INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 15 (2026), No. 1, 5 - 13

CONICS ASSOCIATED WITH APOLLONIUS' PROBLEM

ARNE ERIKSON

Abstract. In this article we consider a theorem that has recently been published and given an analytical proof [1]. The theorem states that the centers of the three given circles of Apollonius' problem lie on a conic, either an ellipse or a hyperbola, and that the centers of the solution circles (in pairs) are the foci of the conic. Here we present a synthetic proof. To aid us with the proof, we will use a construction method by Casey solving Apollonius' problem [2, pp. 121–123, prop. 10].

1. Introduction

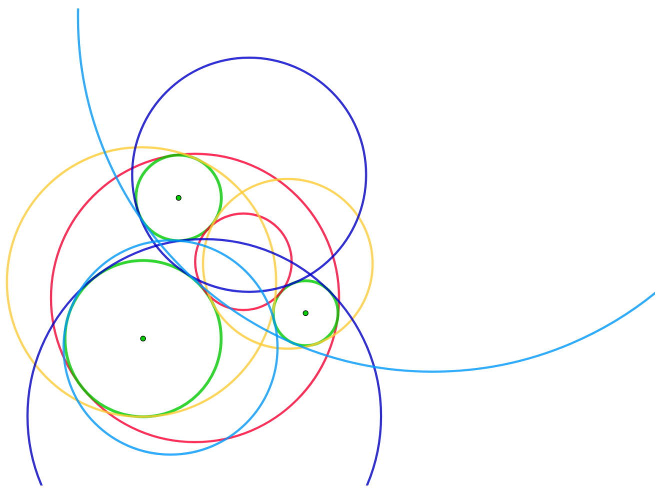


Figure 1. Apollonius' problem. 8 solution circles each tangent to the three given green circles. The solution circles are paired (2 dark blue, 2 light blue, 2 yellow, 2 red), giving four conjugate pairs.

Keywords and phrases: Apollonius' problem, conics, radical circle.
(2020) Mathematics Subject Classification: 51-02, 51-03
Received: 26.06.2025. In revised form: 07.11.2025. Accepted: 03.09.2025.

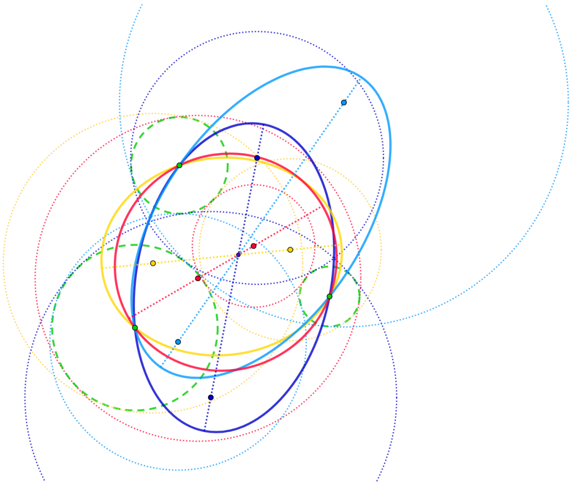
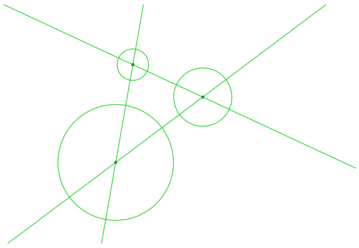


FIGURE 2. Four ellipses whose foci are the centers of the four conjugate solutions shown in figure 1. The centers of the given circles (shown in green dashed lines) are on the conics.

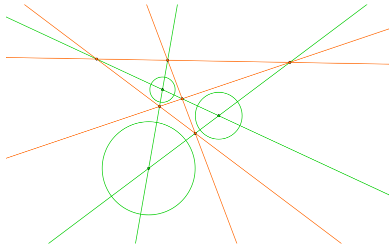
Apollonius' problem is well known and more than two thousand years old: *"To construct the circle or circles tangent to three given circles"*. [7, p. 118]. One or more of the given circles may be replaced by a line or a point and the number of solutions in each case may depend on the relative positions of the given objects. The maximum number of solutions for the case of three given circles, CCC, is eight (see Figure 1), which in turn gives 4 conics (see Figure 2). The solution circles generally occur in pairs [4, p. 450], [5, p. 159] and are often named conjugate pairs. Using straight edge and compass only, the construction method most cited is that by Gergonne {[3, p. 171], [5, p. 160, fig. 18], [6], [7, p.120, fig. 38], [8, p. 189, figs. 4.21 & 4.22], [10, pp. 22–23, fig. 28]}. We however, will use a construction method due to Casey [2, pp. 121–123, prop. 10], and note that Poncelet's method [8, pp. 190–194, figs. 4.23 & 4.24], could equally have served our purpose in proof (I) of theorem 3.1.

2. Construction method.

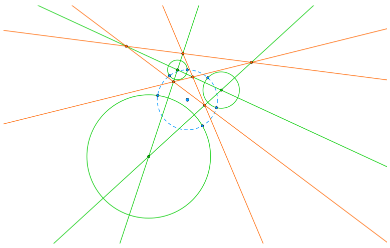
We recreate the CCC construction by Casey in figure 3. The crucial features are the radical circle, its radical axes connected with the three given circles and the intersection points of these axes with the four axes of similitude (see Figures 3c and 3d). A detailed proof can be found in [2, pp. 121–123, prop. 10]. The construction shown in figure 3 is executed thus [2]: "Describe the orthogonal (*radical*) circle of X, Y, Z, (*the three given circles*) and draw the three chords of intersection (*radical axes*) of this circle with X, Y, Z respectively; and from the points where these chords meet the axis of similitude of X, Y, Z draw pairs of tangents to X, Y, Z ; then the two circles described through these six points of contact will be tangential to X, Y, Z." (Words in italics are our additions.)



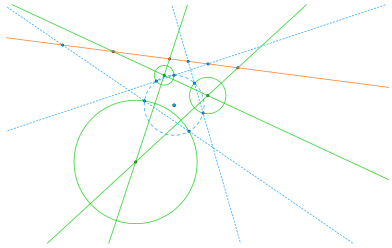
(a) The given three circles



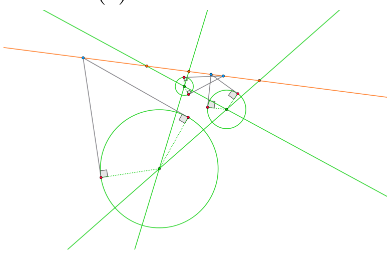
(b) Four axes of similitude



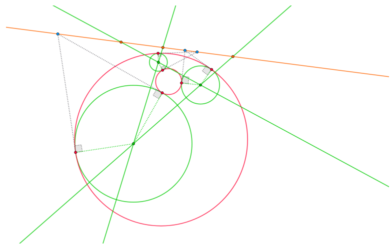
(c) Radical circle



(d) Three radical axes and their intersection points with one axis of similitude



(e) Tangents to given circles from intersection points in (d)



(f) Solution circles in red

FIGURE 3. Construction of Apollonius' problem. Method by Casey [2, pp. 121–123, prop. 10].

As noted above, the solution circles generally occur in pairs. We expand upon this in the following and first quote from [12, p. 584].
(Muirhead): *"It is well known that the contact circles occur in pairs such that each pair has for its radical axis one of the four axes of similitude of the given circles."*
(Plücker): *"The centers of the eight circles which simultaneously touch the same three given circles are distributed pairwise on the perpendiculars dropped from the radical center of the three given circles onto their four axes of similitude."* Thus, if we drop a perpendicular from the radical center to a chosen axis of similitude, we need only one of the intersection points in 3d to find the centers of the solution circles, thereby suggesting a simpler construction. Figure 4 shows the perpendicular τ from the radical center R to the axis of similitude ϵ.

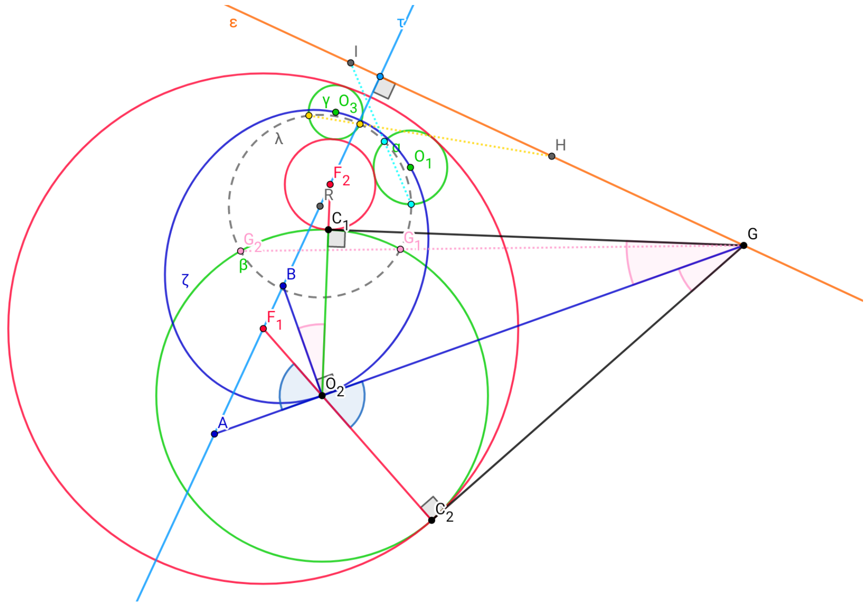


Figure 4. Figure for proof I of theorem 3.1.

3. Conics associated with Apollonius' problem

Theorem 3.1. [1] *The centers of the conjugate solutions to Apollonius' problem are foci of conics. The centers of the given circles are on the conics.*

We present two alternative proofs (I) and (II).

Proof. (I) We will first use the construction discussed in section 2 and focus on the intersection point, G, between the radical axis $\{GG_1G_2\}$ of the circles $\{\beta, \lambda\}$, and one of the axes of similarity, ϵ (see Figure 4).

We may use either of two well known propositions to aid our proof and quote them from [11, p. 83 & p. 92]:
"Proposition V. *The tangent at any point of a central conic makes equal angles with the two focal distances of that point.*"
"Proposition X. *The normal at any point of a central conic bisects the angle between the two focal distances of that point.*"

Make a line from G through the center O_2 of one of the given circles β, and extend it to intersect the line F_1F_2 at A, where $\{F_1, F_2\}$ are the centers of the solution circles. Make a perpendicular to the line GO_2A, at the point O_2 and extend it to meet the line F_1F_2 at B. By the construction in section 2, we have the tangents $\{GC_1, GC_2\}$ to the circle $\beta(O_2)$, giving two congruent right-angled triangles $\{GC_2O_2, GC_1O_2\}$. In figure 4 we see that the angles $\{\widehat{AO_2F_1}, \widehat{GO_2C_2}\}$ are opposite. But these angles are equal to $\widehat{GO_2C_1}$, and thus the line GO_2A makes equal angles with the focal distances $\{O_2F_1, O_2F_2\}$ at the point O_2, and by the proposition V quoted above, is therefore a tangent at the point O_2 to the ellipse ζ (see Figure 4) which is one of the conics of theorem 3.1. Alternatively, we may use proposition X quoted above. We see that the angles $\{\widehat{BO_2F_1}, \widehat{BO_2F_2}\}$ are equal. Therefore

[1] To the best of our knowledge, this theorem is presented for the first time by [1].

the perpendicular O_2B to the line GO_2A at the point O_2, bisects the angle $\widehat{F_1O_2F_2}$ between the two focal distances. Thus the line O_2B is a normal at the point O_2 of the ellipse ζ. In exactly the same way we may prove that the other two given circle centers are on the ellipse ζ, by using the points of intersection H and I between the radical axes (yellow and turquoise dotted lines respectively) and the axis of similitude, ϵ. A similar proof may easily be made if the conic is a hyperbola.

We identify the three points $\{G, H, I\}$ as radical centers respectively of the circle triads $\{(\beta, \lambda, \delta), (\gamma, \lambda, \delta), (\alpha, \lambda, \delta)\}$. Where (α, β, γ) are the three given circles, λ is the radical circle and δ is one of the solution circles (θ being the other, could equally well have been used or included). See figures 4 and 5. The novel and interesting feature we have found in the Casey construction, is that the line joining a radical center, e.g. G, to the center O_2 of its associated given circle β, is tangent to the conic at that center, O_2. (see corollary 3.1). These radical centers also appear in Poncelet's construction [8, pp. 190–194, figs. 4.23 & 4.24], where he identifies them as poles. He uses, not the radical circle λ of the three given circles, but a constructed member of a pencil of circles of which also λ belongs, as do the two solution circles $\{\delta, \theta\}$. The axis of similitude ϵ is their shared radical axis. See also [4, p. 449] regarding this pencil of circles.

Proof. (II) Alternatively, we may use the following definitions:

1. An ellipse is the locus of points for which the sum of their focal distances is constant.

2. A hyperbola is the locus of points for which the difference of their focal distances is constant.

We assume the centers of the conjugate solutions are foci of a conic. In figure 5, we deduce the following sums of focal distances for the three given circles' $\{\alpha(a), \beta(b), \gamma(c)\}$ centers, where $\{R, r\}$ are the radi of the conjugate solutions (red):

$$(R-a)+(r+a) = R+r, \quad (R-b)+(r+b) = R+r, \quad (R-c)+(r+c) = R+r.$$

Thus definition 1 is verified. Similarily for figure 6 we deduce the following differences of focal distances for the three given circles' centers:

$$(a+R)-(a+r) = R-r, \quad (b-r)-(b-R) = R-r, \quad (c+R)-(c+r) = R-r.$$

And we have verified definition 2.

Corollary 3.1. *Let the three given circles $\{\alpha, \beta, \gamma\}$ with respective centers $\{O_1, O_2, O_3\}$ of Apollonius' problem intersect their radical circle λ. Through the six intersection points, draw the three radical axes of the circle pairs $\{(\alpha, \lambda), (\beta, \lambda), (\gamma, \lambda)\}$ to intersect any one of the four axes of similitude of the given circles. Call these intersection points I, G, H respectively. The lines $\{IO_1, GO_2, HO_3\}$, will then be tangent to the conic associated with Apollonius' problem at the given circle's centers $\{O_1, O_2, O_3\}$ respectively.*

This is shown in proof (I) of theorem 3.1.

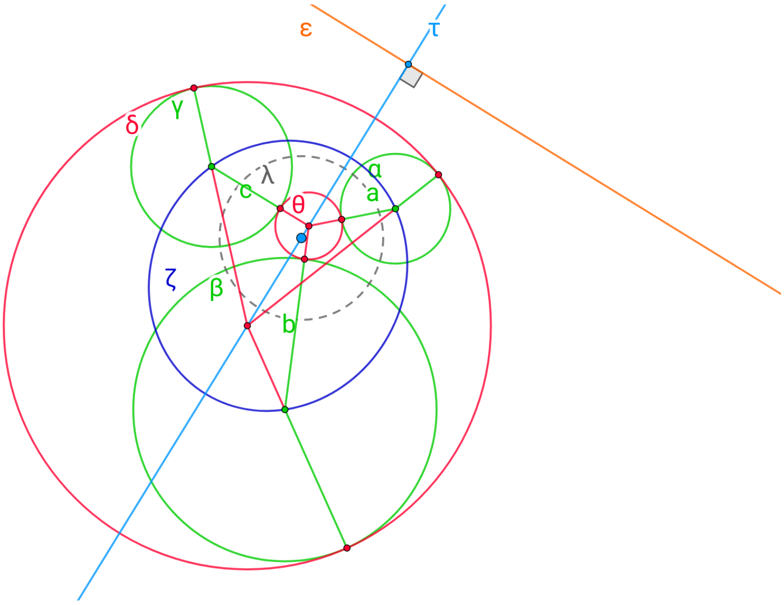


FIGURE 5. Figure for proof (II). Case: ellipse.

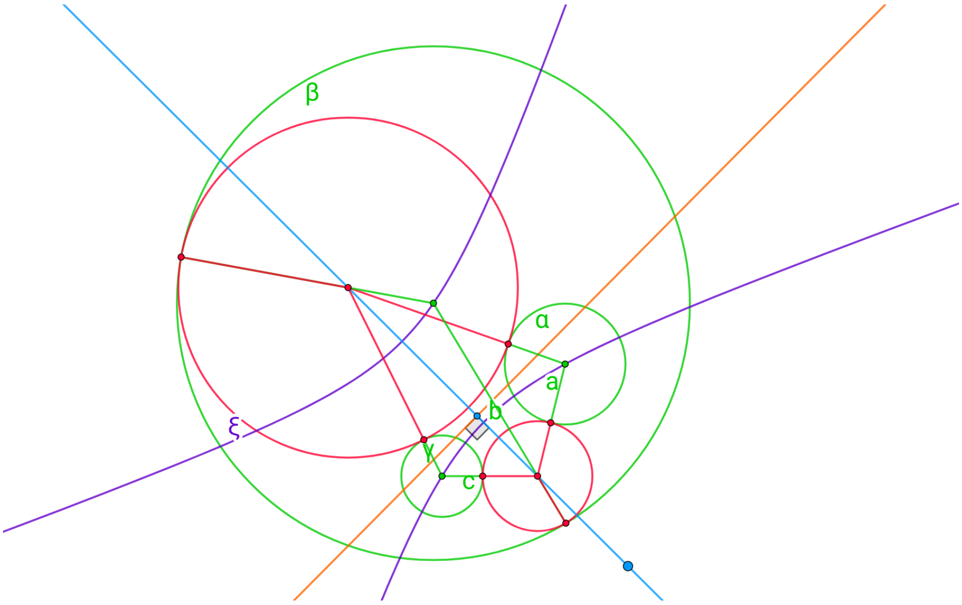
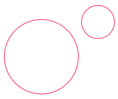
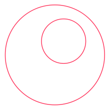
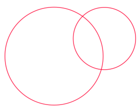
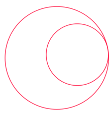
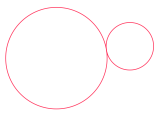


FIGURE 6. Figure for proof (II). Case: hyperbola.

4. Circles tangent to two given circles. Their centers' loci.

We next consider circles tangent to two given circles and their centers' loci. We will use a conjugate pair from the solution to Apollonius' problem as the given two circles. There are five possible configurations of two circles which we show in table 1. Our aim is to categorize which associated conics arise for the different configurations and the result is shown in table 1, which we now proceed to explain. First we quote some well known theorems.

Table 1. Conics associated with the three given circles of Apollonius' problem for all five possible configurations of the conjugate solutions and their centers of similitude.

Conjugate solution	Internal center of similitude	External center of similitude
	Conic	
	Hyperbola	Hyperbola
	Ellipse	Ellipse
	Ellipse	Hyperbola
	Ellipse	Degenerate
	Degenerate	Hyperbola

Theorem 4.1. [7, p. 111] *"If a circle has like contact with two circles, the points of tangency are collinear with the external homothetic center of the two circles; if unlike contact, with the internal center...".* See figure 7.

Regarding the five possible configurations:

Theorem 4.2. [7, p. 112] *"In every case, the circles tangent to two given circles fall into two series, according as they have like or unlike contact with the given circles...".* See figure 8.

Theorem 4.3. [7, p. 113] *"If two circles touch two others, and belong to the same series, the radical axis of either pair passes through the corresponding center of similitude of the other pair".* [7, p. 113].

Theorem 4.4. [9, p. 59] *"The centers of all circles tangent to two fixed circles lie on a conic".*

We note that theorem 4.4 has allusions to theorem 3.1. In figures 7 and 8, we have shown the case when the two fixed circles are intersecting each other. The two different series $\{u_i, v_i\}$ clearly lie on different conics Φ, Ψ, a hyperbola and an ellipse respectively. In table 1 we see cases where both series lie on the same type of conic, but these will be different conics nonetheless. It may easily be shown that the constant sums or differences of the focal distances from the circle centers will be different for each series.

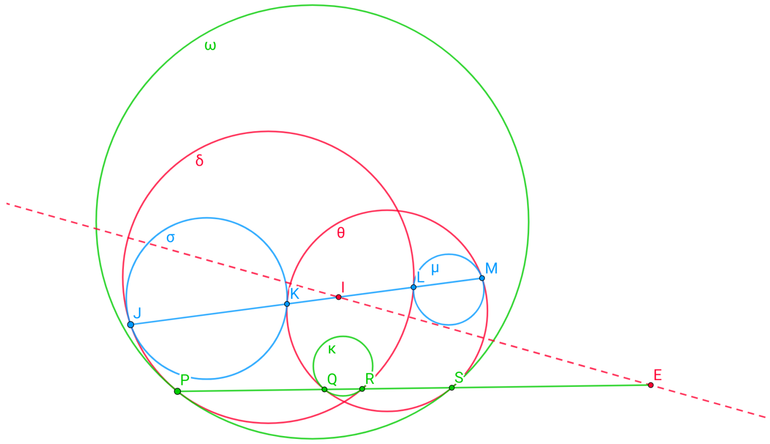


FIGURE 7. Tangent circles $\{\omega, \kappa\}$ and $\{\sigma, \mu\}$ have like and unlike tangencies respectively to the given two circles $\{\delta, \theta\}$. Their tangency points (P, Q, R, S) and (J, K, L, M) are collinear respectively with the external and internal homothetic centers E and I.

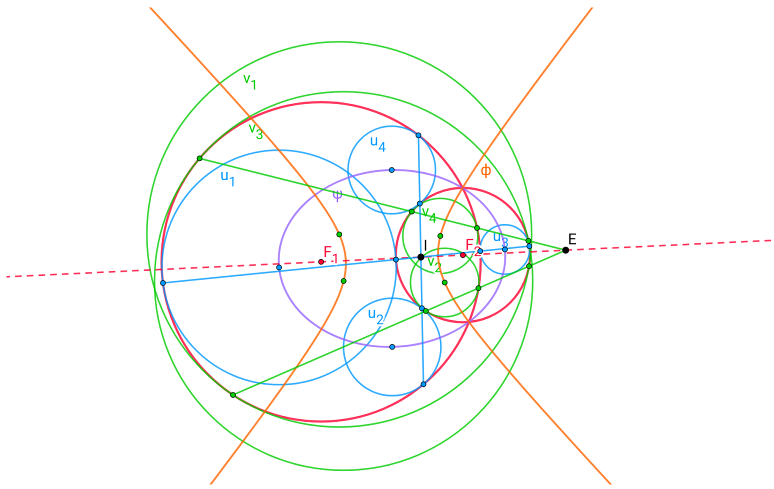


FIGURE 8. Two different series $\{u_i, v_i\}$ of tangent circles to two given intersecting circles. $\{u_i, v_i\}$ = {unlike, like} contact.

Proposition 4.1. [2] *The radical center of the three given circles of Apollonius' problem is a center of similitude for conjugate pairs of solution circles.*

Proof. We have proven that the circle centers of the three given circles lie on the same conic (Theorem 3.1). As a consequence, the circles must then all belong to the same series. Consider pairing two of the three given circles α, β, γ with the solution circles δ, θ, in succession. The radical axis of each pair $\{(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha)\}$ will pass through a center of similitude of the pair

[2]We note that [7, p. 121] mentions this, but not as a theorem or proposition.

(δ, θ) of which there exist only two, one external and one internal. Being of the same series, all three radical axes must necessarily pass through the same center of similitude, and this must also be the radical center of the three given circles. In this proof we have used the the theorems 4.1, 4.2, 4.3, 4.4 above and the subsequent remarks.

Theorem 3.1 and proposition 4.1 establish the relations shown in table 1.

References

[1] Amsterdam, C.M., *Solving Apollonius' Problem Through Relativity*, Journal of Humanities & Social Sciences, **7(8)(2024)** 1–5.
[2] Casey, J., *A sequel to the first six books of the Elements of Euclid, containing an easy introduction to modern geometry, with numerous examples*, London: Longmans, Green, & Co.,1886.
[3] Coolidge, J.L., *A Treatise on the Circle and Sphere*, Clarendon Press, 1916.
[4] Court, N.A., *Historically speaking,- The problem of Apollonius*, The Mathematics Teacher, **54(1961)**, 444–452.
[5] Dörrie, H., *100 Great Problems of Elementary Mathematics, Their History and Solution*, Dover Publications, Inc., New York, 1965.
[6] Gergonne, J.D., *Géométrie analitique. Recherche du cercle qui en touche trois autre sur un plan*, Annales de mathématiques pures et appliquées, **7(1816-1817)** 289–303.
[7] Johnson, R.A., *Advanced Euclidean Geometry*, Dover Publications, Inc. Mineola, New York 2007.
[8] Lorenat, J., *"Die Freude an der Gestalt" : methods, figures and practices in early nineteenth century geometry*, Université Pierre et Marie Curie - Paris VI; Simon Fraser university (Burnaby, Canada), 2015.
[9] Ogilvy, C. S., *Excursions in geometry*, Oxford University Press, New York, 1969.
[10] Pedoe, D., *Circles A Mathematical View*, The Mathematical Association of America (Incorporated), Washington D.C., 1995.
[11] Taylor, C., *An Introduction to the Ancient and Modern Geometry of Conics*, Deighton Bell and Co, Cambridge, 1881.
[12] Winterrink, J., *The Apollonius Tangency Problem*, Pi Mu Epsilon Journal, **10(1979)**, 582–592.

BRATTBAKKEN 124
9018 TROMSØ
NORWAY
E-mail address: `arerikso@gmail.com`