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TANGENTS TO A CUBIC CURVE II

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Abstract. The most celebrated result on the tangents to cubic curves is due to Maclaurin and is concerned with the four tangents that may be drawn from a point X on a cubic curve U so as to touch U at P_i $(1 \leq i \leq 4)$. He showed that the lines P_1P_2 and P_3P_4 meet at a point on U. This result has inspired an earlier article in which X is an arbitrary point (although much of the elegance is lost). There are now six tangents which define not only six points of contact but they meet U again at six new points (labelled D_i). It is shown here that many of the properties of the P-points can be applied to these D-points. Also the importance of the polar line of X in U becomes very apparent.

1. Introduction.

1.1. **Background.** Conics (that is, plane curves defined by polynomials of degree two) have been studied since antiquity and remain an essential part of any first course in mathematics. By contrast, cubic curves (defined by polynomials of degree three) have never been a recognised part of a mathematician's training. That is not to say that they have been completely neglected; in the 19th century several prominent mathematicians ([2] gives an impressive list) investigated cubic and higher order curves. At the present time, classical geometry has faded from school curricula and shows little sign of being re-instated. And yet geometry is such a natural way of introducing students to the notions of mathematics and the results can be demonstrated in the most accessible way - through pictures. Cubic curves may not assist in the design of better mouse-traps but they do have a wealth of interesting properties, some being related to those of conics while others are quite separate.

1.2. **Technicalities.** All of the points, lines and curves considered here lie in a plane. Any three non-collinear points in this plane may be used to define a triangle of reference and hence a system of trilinear coordinates. There are various kinds of such coordinates (e.g. areal, homogeneous, barycentric) but the basic idea is the same in each case. For those unfamiliar with such

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coordinates, the Wikipedia article *Homogeneous Coordinates* is a good place to start as are [4] (the classic account) and [1] (which was once a standard school text-book).

Only algebraic curves will be encountered. Any homogeneous polynomial Θ of degree n in $\mathbf{r} = (r_1, r_2, r_3)$ will define a plane curve (an n-curve) also denoted by Θ, where (r_1, r_2, r_3) are used as the trilinear coordinates. With $X(\boldsymbol{\alpha})$ any point, the (first) polar curve of X in Θ is the $(n-1)$-curve Θ_X defined by

$$n\Theta_X \equiv \alpha_1 \frac{\partial \Theta}{\partial r_1} + \alpha_2 \frac{\partial \Theta}{\partial r_2} + \alpha_3 \frac{\partial \Theta}{\partial r_3} \equiv \boldsymbol{\alpha} \cdot \nabla\Theta = 0.$$

The most significant property of the polar curve is that the common points of Θ and Θ_X are the points of contact of the tangents from X to Θ.

Some of the results of [5] will be used here and the remainder of this introductory section gives a summary of the findings of that paper.

1.3. **Geometry.** The most celebrated result on planar cubic curves is due to Maclaurin. Let U be a cubic curve and X a general point on U. Four tangents may be drawn from X to touch U at P_i $(1 \le i \le 4)$. The principal result of Maclaurin is that the lines P_1P_2 and P_3P_4 meet at a point on U. Thus the three diagonal points of the quadrangle $P_1P_2P_3P_4$ lie on U. Furthermore, the tangents at these diagonal points are concurrent at yet another point of U. This was the inspiration for [5] and the key results of that investigation are given next but it has to be admitted that elegance has been lost at the expense of generality.

Let U be a planar cubic curve and X any point not on U. The points of contact of the six tangents from X to U are labelled P_i $(1 \le i \le 6)$. The line P_1P_2 will meet U again; label this point Q_{12}. There will be fifteen such Q-points. The lines

$$P_1Q_{23},\ P_2Q_{31},\ P_3Q_{12}$$

are concurrent and their common point is named R_{123}. Although one might expect there to be twenty such R-points, in fact there are only ten because R_{123} and R_{456} coincide. Furthermore, there exists a cubic curve V which passes through these ten points and also through X.

1.4. **Algebra.** Let $U(\mathbf{r})$ be a homogeneous polynomial of degree three so that U is a cubic curve. The first polar curve of $X(\boldsymbol{\alpha})$, U_X, is a conic and the second polar curve, U_{XX}, is a line - the polar line of X in U. Also $U_{XXX} \equiv U(\boldsymbol{\alpha})$. The six points of contact of the tangents from X to U are $P_i(\mathbf{p}_i)$ $(1 \le i \le 6)$ and their coordinates satisfy

$$\nabla U = \mathbf{r} \times \boldsymbol{\alpha}. \tag{1}$$

Using $[..]_i$ to denote the value of the expression in square brackets when evaluated at P_i, this last equation can be written as

$$[\nabla U]_i = \mathbf{p}_i \times \boldsymbol{\alpha}\ (1 \le i \le 6). \tag{2}$$

The use of vectorial notation (specifically scalar and vector products) is very convenient but only coordinates are involved and not vectors. It is also to be emphasised that equation (2) is not homogeneous in $\mathbf{p}_i$, it implies some special scaling of the coordinates $\mathbf{p}_i$.

The coordinates of the point Q_{ij} are $(\mathbf{p}_i + \mathbf{p}_j)$ and R_{ijk} has coordinates $(\mathbf{p}_i + \mathbf{p}_j + \mathbf{p}_k)$. The fact that R_{123} and R_{456} coincide follows from the result

$$\sum_{i=1}^{6} \mathbf{p}_i = \mathbf{0}. \tag{3}$$

The Hessian of U is the cubic curve given by

$$H \equiv \left| \frac{\partial^2 U}{\partial r_i \partial r_j} \right| = 0. \tag{4}$$

Define $\boldsymbol{\beta}$ by

$$\boldsymbol{\beta}(\mathbf{r}) \equiv \nabla U \times \nabla H, \tag{5}$$

the quartic curve W and the cubic curve V by

$$W \equiv -\tfrac{2}{3}\boldsymbol{\beta} \cdot [\nabla U]_X = 0 \tag{6}$$

$$\text{and} \quad V \equiv W_X = 0. \tag{7}$$

Then V is the curve that passes through the ten R-points. These are the common points of W and V (other than X) and so are the touching points of the tangents from X to W.

The following results, also given in [5], will be used later:

$$W(\mathbf{p}_1) + W(\mathbf{p}_1 + \mathbf{p}_2) + W(\mathbf{p}_2) = 0, \tag{8}$$

$$H(\mathbf{p}_1) - H(\mathbf{p}_1 + \mathbf{p}_2) + H(\mathbf{p}_2) = 0, \tag{9}$$

$$W(\mathbf{p}_i) = -4H(\mathbf{p}_i)U(\boldsymbol{\alpha}), \tag{10}$$

$$3H_X(\mathbf{p}_i) = \boldsymbol{\alpha} \cdot [\nabla H]_i = -24U(\boldsymbol{\alpha}), \tag{11}$$

$$8U(\boldsymbol{\beta}(\mathbf{r})) = -U(\mathbf{r}) \left| \frac{\partial \beta_i}{\partial r_j} \right| \quad \text{and} \quad 8H(\boldsymbol{\beta}(\mathbf{r})) = -H(\mathbf{r}) \left| \frac{\partial \beta_i}{\partial r_j} \right|. \tag{12}$$

Another set of interesting points is S_{ij} with coordinates $(\mathbf{p}_i - \mathbf{p}_j)$. These points obviously lie on the line P_iP_j and are such that $P_iQ_{ij}P_jS_{ij}$ form an harmonic range. Furthermore, the S-points lie on W.

2. $\mathscr{A}$ and $\mathscr{B}$ points

Let $A(\mathbf{a})$ be any point on the cubic curve U. The tangent to U at A will meet U again, call this point B. It is assumed here and throughout this work that all the points considered on U have a unique tangent and that this tangent meets U again at a different point. There does not seem to be an appropriate name for the point(s) at which a tangent meets its curve again. In [3] they are referred to as 'tangential points' which is not very satisfactory. In [6] the terms $\mathscr{A}$ and $\mathscr{B}$ points were introduced (so that the tangent at an $\mathscr{A}$ point meets the same curve again at a $\mathscr{B}$ point) and this is used here.

It may be verified algebraically that (confining attention to cubic curves) the tangent at the point $A(\mathbf{a})$ meets U again at the point with coordinates $\boldsymbol{\beta}(\mathbf{a})$. That this point lies on U follows from (12). More generally, let $A(\mathbf{a})$ be any point and let the member of the pencil of cubic curves $(U + \lambda H)$ that passes through A be U^*. Then $\boldsymbol{\beta}(\mathbf{a})$ gives the coordinates of the point at which the tangent to U^* at A meets U^* again. Thus A and B are $\mathscr{A}$ and $\mathscr{B}$ points for U^*. When $A(\mathbf{a})$ is a point on W, $B(\boldsymbol{\beta}(\mathbf{a}))$ will lie on the line

U_{XX}, see (6). Also, if A is one of the twelve common points of U and W, then B is one of the three common points of U and U_{XX}.

It follows directly from the definition of $\boldsymbol{\beta}(\mathbf{r})$ that if $A(\mathbf{a})$ is any point then $\boldsymbol{\beta}(\mathbf{a})$ gives the coordinates of the common point of the polar lines of A in the curves U and H i.e. the meet of the lines U_{AA} and H_{AA}. Alternatively, with $A(\mathbf{a})$ any point, the polar line of A in each of the cubics in the pencil $(U + \lambda H)$ passes through $B(\boldsymbol{\beta}(\mathbf{a}))$.

Since the points $P_1(\mathbf{p}_1)$, $P_2(\mathbf{p}_2)$, $Q_{12}(\mathbf{p}_1 + \mathbf{p}_2)$ are collinear and lie on U, the points with coordinates $\boldsymbol{\beta}(\mathbf{p}_1)$, $\boldsymbol{\beta}(\mathbf{p}_2)$, $\boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2)$ not only lie on U, they are also collinear. The most elegant proof of this is by using Cayley residuals, see [3].

2.1. **Notation.** Reminder: the points with coordinates

$$\begin{array}{lcccc} & \boldsymbol{\alpha}, & \mathbf{p}_1, & \mathbf{p}_1 + \mathbf{p}_2, & \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3, \\ \text{are} & X, & P_1, & Q_{12}, & R_{123}. \end{array}$$

Now denote the points

$$\begin{array}{lcccc} & \boldsymbol{\beta}(\boldsymbol{\alpha}), & \boldsymbol{\beta}(\mathbf{p}_1), & \boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2), & \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_2) \\ \text{by} & Y, & D_1, & E_{12}, & F_{12}. \end{array}$$

and

$$\begin{array}{lcc} & \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_2), & \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_3) \\ \text{by} & G_{12}, & K_{123}. \end{array}$$

Figure 1 shows schematically the relative positions of some of these points.

3. Theorems

As mentioned in Section 1.3, the lines $P_1Q_{23},\ P_2Q_{31},\ P_3Q_{12}$ are concurrent and since the action of $\boldsymbol{\beta}$ is to send P_i to D_i and Q_{ij} to E_{ij} it is natural to enquire whether this concurrency result applies to the D and E-points. It does not. But it is true for the D and F-points, i.e. the lines

$$D_1F_{23},\ D_2F_{31},\ D_3F_{12}$$

are concurrent, their common point being K_{123}. The first theorem shows that there are only ten K-points (which mirrors the R-points).

Theorem 1. $\displaystyle\sum_{i=1}^{6} \boldsymbol{\beta}(\mathbf{p}_i) = \mathbf{0}.$

Proof. From the definition of $\boldsymbol{\beta}(\mathbf{r})$,

$$\begin{aligned} \boldsymbol{\beta}(\mathbf{p}_i) &= [\nabla U]_i \times [\nabla H]_i, \\ &= (\mathbf{p}_i \times \boldsymbol{\alpha}) \times [\nabla H]_i \\ &= (\mathbf{p}_i \cdot [\nabla H]_i)\boldsymbol{\alpha} - (\boldsymbol{\alpha} \cdot [\nabla H]_i)\mathbf{p}_i \\ &= 3H(\mathbf{p}_i)\boldsymbol{\alpha} - 3H_X(\mathbf{p}_i)\mathbf{p}_i. \end{aligned}$$

But, from equation (11), $H_X(\mathbf{p}_i) = -8U(\boldsymbol{\alpha})$ and so

$$\boldsymbol{\beta}(\mathbf{p}_i) = 3H(\mathbf{p}_i)\boldsymbol{\alpha} + 24U(\boldsymbol{\alpha})\mathbf{p}_i. \tag{13}$$

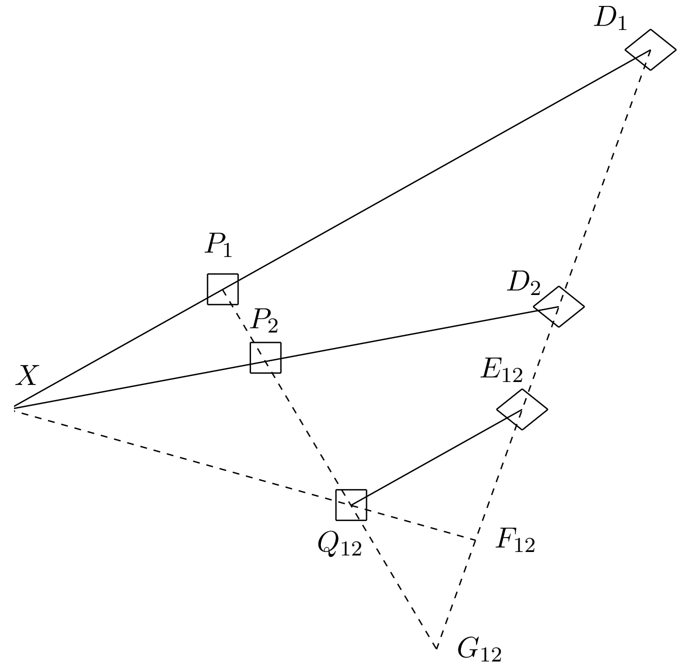

FIGURE 1. A schematic diagram to show the relationships between the principal points. Tangents to U are shown as solid lines with their $\mathscr{A}$-points shown as squares and $\mathscr{B}$-points as diamonds. G_{12} also lies on the line U_{XX}.

From the definition of $W(\mathbf{r})$ it follows that

$$\begin{aligned} W(\mathbf{p}_i) &= -\tfrac{2}{3}\boldsymbol{\beta}(\mathbf{p}_i)\cdot[\nabla U]_X \\ &= -\{2H(\mathbf{p}_i)\boldsymbol{\alpha}+16U(\boldsymbol{\alpha})\mathbf{p}_i\}\cdot[\nabla U]_X \\ &= -6H(\mathbf{p}_i)U(\boldsymbol{\alpha})-16U(\boldsymbol{\alpha})\mathbf{p}_i\cdot[\nabla U]_X. \end{aligned}$$

Now use equation (10) to obtain

$$H(\mathbf{p}_i) = -8\mathbf{p}_i\cdot[\nabla U]_X. \tag{14}$$

Since $\sum_{i=1}^{6}\mathbf{p}_i=\mathbf{0}$, it follows that

$$\sum_{i=1}^{6}H(\mathbf{p}_i)=0,\ \sum_{i=1}^{6}W(\mathbf{p}_i)=0\ \text{ and }\ \sum_{i=1}^{6}\boldsymbol{\beta}(\mathbf{p}_i)=\mathbf{0}. \tag{15}$$

This shows that the points K_{123} and K_{456} coincide so that there are only ten distinct K-points. By analogy with the R-points, it may be surmised that these K-points also lie on a cubic curve. The following theorem shows that this is indeed the case.

Theorem 2. *The ten K-points lie on a cubic curve which touches V at X.*

[Only a brief account of the algebraic contortions will be given. The process of finding the expression for ϕ given below is far too tedious to include and is of no importance.]

Proof. Consider the cubic curve

$$\begin{array}{rcl} \phi & \equiv & 4VU(\boldsymbol{\alpha})^2 - 18V_XU(\boldsymbol{\alpha})U_{XX} + 27V_{XX}U_{XX}^2 = 0, \\ \Rightarrow \quad \phi_X & = & 2U(\boldsymbol{\alpha})\{3V_{XX}U_{XX} - V_XU(\boldsymbol{\alpha})\}, \\ \Rightarrow \quad \phi_{XX} & = & U(\boldsymbol{\alpha})^2V_{XX}. \end{array}$$

Evidently ϕ touches V at X. There is no loss of generality in taking X to have coordinates $(1,1,1)$ and three of the six points of contact of the tangents from X to U to be the vertices of the triangle of reference. With these choices, U becomes

$$U \equiv ar_1^2(r_2 - r_3) + br_2^2(r_3 - r_1) + cr_3^2(r_1 - r_2) + 2dr_1r_2r_3 = 0.$$

In order to satisfy equation (1),

$$\mathbf{p}_1 = (-1/a, 0, 0), \ \mathbf{p}_2 = (0, -1/b, 0), \ \mathbf{p}_3 = (0, 0, -1/c).$$

Writing $\mathbf{p}_4 = (x, y, z)$ for one of the other three points of contact, it was shown in [5] that

$$a = \frac{1}{x} - \frac{dyz(xz + xy - yz)}{x(y-z)(z-x)(x-y)} \tag{16}$$

with corresponding expressions for b and c. It is now a straightforward task to find the Hessian H of U, $\boldsymbol{\beta}(\mathbf{r})$ at the four points $\mathbf{p}_i$, $1 \le i \le 4$ and thus expressions for

$$\boldsymbol{\Psi} = \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_3) \quad \text{and} \quad \boldsymbol{\Omega} = \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_4).$$

It is sufficient to consider just these two points because of Theorem 1. When each of $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$ is substituted into ϕ, the result, after using equation (16) (and its b and c friends), is zero. Hence the ten K points lie on ϕ.

4. Concurrent Lines and Collinear Points

The equation (8) implies that

$$\{\boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_2)\} \cdot [\nabla U]_X = 0$$

and so G_{12} lies on the line U_{XX} as well as the line D_1D_2.

For convenience, let $\mathbf{s}$ be the coordinates of G_{12} so that

$$\mathbf{s} = \boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_2).$$

Since G_{12} lies on D_1D_2 we may set

$$\mathbf{s} = \mu\boldsymbol{\beta}(\mathbf{p}_1) + \nu\boldsymbol{\beta}(\mathbf{p}_2)$$

(for some numbers μ and ν) and, since G_{12} also lies on U_{XX},

$$\mathbf{s} \cdot [\nabla U]_X = 0.$$

Now from equations (13) and (14)

$$\begin{array}{rcl} \boldsymbol{\beta}(\mathbf{p}_i) \cdot [\nabla U]_X & = & 3H(\mathbf{p}_i)\boldsymbol{\alpha} \cdot [\nabla U]_X + 24U(\boldsymbol{\alpha})\mathbf{p}_i \cdot [\nabla U]_i, \\ & = & 9H(\mathbf{p}_i)U)\boldsymbol{\alpha}) - 3U(\boldsymbol{\alpha})H(\mathbf{p}_i), \\ & = & 6U(\boldsymbol{\alpha})H(\mathbf{p}_i). \end{array}$$

Thus

$$\mu H(\mathbf{p}_1) + \nu H(\mathbf{p}_2) = 0 \ \Rightarrow \ \mu = \lambda H(\mathbf{p}_2) \quad \text{and} \quad \nu = -\lambda H(\mathbf{p}_1)$$

for some number λ and so

$$\begin{aligned}\mathbf{s} &= \lambda H(\mathbf{p}_2)\boldsymbol{\beta}(\mathbf{p}_1) - \lambda H(\mathbf{p}_1)\boldsymbol{\beta}(\mathbf{p}_2),\\ &= \lambda H(\mathbf{p}_2)\{3H(\mathbf{p}_1)\boldsymbol{\alpha} + 24U(\boldsymbol{\alpha})\mathbf{p}_1\}\\ &\qquad -\lambda H(\mathbf{p}_1)\{3H(\mathbf{p}_2)\boldsymbol{\alpha} + 24U(\boldsymbol{\alpha})\mathbf{p}_2\},\\ &= 24\lambda U(\boldsymbol{\alpha})\{H(\mathbf{p}_2)\mathbf{p}_1 - H(\mathbf{p}_1)\mathbf{p}_2\}\end{aligned}$$

and so G_{12} lies on the line P_1P_2. The three lines P_1P_2, D_1D_2 and U_{XX} are therefore concurrent at the point G_{12}.

This proves the following result.

Theorem 3. *The points A_1 and A_2 lie on the cubic curve U. The tangents to U at A_1 and A_2 meet at X (not on U). These tangents also meet U again at B_1 and B_2. The lines A_1A_2, B_1B_2 and U_{XX} are concurrent.*

This may be expressed somewhat differently as follows.

Theorem 4. *The line $\mathscr{L}$ meets the cubic curve U at A, B, C. The tangents to U at these points, taken in pairs, meet at X, Y, Z. The polar lines of X, Y, Z are concurrent at a point on $\mathscr{L}$.*

4.1. **Points on U_{XX}.** It has been remarked earlier that for any point $A(\mathbf{a})$ on W, the point $B(\boldsymbol{\beta}(\mathbf{a}))$ lies on U_{XX}. In particular, the points

$$\boldsymbol{\beta}(\boldsymbol{\alpha}),\ \boldsymbol{\beta}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \ \text{ and } \ \boldsymbol{\beta}(\mathbf{p}_1 - \mathbf{p}_2)$$

are on U_{XX} (because X, R_{123} and S_{12} lie on W). But there are also other points as given in the following table (which also indicates the other line(s) on which they lie):

$$\begin{aligned} XP_1: &\quad H(\mathbf{p}_1)\boldsymbol{\alpha} + 24U(\boldsymbol{\alpha})\mathbf{p}_1\\ Q_{12}E_{12}: &\quad \boldsymbol{\beta}(\mathbf{p}_1+\mathbf{p}_2) - 48U(\boldsymbol{\alpha})(\mathbf{p}_1+\mathbf{p}_2)\\ XE_{12}: &\quad \boldsymbol{\beta}(\mathbf{p}_1+\mathbf{p}_2) + 2H(\mathbf{p}_1+\mathbf{p}_2)\boldsymbol{\alpha}\\ XQ_{12}: &\quad H(\mathbf{p}_1+\mathbf{p}_2)\boldsymbol{\alpha} + 24U(\boldsymbol{\alpha})(\mathbf{p}_1+\mathbf{p}_2)\\ P_1P_2 \text{ and } D_1D_2: &\quad \begin{cases}\boldsymbol{\beta}(\mathbf{p}_1) + \boldsymbol{\beta}(\mathbf{p}_1+\mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_2)\\ \quad H(\mathbf{p}_1)\mathbf{p}_2 - H(\mathbf{p}_2)\mathbf{p}_1\end{cases}\end{aligned}$$

This by no means exhausts the list of curious points. For example, because $\sum_{i=1}^{6}\mathbf{p}_i = \mathbf{0}$ and G_{12} lies on U_{XX} it follows that

$$\boldsymbol{\beta}(\mathbf{p}_1+\mathbf{p}_2) + \boldsymbol{\beta}(\mathbf{p}_3+\mathbf{p}_4) + \boldsymbol{\beta}(\mathbf{p}_5+\mathbf{p}_6)$$

also lies on U_{XX} (fifteen points in total). Similarly

$$\mathbf{p}_1+\mathbf{p}_2,\ \mathbf{p}_3+\mathbf{p}_4,\ \mathbf{p}_5+\mathbf{p}_6$$

all lie on U and are collinear. This implies that

$$\boldsymbol{\beta}(\mathbf{p}_1+\mathbf{p}_2),\ \boldsymbol{\beta}(\mathbf{p}_3+\mathbf{p}_4),\ \boldsymbol{\beta}(\mathbf{p}_5+\mathbf{p}_6)$$

are also collinear (and lie on U). But the geometrical significance of these last results does not seem to be of great interest.

5. Final Observations

In the course of these investigations, two unexpected results were encountered which involve the Hessian:

- With η and ζ any cubics, the cubic ω defined by

$$\omega \equiv \eta_{XXX}\zeta - 3\eta_{XX}\zeta_X + 3\eta_X\zeta_{XX} - \eta\zeta_{XXX} = 0$$

  will give three straight lines with common point X. For example, when η and ζ are V and ϕ, the three lines forming ω are the tangent to V at X and the pair of lines joining X to the common points of the conic V_X and the line U_{XX}.

  However, if ζ is the Hessian of η, then ω is identically zero for all $\mathbf{r}$ and all points X. Furthermore, if η is a given cubic and ω is known to be zero for all $\mathbf{r}$ and X then ζ is a linear combination of η and its Hessian.
- For a given cubic U, let W be the quartic defined by equation (6). It may be verified algebraically that

$$U_{XXX}W - 4U_{XX}W_X + 6U_XW_{XX} - 4UW_{XXX} \equiv 0 \;\forall\; \mathbf{r}.$$

The geometrical interpretations of these results eludes the writer.

References

[1] Durell, C. V., *Homogeneous Coordinates*, G. Bell and Sons, 1961.

[2] Plauman, D., Sturmfels, B. and Vinzant, C., *Quartic curves and their bitangents*, J. Symb. Comp., **46(2011)**, 712-733.

[3] Salmon, George, *A Treatise on the Higher Plane Curves*, 3rd ed., Hodges, Foster and Figgis, Dublin, 1879.

[4] Semple, J. G. and Kneebone, G. T., *Algebraic Projective Geometry*, O.U.P., Oxford, 1960.

[5] Vickers, G.T., *Tangents to Cubic Curves*, Int. J. Geom., **11(2022)**, 75-85.

[6] Vickers, G.T., *A Note on Quartic Curves*, Int. J. Geom., **14(2025)**, 54-46.

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