, but triangles ABP and CDP have an equal angle at P. Assume without loss of generality that $BP < DP$ and $AP \le CP$. Then there are points A' and B' on CP and DP respectively such that $AP = A'P$ and $BP = B'P$ (see Figure 25). Thus $ABA'B'$ is

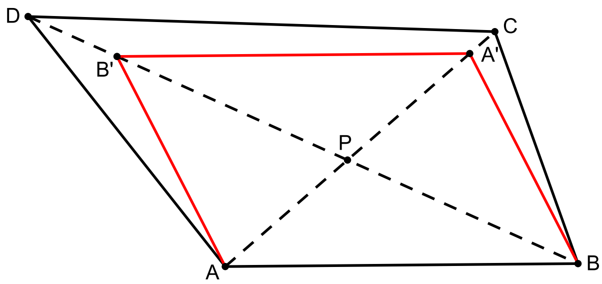

FIGURE 25. Comparison of opposite quarter-triangles

a parallelogram, so the incircles in triangles ABP and $A'B'P$ have equal inradii. Next we prove that the incircles in triangles CDP and $A'B'P$ do not have equal inradii. We get (deriving the applied triangle formula is left as an exercise for the reader)

$$r_c = \frac{CP}{\cot\frac{P}{2} + \cot\frac{\angle DCP}{2}} > \frac{A'P}{\cot\frac{P}{2} + \cot\frac{\angle B'A'P}{2}} = r_{B'A'P} = r_a$$

since we can assume without loss of generality that $\angle DCP > \angle B'A'P$, which implies $\cot\frac{\angle DCP}{2} < \cot\frac{\angle B'A'P}{2}$. We have reached a contradiction: $r_c = r_a$ and $r_c > r_a$. Hence we cannot have that $r_a = r_c$ and $r_b = r_d$ hold in a convex quadrilateral that is not a parallelogram, so $ABCD$ is indeed a parallelogram.

(h) That $\rho_a = \rho_c$ and $\rho_b = \rho_d$ hold in a parallelogram is a direct consequence of the fact that each pair of triangles ABP, CDP and BCP, DAP are congruent (via SSS or SAS or ASA), see Figure 26.

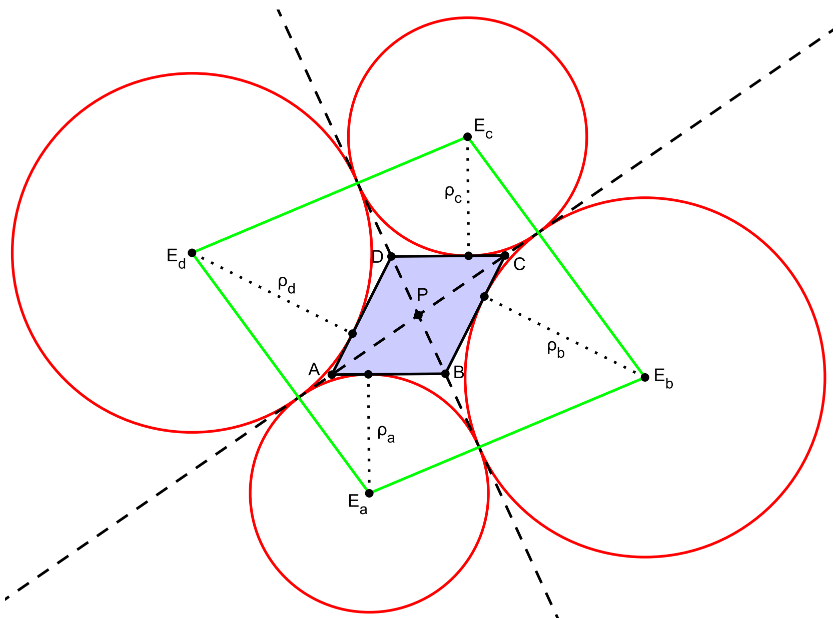

FIGURE 26. Excircles to the four quarter-triangles

The proof of the converse is very similar to that of *(g)* and we again use contradiction. Assume that $\rho_a = \rho_c$ and $\rho_b = \rho_d$ hold in a convex quadrilateral that is not a parallelogram. Then at least one of the diagonals is not bisected, but triangles ABP and CDP have an equal angle at P. Assume without loss of generality that $BP < DP$ and $AP \le CP$. Then there are points A' and B' on CP and DP respectively such that $AP = A'P$ and $BP = B'P$. Thus $ABA'B'$ is a parallelogram, so the excircles to triangles ABP and $A'B'P$ have equal exradii. Next we prove that the excircles to triangles CDP and $A'B'P$ do not have equal inradii. We get (deriving the applied triangle formula is left as an exercise for the reader)

$$\rho_c = \frac{DP}{\tan\frac{P}{2} + \tan\frac{\angle CDP}{2}} > \frac{B'P}{\tan\frac{P}{2} + \tan\frac{\angle A'B'P}{2}} = \rho_{A'B'P} = \rho_a$$

since we can assume without loss of generality that $\angle CDP < \angle A'B'P$ (see Figure 25), which implies $\tan\frac{\angle CDP}{2} < \tan\frac{\angle A'B'P}{2}$. We have reached a contradiction: $\rho_c = \rho_a$ and $\rho_c > \rho_a$. Hence we cannot have that $\rho_a = \rho_c$ and $\rho_b = \rho_d$ hold in a convex quadrilateral that is not a parallelogram, so we conclude that $ABCD$ must be a parallelogram. □

8. Rhombi

Next we have three necessary and sufficient conditions for when a convex quadrilateral is a parallelogram expressed in terms of the formation of different rhombi. Characterization *(a)* was proved in the paper [38, pp. 17–20]. The sufficient condition in *(b)* is due to the author, but to prove the necessary part was given as a problem in [26, pp. 61–62], and the similar *(c)* is also due to the author.

Theorem 8.1. *A convex quadrilateral $ABCD$ with diagonal intersection P satisfies any one of:*

- *(a)* *$S_1S_2S_3S_4$ is a rhombus, where S_1, S_2, S_3, S_4 form exterior isosceles triangles with the sides of $ABCD$ as bases such that AS_1BP, BS_2CP, CS_3DP, and DS_4AP are cyclic quadrilaterals*
- *(b)* *$I_aI_bI_cI_d$ is a rhombus, where I_a, I_b, I_c, I_d are the incenters of quarter-triangles ABP, BCP, CDP, DAP respectively*
- *(c)* *$E_aE_bE_cE_d$ is a rhombus, where E_a, E_b, E_c, E_d are the excenters of quarter-triangles ABP, BCP, CDP, DAP respectively that are tangent to the sides AB, BC, CD, DA*

if and only if it's a parallelogram.

Proof. *(a)* A convex quadrilateral is a rhombus if and only if its diagonals bisect each other at right angles. We know from Theorem 7.1 *(f)* that $ABCD$ is a parallelogram if and only if $S_1P = S_3P$ and $S_2P = S_4P$. What remains to prove is that the diagonals of $S_1S_2S_3S_4$ always intersect each other at right angles at P.

In cyclic quadrilateral AS_1BP, we get $\angle APS_1 = \angle ABS_1$ and $\angle BAS_1 = \angle BPS_1$ (see Figure 24). Triangle ABS_1 is isosceles, so $\angle ABS_1 = \angle BAS_1$, and we conclude that $\angle APS_1 = \angle BPS_1$ always holds. This means that PS_1 is an angle bisector to angle APB. In the same way we conclude that

PS_3 is an angle bisector to the same angle, so S_1, P, S_3 are collinear. By symmetry, S_2, P, S_4 are also collinear. Hence the diagonals of $S_1S_2S_3S_4$ and $ABCD$ are concurrent at P. We also conclude that PS_2 is an exterior angle bisector to triangle APB, and since an exterior and an interior angle bisector are always perpendicular, this proves that the diagonals of $S_1S_2S_3S_4$ always intersect each other at right angles at P.

(b) Since I_aI_c and I_bI_d are angle bisectors to the angles between the diagonals, it is always true that $I_aI_c \perp I_bI_d$. Hence $I_aI_bI_cI_d$ is a rhombus if and only if $I_aP = I_cP$ and $I_bP = I_dP$. These are satisfied in a parallelogram $ABCD$ since $\triangle ABP \cong \triangle CDP$ and $\triangle BCP \cong \triangle DAP$ (see Figure 27).

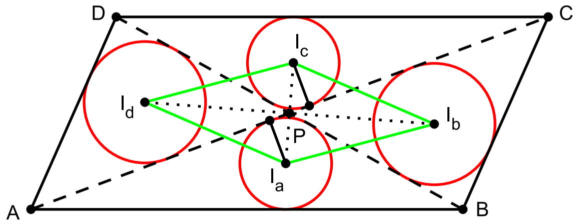

FIGURE 27. Incircles in the four quarter-triangles

Conversely, if $I_aI_bI_cI_d$ is a rhombus, then $I_aP = I_cP$ and $I_bP = I_dP$. By congruent triangles (AAS), this implies $r_a = r_c$ and $r_b = r_d$, where r_a, r_c, r_b, r_d are the inradii of triangles ABP, CDP, BCP, DAP respectively. The two inradii equalities imply that $ABCD$ is a parallelogram according to Theorem 7.1 *(g)*.

(c) Since E_aE_c and E_bE_d are angle bisectors to the angles between the diagonals, it is always true that $E_aE_c \perp E_bE_d$. Hence $E_aE_bE_cE_d$ is a rhombus if and only if $E_aP = E_cP$ and $E_bP = E_dP$. These are satisfied in a parallelogram $ABCD$ since $\triangle ABP \cong \triangle CDP$ and $\triangle BCP \cong \triangle DAP$.

Conversely, if $E_aE_bE_cE_d$ is a rhombus, then $E_aP = E_cP$ and $E_bP = E_dP$ (see Figure 26). By congruent triangles (AAS), this implies $\rho_a = \rho_c$ and $\rho_b = \rho_d$, where ρ_a, ρ_c, ρ_b, ρ_d are the exradii to triangles ABP, CDP, BCP, DAP respectively that are tangent to the sides AB, BC, CD, DA. Hence $ABCD$ is a parallelogram according to Theorem 7.1 *(h)*. □

9. DIAGONALS

The last theorem deals with five characteristic properties of parallelograms that are related to the diagonals. The first four are due to the author, but the necessary condition in *(a)* is from [36, p. 66], and the necessary condition in *(c)* and *(d)* were stated in [12]. The sufficient condition in *(e)* is from the third Croatia Girls Mathematical Olympiad in 2022 [33].

It's noteworthy that *(c)* and *(d)* in the following theorem together with Theorem 3.1 *(a)* in Part 1 can be merged into the following characterization: *A convex quadrilateral is a parallelogram if and only if the midpoints of any two of AC, BD, $A'C'$, $B'D'$ coincide.*

We denote by T_{XYZ} the area of triangle XYZ.

Theorem 9.1. *A convex quadrilateral $ABCD$ with area K, diagonal intersection P, and where A', B', C', D' are the centers of external squares $BAEF$, $CBGH$, $DCIJ$, $ADLM$ erected on the sides, satisfies any one of:*

(a) $AC \perp EM$ *and* $BD \perp FG$

(b) $G_1 \vee G_3 \in AC$ *and* $G_2 \vee G_4 \in BD$, *where* G_1, G_2, G_3, G_4 *are the centroids of triangles* BCD, CDA, DAB, ABC *respectively*

(c) the midpoint of AC *(or* BD*) coincides with the midpoint of* $A'C'$ *(or* $B'D'$ *respectively)*

(d) the midpoint of $A'C'$ *coincides with the midpoint of* $B'D'$

(e) $T_{ADN} = T_{CDO} = \frac{1}{4}K$ *and diagonal* AC *is trisected by* DN *and* DO, *where* $N \in AB$ *and* $O \in BC$

if and only if it's a parallelogram.

Proof. *(a)* In a parallelogram $ABCD$, $\angle CDA = \angle EAM$ since they are both supplementary angles to angle DAB. Thus triangles CDA and EAM are congruent (SAS) due to equal opposite sides in a parallelogram and the constructed squares (see Figure 28). Now let the extension of diagonal AC intersect EM at N. Then $\angle DAC = \angle AMN$ due to this congruence, and since angles DAC and MAN are complementary, so are angles AMN and MAN. This proves that the third angle MNA in triangle MNA is a right angle, confirming that $AC \perp EM$. In the same way we prove that $BD \perp FG$.

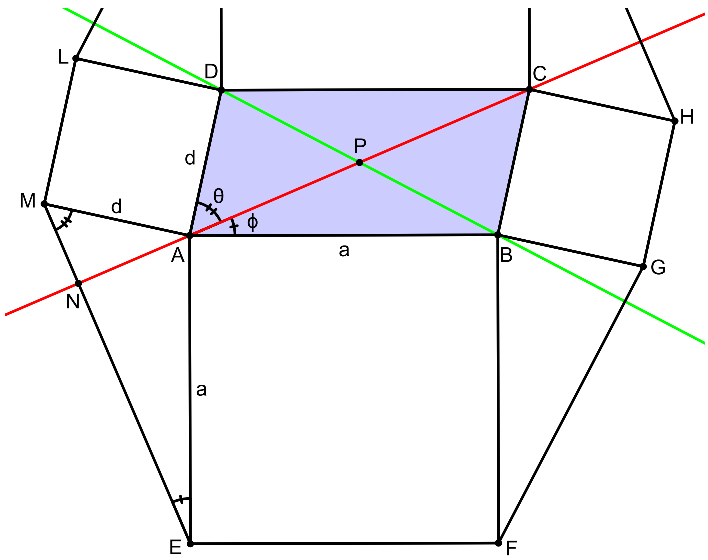

FIGURE 28. External squares on the sides

For the converse, in a convex quadrilateral where $AC \perp EM$ and $BD \perp FG$, let $\theta := \angle DAP$ and $\phi := \angle PAB$ (see Figure 28). Then we have $\angle AMN = \theta$ and $\angle AEN = \phi$, where N is the intersection between the extensions of AC and EM, since angles DAP and AMN are both complementary angles to angle MAN, and angles PAB and AEN are both complementary angles to angle EAN. Applying the Law of Sines in triangles DAP and BPA, we get

$$\frac{DP}{\sin\theta} = \frac{d}{\sin P} \quad \text{and} \quad \frac{BP}{\sin\phi} = \frac{a}{\sin(\pi - P)}$$

and from the definition of sines in triangles AMN and AEN, it holds

$$\sin\theta = \frac{AN}{d} \quad \text{and} \quad \sin\phi = \frac{AN}{a}.$$

Hence we get

$$\frac{DP}{BP} = \frac{\frac{d\sin\theta}{\sin P}}{\frac{a\sin\phi}{\sin(\pi-P)}} = \frac{AN}{AN} = 1$$

proving that $AC \perp EM$ implies $DP = BP$. In the same way it's proved that $BD \perp FG$ implies $AP = CP$, so we have that the diagonals bisect each other. Then $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in Part 1.

(b) We assume the quadrilateral is placed in a coordinate system such that $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, and $D = (d,e)$, see Figure 7. Then

$$G_3 = \left(\frac{0+a+d}{3}, \frac{0+0+e}{3}\right) = \left(\frac{a+d}{3}, \frac{e}{3}\right).$$

The line AC has the equation $y = \frac{c}{b}x$ and we get that G_3 lies on this line if and only if

$$\frac{e}{3} = \frac{c}{b} \cdot \frac{a+d}{3} \quad \Rightarrow \quad be = c(a+d).$$

The diagonal BD has the equation

$$y = \frac{e}{d-a}x - \frac{ae}{d-a}$$

and G_4 lies on this line if and only if

$$\frac{c}{3} = \frac{e}{d-a} \cdot \frac{a+b}{3} - \frac{ae}{d-a}$$

which is equivalent to $cd - ac = be - 2ae$. Thus we have the system of equations

$$\begin{cases} be = c(a+d) \\ cd - ac = be - 2ae \end{cases}$$

and substituting the first equation into the second yields $2a(e-c) = 0$. Then $c = e$ since $a \neq 0$, and we get $b = a + d$ from the first equation. These two equalities are equivalent to that $ABCD$ is a parallelogram according to Theorem 1.1.

The other possibilities are proved in the same way (for instance $G_1 \in AC$ and $G_2 \in BD$ imply that $ABCD$ is a parallelogram).

(c) Let quadrilateral $ABCD$ be placed in an Argand plane with vertices given by the complex numbers A, B, C, D (see Figure 29). Then the vector $\mathbf{BA} = A - B$ so $\mathbf{BF} = (A-B)i$, and we get $F = B + (A-B)i$. Hence the center A' is represented by the complex number

$$A' = \tfrac{1}{2}(A+F) = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)i.$$

In the same way

$$C' = \tfrac{1}{2}(C+D) + \tfrac{1}{2}(C-D)i$$

and the midpoint of $A'C'$ is

$$\tfrac{1}{2}(A'+C') = \tfrac{1}{4}[(A+B+C+D) + (A-B+C-D)i].$$

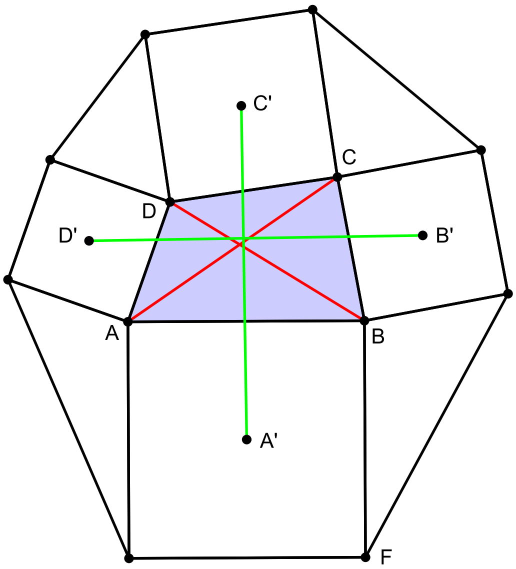

FIGURE 29. External squares in the Argand plane

The midpoint of AC is $\frac{1}{2}(A+C)$, so the midpoints of $A'C'$ and AC coincide if and only if

$$\tfrac{1}{4}[(A+B+C+D)+(A-B+C-D)i] = \tfrac{1}{2}(A+C)$$

which is equivalent to

$$(A-B+C-D)(-1+i) = 0$$

and since $-1+i \neq 0$, the only possible solution is $A+C = B+D$, which according to Theorem 3.1 *(e)* in Part 2 characterize parallelograms.

There are three other possible coincidences stated, but the proof of all of them are the same (only a few changes of letters or signs are required).

(d) With the same notations as in *(c)*, the midpoint of $B'D'$ is

$$\tfrac{1}{4}[(A+B+C+D)+(-A+B-C+D)i]$$

so the midpoint of $A'C'$ coincides with the midpoint of $B'D'$ if and only if

$$\tfrac{1}{4}[(A+B+C+D)+(A-B+C-D)i]$$
$$= \tfrac{1}{4}[(A+B+C+D)+(-A+B-C+D)i]$$

which is equivalent to $2i(A-B+C-D) = 0$. Hence $A+C = B+D$ since $2i \neq 0$, and this is equivalent to $ABCD$ being a parallelogram according to Theorem 3.1 *(e)* in Part 2.

(e) In a parallelogram $ABCD$, one median in triangle ABD lies along diagonal AC, and since AC is trisected by DN, $AQ = \frac{2}{3}AP$ where P is the midpoint of AC. This is the ratio in which the medians in a triangle divide each other, implying that DN is a median in triangle ABD, so N is the midpoint of AB. In the same way, O is the midpoint of BC. Then the areas of the pairs of triangles AND, BND and COD, BOD are equal, and each pair is equal to half the area K of parallelogram $ABCD$, so $T_{ADN} = T_{CDO} = \frac{1}{4}K$.

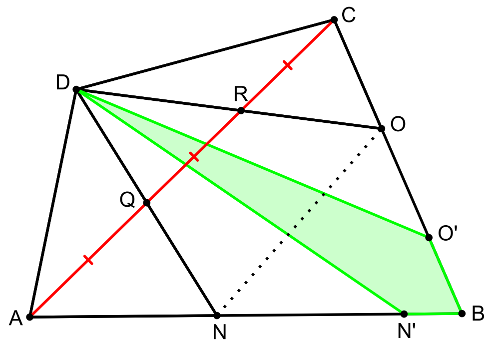

Figure 30. A trisected diagonal

Conversely, the trisection of AC at Q and R implies that triangles ADQ, QDR, CDR have equal area, so by $T_{ADN} = T_{CDO} = \frac{1}{4}K$, we get that triangles AQN and CRO have equal area (see Figure 30). This implies that N and O have equal distance from AC, so $NO \parallel AC$. Hence N and O divide BA and BC in the same ratio.

Next we shall show that N and O are the midpoints of AB and BC respectively. By way of contradiction, assume the opposite, which without loss of generality can be taken as $AN < \frac{1}{2}AB$ and $CO < \frac{1}{2}CB$. Then there are points $N' \in AB$ and $O' \in BC$ such that $AN = NN'$ and $CO = OO'$. Thus $T_{AND} = T_{NN'D}$ and $T_{COD} = T_{OO'D}$. But then we get

$$\begin{aligned} K &= T_{AND} + T_{NN'D} + T_{COD} + T_{OO'D} + K_{DN'BO'} \\ &= \tfrac{1}{4}K + \tfrac{1}{4}K + \tfrac{1}{4}K + \tfrac{1}{4}K + K_{DN'BO'} \\ &= K + K_{DN'BO'} > K \end{aligned}$$

which is a contradiction, so the assumption that N and O are not the midpoints of AB and BC is wrong (the other case is similar). Hence we conclude that N and O are the midpoints of AB and BC respectively. The proof is now completed in the same way as for Theorem 3.1 *(i)* in Part 1. □

10. Twofold characterizations

We note that convex quadrilaterals satisfying at least one of the two equalities

$$a^2 + b^2 = c^2 + d^2 \quad \text{or} \quad d^2 + a^2 = b^2 + c^2,$$

which appeared in Theorem 5.1 *(a)*, were called Pythagorean quadrilaterals in [17]. Hence parallelograms are the only *twofold* Pythagorean quadrilaterals according to that theorem.

There are several other examples of twofold characterizations of parallelograms. We summarize these results in Table 1, where Theorem and Part refer to our three papers on characterizations of parallelograms, but Ref. is a reference where we studied characterizations of all but one of the quadrilaterals in the table. Here we consider convex quadrilaterals $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, diagonal intersection P, and semidiagonals $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$.

Original quadrilateral	**Twofold parallelogram characterization**	**Theorem**	**Part**	**Ref.**
Quasi-isosceles quadrilateral	$\begin{cases} a = c \\ b = d \end{cases}$	2.1 *(a)*	1	[20]
Tilted kite	$\begin{cases} \angle A = \angle C \\ \angle B = \angle D \end{cases}$	4.1 *(a)*	1	[19]
Bisect-diagonal quadrilateral	$\begin{cases} a' = c' \\ b' = d' \end{cases}$	3.1 *(a)*	1	[18]
Extangential quadrilateral	$\begin{cases} a + b = c + d \\ d + a = b + c \end{cases}$	3.1 *(d)*	1	[14]
Trapezoid	$\begin{cases} \angle A + \angle B = \angle C + \angle D \\ \angle D + \angle A = \angle B + \angle C \end{cases}$	4.1 *(b)*	1	[16]
Semidiagonal quadrilateral	$\begin{cases} a' + b' = c' + d' \\ d' + a' = b' + c' \end{cases}$	3.1 *(f)*	1	
Pythagorean quadrilateral	$\begin{cases} a^2 + b^2 = c^2 + d^2 \\ d^2 + a^2 = b^2 + c^2 \end{cases}$	5.1 *(a)*	3	[17]

TABLE 1. Twofold characterizations of parallelograms

Quasi-isosceles quadrilaterals are defined to have at least one pair of opposite equal sides, *tilted kites* are defined to have at least one pair of opposite equal angles, and *bisect-diagonal quadrilaterals* are defined as quadrilaterals where at least one diagonal bisects the other diagonal.

Semidiagonal quadrilaterals is a new type of quadrilateral that was used in [23] to state a few characterizations of rectangles. The concept was coined by my friend Mario Dalcín from Uruguay as quadrilaterals satisfying at least one of $a' + b' = c' + d'$ or $d' + a' = b' + c'$ and used in [6] to study a new classification of convex quadrilaterals based on three types of symmetry.

Theorem 4.1 *(b)* in Part 1 was stated somewhat differently than in this table: a convex quadrilateral $ABCD$ is a parallelogram if and only if any pair of adjacent angles are supplementary, for example

$$\angle A + \angle B = \pi = \angle B + \angle C.$$

The two formulations are equivalent via the angle sum of a quadrilateral and are direct consequences of Propositions I.28 and I.29 in Euclid's *Elements*.

To clarify what Table 1 shows, we comment on the extangential quadrilaterals. They are characterized as convex quadrilaterals satisfying at least one of $a + b = c + d$ or $d + a = b + c$ (see [14, p. 64]). If both of these equalities are satisfied in a quadrilateral, then it is a parallelogram (according to Theorem 3.1 *(d)* in Part 1), so parallelograms are the only twofold extangential quadrilaterals. In the same way *parallelograms are the only twofold type of each of the quadrilaterals that are present in this table*, and these quadrilaterals are each characterized (or defined) to satisfy at least one of the pair of equalities next to their names.

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