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A REFLECTION ON GEOMETRIC CONSTRUCTIONS AND SYMMETRY. THE CASE OF VARIGNON'S THEOREM

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We dedicate this work to our late colleague and friend Michel Alessandri (1962-2023). He loved mathematics and especially groups in geometry[1]. *We hope he would have enjoyed this work. One of his favourite jokes was the following: "A mathematics teacher asks his students: are the groups S_3 and $\mathbb{Z}/6\mathbb{Z}$ isomorphic? One of the students replies: I think the first is, but not the second". But the funniest thing for Michel was to add: "This is a typical joke that only mathematicians laugh at!"*

Abstract. Throughout this work, Varignon's theorem is used as a guideline to better understand what a geometric construction is, and why symmetry can emerge from a general figure after a prescribed construction. We highlight the role played by barycentric maps in such geometric constructions, and attempt to demystify the emergence of symmetry in a result like Varignon's.

1. Introduction

In geometry, it is not uncommon that, starting from an ordinary element of a certain family (triangle, quadrilateral, etc.) and following a prescribed construction, one obtains a highly symmetrical element from the same family. In other words, an *input* configuration with no particular characteristics can give birth, after some construction, to a very particular *output* configuration. This emergence of *symmetry* [18] from nothing is amazing. Varignon's, Wittenbauer's and Napoleon's theorems [2, 3, 4, 7, 14, 17, 19] are striking examples of this type of phenomenon. In this work on affine geometry, we will focus on Varignon's result (and say a few words about Wittenbauer's). Varignon's theorem [17], a very elementary result at secondary school level, may not be particularly interesting, but if we try to understand what it hides, it contains something very profound. In any case, it is a very classical theorem that is widely quoted [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16].

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Our aim in this work is to highlight what is hidden deep down in Varignon's result and to better understand the emergence of symmetry in it. It seems that, blinded by symmetry - beauty -, we fail to see that any quadrilateral prescribed in advance could emerge from a Varignon's kind construction - even the ugliest, assuming that this aesthetic judgement has any meaning. Metaphorically speaking, parallelograms play the same role among quadrilaterals as particularly remarkable people in a crowd. We would only see these particular people, without seeing the others, even though they are probably more numerous in this crowd. Moreover, this phenomenon is true not only for quadrilaterals, but for any polygon.

This article is divided into four sections, the first of which being this introduction. The second introduces notions of equivariant geometric construction and barycentric map and study their link. It is illustrated by two classical theorems in affine geometry Varignon's [17] and Wittenbauer's [2] theorems. In the third section, after some elements concerning affine equivalence, we present two of our main results. The first and most general result consists in proving that, given a polygon prescribed in advance, there exists a barycentric map that transforms any polygon into another that is affinely equivalent to the prescribed polygon; this is a sort of ultimate extension of Varignon's theorem, in which symmetry has been completely banished. Nevertheless, since symmetrical figures are more attractive, we examine the special case where the output figure is symmetrical and give an exhaustive list of all the barycentric maps that allow us to obtain a Varignon-type result in the generalized context of p-grams instead of just parallelograms. Of course, as a corollary, we obtain all possible Varignon constructions that allow us to obtain a parallelogram as an output figure. The fourth and final section introduces the notion of regular affine polygon and proposes to extend some of the previous results to the case of star polygons.

All these results will be illustrated by numerous figures.

2. Equivariant geometric constructions and barycentric maps

2.1. A natural condition imposed on a geometric construction.

In geometry we meet a lot of different constructions and, actually, geometry is founded on its constructions. Among them, the most amazing are those which create symmetry from nothing. We can cite as examples of such constructions Varignon's, Wittenbauer's, Napoléon's ones. Before to look at symmetry we would like to introduce a condition which seems to be satisfied in many geometric constructions. This condition is that a construction must respect the changes of frame in the sense that two observers which apply the same construction in two different frames must observe the same result. The formalization of that idea is that the construction must be equivariant under the natural diagonal actions of the group which is relevant in the geometry we consider. Because we are only interested in this work by affine geometry this group will be the affine group.

So if $\mathcal{E}$ is an affine space and $GA(\mathcal{E})$ is the affine group consisting in affine transformations of $\mathcal{E}$ we define *a construction* as

Definition 2.1. *A (geometric) construction in the affine plane $\mathcal{E}$ is a map $f : \mathcal{E}^k \to \mathcal{E}^l$ which is equivariant relatively to the natural diagonal actions of* *the affine group respectively on $\mathcal{E}^k$ and $\mathcal{E}^l$:*

$$\forall g \in GA(\mathcal{E}),\ \forall A \in \mathcal{E}^k,\ f(g.A) = g.f(A),$$

so if $f : A = (A_1, \cdots, A_k) \mapsto (f_1(A), \cdots, f_l(A))$,

$$f\Big(g(A_1), \cdots, g(A_k)\Big) = \Big(g(f_1(A)), \cdots, g(f_l(A))\Big)\ (*)$$

which must be satisfied for any k-tuple $A = (A_1, \cdots, A_k)$ of points of $\mathcal{E}$ and for any affine transformation g of $\mathcal{E}$. We saw above that Varignon's constructions verify this property.

We can characterize such constructions in the case where they are assumed to be affine maps. Before that, let us introduce the following definition:

Definition 2.2. *A map $f : \mathcal{E}^k \to \mathcal{E}$ will be called a barycentric map if there is a k-tuple $(\alpha_1, \cdots, \alpha_k)$ of real numbers, with the condition $\alpha_1 + \cdots + \alpha_k = 1$, such that for all*
$A = (A_1, \cdots, A_k) \in \mathcal{E}^k$, $f(A) = \alpha_1 A_1 + \cdots + \alpha_k A_k$. Such a k-tuple $(\alpha_1, \cdots, \alpha_k)$ will be called a weight associated to the barycentric map f.

Proposition 2.1. *Let us assume that $k > n+1$, where $n = dim\ \mathcal{E}$. Then an affine map $f : \mathcal{E}^k \to \mathcal{E}^l$ is a construction if and only if each of its components is a barycentric map for some weight.*

Proof.
Let us denote E the vector space which is the direction of the affine space $\mathcal{E}$ and $GA(\mathcal{E})$ the affine group of $\mathcal{E}$.
Let $A = (A_1, \cdots, A_k)$ and $B = (B_1, \cdots, B_k)$ be points in $\mathcal{E}^k$. Because f is assumed to be affine from the product affine space $\mathcal{E}^k$ to the affine space $\mathcal{E}$, we can write

$$\overrightarrow{f(A)f(B)} = \overrightarrow{f}(\overrightarrow{AB}) = \overrightarrow{f}(\overrightarrow{A_1B_1}, \cdots, \overrightarrow{A_kB_k}).$$

Now the assumption that f is a construction implies that for any $g \in GA(\mathcal{E})$ and any $(v_1, \cdots, v_k) \in E^k$,

$$\overrightarrow{g} \circ \overrightarrow{f}(v_1, \cdots, v_k) = \overrightarrow{f}(\overrightarrow{g}(v_1), \cdots, \overrightarrow{g}(v_k)).$$

So, denoting for each $i \in [\![1, k]\!]$ $\overrightarrow{f_i}$ the linear map defined by

$$\overrightarrow{f_i}(v) = \overrightarrow{f}(0_E, \cdots, v, \cdots, 0_E)$$

where vector v is at the ith argument, we get that each of these maps belongs to the center of the linear group $GL(E)$ and so is an homothety. It results the existence of k real numbers $\alpha_1, \cdots, \alpha_k$ such that for all k-tuples $A = (A_1, \cdots, A_k) \in \mathcal{E}^k$ and $B = (B_1, \cdots, B_k) \in \mathcal{E}^k$,

$$\overrightarrow{f(A)f(B)} = \sum_{i=1}^{k} \alpha_i \overrightarrow{A_iB_i}, \text{ or equivalently, } f(B) = f(A) + \sum_{i=1}^{k} \alpha_i \overrightarrow{A_iB_i}.$$

Now, using the property of constructions in the specific case of a translation by vector $U = (u, \cdots, u)$, $u \in E$, $u \neq 0_E$ and applying the above formula to $B = A + U$, we get that $f(A + U) = f(A) + u$. So we get the equality $f(A) + u = f(A) + \sum_{i=1}^k \alpha_i u$, and so $\displaystyle\sum_{i=1}^{k} \alpha_i = 1$. Now, applying the previous formula to any A and $B = g(A)$, g being an arbitrary element of $GA(\mathcal{E})$, we get that

$$g(f(A)) = f(g(A)) = f(A) + \sum_{i=1}^{k} \alpha_i \overrightarrow{A_i g(A_i)}.$$

Let us denote G_A the barycentre of weighted points $(A_1, \alpha_1), \cdots, (A_k, \alpha_k)$ so $G_A := \sum_{i=1}^{k} \alpha_i A_i$. We assert that $f(A) = G_A$ which will complete our proof. Indeed, for all $A \in \mathcal{E}^k$ and $g \in GA(\mathcal{E})$,

$$\overrightarrow{g}(\overrightarrow{G_A f(A)}) = \overrightarrow{g(G_A) g(f(A))} = \overrightarrow{g(G_A) f(A)} + \sum_{i=1}^{k} \alpha_i \overrightarrow{A_i g(A_i)},$$

according to the previous formula. So, since $g(G_A) = \sum_{i=1}^{k} \alpha_i g(A_i)$ we can continue and obtain

$$\overrightarrow{g}(\overrightarrow{G_A f(A)}) = \sum_{i=1}^{k} \alpha_i \overrightarrow{g(A_i) f(A)} + \sum_{i=1}^{k} \alpha_i \overrightarrow{A_i g(A_i)} = \sum_{i=1}^{k} \alpha_i \overrightarrow{A_i f(A)} = \overrightarrow{G_A f(A)}.$$

Therefore vector $\overrightarrow{G_A f(A)}$ is fixed by any element of the linear group $GL(E)$ so it is zero and therefore $f(A) = G_A = \sum_{i=1}^{k} \alpha_i A_i$ as announced.

Warning: In this work we will consider quadrilaterals and more generally polygons. We highlight the fact that there is a deep difference between the k-tuple consisting in the vertices of a k-gon and the k-gon itself. Indeed, let us recall the definition given by Coxeter in [3]:

"A *polygon* may be defined as consisting of a number of points (called vertices) and an equal number of line segments (called sides), namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points. In other words, a polygon is a closed broken line lying in a plane."

From this definition, a k-gon with vertices $A_1, \cdots, A_k$, that we will denote $A_1 \cdots A_k$, corresponds to the k-tuple $(A_1, \cdots, A_k)$ read in this cyclic order.

In what follows, we will insist on the "no three successive collinear points" condition, which is fundamental to our work, by calling such polygons "non-degenerate polygons".

2.2. **The examples of Varignon's and Wittenbauer's constructions.**

Varignon's theorem claims that if $ABCD$ is a quadrilateral then the quadrilateral $IJKL$, where I, J, K, L are the midpoints of sides $[AB]$, $[BC]$, $[CD]$, $[DA]$, is a parallelogram. Varignon's construction can be written as the barycentric map

$$(A, B, C, D) \mapsto \left(I = \frac{A+B}{2}, J = \frac{B+C}{2}, K = \frac{C+D}{2}, L = \frac{D+A}{2} \right).$$

We can give two other barycentric maps associating to any quadrilateral a parallelogram, for instance:

$(A, B, C, D) \mapsto (I = A, J = B, K = C, L = A - B + C)$, or

$$(A, B, C, D) \mapsto \left(I = \frac{A+2B}{3}, J = \frac{2B+C}{3}, K = \frac{C+2D}{3}, L = \frac{A+2D}{3}\right).$$

These three barycentric constructions are represented on the figure below.

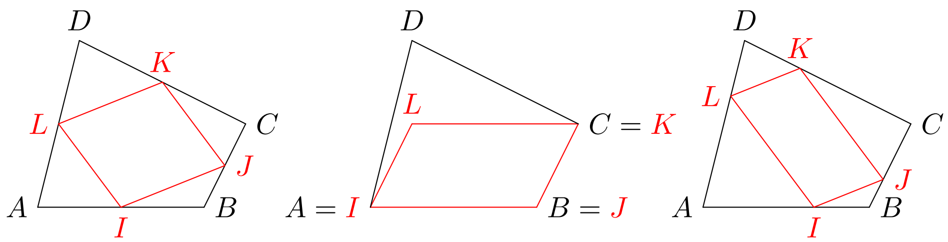

FIGURE 1. The original Varignon's construction and two other barycentric constructions

Remark 2.1. *The original Varignon's map has the following property: a cyclic permutation of points A, B, C, D induces the same cyclic permutation of points I, J, K, L. This property is not verified by the other two maps.*

It is easy to prove that, actually, there is a infinity of ways to provide barycentric maps including Varignon's one which allow to get a parallelogram from any quadrilateral. Indeed, for any real numbers s and t, the barycentric map $f : (A, B, C, D) \mapsto (A', B', C', D')$, where

$$A' = sA+tB+(1/2-s)C+(1/2-t)D,\ B' = tA+(1/2-s)B+(1/2-t)C+sD,$$

$$C' = (1/2-s)A+(1/2-t)B+sC+tD,\ D' = (1/2-t)A+sB+tC+(1/2-s)D),$$

maps any quadrilateral onto a parallelogram.

Another theorem that closely resembles to Varignon's one is due to Wittenbauer [2, 19]. It states that if you divide the sides of a quadrilateral $ABCD$ into three equal parts and draw lines joining the two closest points to A, the two closest points to B, etc., then the four points I, J, K, L, intersection points of these lines, form a parallelogram. The proof is simple (using Thales's intercept theorem).

Nevertheless, to better understand the difference between Wittenbauer's and Varignon's theorems, it may be useful to give the points I, J, K, L as functions of A, B, C, D.

Let $ABCD$ be a quadrilateral[1] and let us denote (b, d) the coordinates of C in the frame $(A; \overrightarrow{AB}, \overrightarrow{AD})$. Easily $\overrightarrow{AC} = b\overrightarrow{AB} + d\overrightarrow{AD}$, or $C(b, d)$. With the notations used in the figure, we clearly get $A'(1/3, 0)$, $A''(2/3, 0)$, $D''(0, 1/3)$ and $B'((2+b)/3, d/3)$. It results that the line equations of $(A'D'')$ and $(B'A'')$ are respectively $x + y = 1/3$ and $by = d(x - 2/3)$. So the point I

[1] We assume that any three of its vertices are non-collinear; we will come back to this important hypothesis in next section.

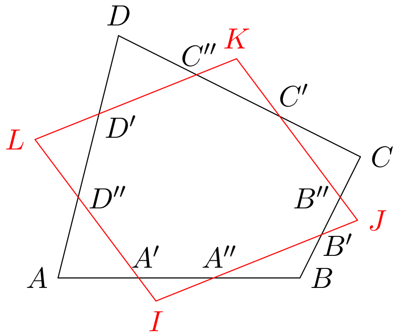

FIGURE 2. Wittenbauer's construction.

defined by $(A'D'') \cap (B'A'')$ is $I\left(\frac{b+2d}{3(b+d)}, \frac{-d}{3(b+d)}\right)$. From I we easily obtain the three other points J, K, L. Indeed, an easy calculation gives that

$$J = I + \frac{2}{3}\overrightarrow{AC},\ K = J + \frac{2}{3}\overrightarrow{BD},\ L = K + \frac{2}{3}\overrightarrow{CA},\ I = L + \frac{2}{3}\overrightarrow{DB}.$$

A number of comments are in order when comparing this result with Varignon's. As in the original Varignon's case, a cyclic permutation of the points A, B, C, D induces the same permutation of the points I, J, K, L. On the other hand, there is one major difference with previous cases involving a Varignon-type construction: here point I, and the others J, K, L, does not depend affinely on the initial points A, B, C, D. However, if g is an affine transformation, then the Wittenbauer's construction applied to points $g(A), g(B), g(C), g(D)$ gives the points $g(I), g(J), g(K), g(L)$, where I, J, K, L are associated to A, B, C, D, and so the equivariance condition $(*)$ is satisfied.

Before concluding these examples, let us explain one last thing. Just as we have seen a few variants of Varignon's original construction, we can easily obtain various Wittenbauer-type constructions. Indeed, instead of cutting each side into three equal parts, we could cut them according to a different barycentric distribution. Namely, for $t \in [0,1]$, let us define

$$A' = (1-t)A+tB,\ A'' = tA+(1-t)B,\ B' = (1-t)B+tC,\ B'' = tB+(1-t)C,$$

$$C' = (1-t)C+tD,\ C'' = tC+(1-t)D, D' = (1-t)D+tA,\ D'' = tD+(1-t)A.$$

It is easy to check that the four points I, J, K, L defined respectively by $(D''A') \cap (A''B')$, $(A''B') \cap (B''C')$, $(B''C') \cap (C''D')$ and $(C''D') \cap (D''A')$ define a parallelogram; the original Wittenbauer's theorem is for $t = 1/3$. In the frame $(A; \overrightarrow{AB}, \overrightarrow{AD})$, if $C(b,d)$, then $I\left(\frac{tb+(1-t)d}{b+d}, \frac{(2t-1)d}{b+d}\right)$. It is clear that Varignon's original construction is nothing more than the case $t = 1/2$ of this Wittenbauer's family constructions.

Remark 2.2. *In the Wittenbauer construction, there are two steps: first, eight points are selected according to a barycentric map, and then lines joining some of these points are cut. The resulting constructed points do not depend affinely on the initial points. The construction is equivariant but not affine, and is therefore not given by a barycentric map.*

3. General Varignon's constructions for polygons

We have already seen that the construction used in Varignon's theorem is given by a barycentric map, and we have given other barycentric maps producing the same result: a parallelogram as the output figure, whatever the quadrilateral chosen as input. In this section, devoted to our main results, we will examine two situations in turn. The first demystifies the emergence of symmetry in Varignon's theorem, as we will prove that any figure can appear in the output, at least when viewed with affine equivalence. The second, on the other hand, determines all the barycentric maps leading to a symmetrical output figure. All these results will be considered in the general case of polygons, not just quadrilaterals. In this generalized framework, figures playing the role of parallelograms will be special polygons called *p-grams*.

From now, we will only work in a affine plane that we will denote $\mathcal{P}$.

3.1. **A few words on affine equivalence.**

Different figures in the plane can be considered identical if you forget their non-essential peculiarities in relation to your problem and instead focus on the essential commonalities they share for your question. For example, all non-degenerate triangles can be considered identical or, more precisely, as avatars of one of them, in the sense that two of them can be exchanged using an affine transformation. Of course this property does not go on with a k-gon, $k \geq 4$.

Let us be more precise. If k is some fixed positive integer, we will say that two k-tuples $A = (A_1, \cdots, A_k)$ and $B = (B_1, \cdots, B_k)$ of $\mathcal{P}^k$ are *affinely equivalent* if there exists an affine transformation g such that for any $i \in [\![1, k]\!]$, $g(A_i) = B_i$. Of course, this property does not depend on the cyclic order of the points and can therefore be extended to non-degenerate polygons. We can summarize by the following definition:

Definition 3.1. *Two non-degenerate k-gons $A_1 \cdots A_k$ and $B_1 \cdots B_k$ will be said affinely equivalent if there exists an affine tranformation g such that $B_1 \cdots B_k = g(A_1) \cdots g(A_k)$.*

The following proposition gives an easy result on affine equivalence.

Proposition 3.1. *Let (A, B, C, D) be a non-degenerate 4-tuple. Then it is affinely equivalent to (B, C, D, A) if and only if A, B, C, D are the vertices of a parallelogram.*

Proof. Let us assume that (B, C, D, A) is affinely equivalent to (A, B, C, D). We can work with coordinates in the frame $(A; \overrightarrow{AB}, \overrightarrow{AD})$ where $A(0, 0)$, $B(1, 0)$, $D(0, 1)$, $C(x_0, y_0)$. Using that there exists an affine transformation g (given here analytically by $(x, y) \mapsto (ax + by + \alpha, cx + dy + \beta)$) such that $g(A) = B, g(B) = C, g(C) = D, g(D) = A$, one easily gets that $x_0 = y_0 = 1$ and so $C(1, 1)$. It results that the diagonals $[AC]$ and $[BD]$ have the same midpoint, namely $I(1/2, 1/2)$. Conversely, let us assume that (A, B, C, D) is a parallelogram and let us denote I the common midpoint of its diagonals. Again we can work with coordinates and choose a frame $(I; \overrightarrow{u}, \overrightarrow{v})$ such that $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$. In this case, the affine transformation (given in coordinates), $(x, y) \mapsto (-y, x)$ realizes an affine equivalence between (A, B, C, D) and (B, C, D, A).

It results from this statement that, parallelograms have a symmetry group of order 8 (in the affine group). More precisely, cyclic vertices permutation is one of the affine transformations that leaves the parallelogram globally invariant. It, and its inverse, are special in that the six others are identity and five affine symmetries (four are axial symmetries and one is a point reflection). As already mentioned in the introduction these eight elements form a subgroup of the affine group, which is isomorphic to the dihedral group D_8.

What about the set of non-degenerate quadrilaterals that we are particularly interested in here? It can be divided into three parts: parallelograms, non-parallelogram trapezoids and others (general quadrilaterals). All parallelograms are affinely equivalent. Of course, it is not the case for trapezoids. We will precise this in what follows.

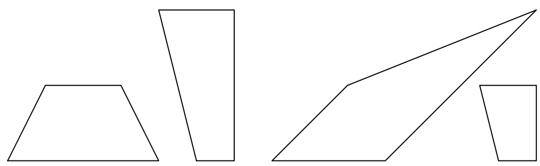

FIGURE 3. What trapezoids are affinely equivalent?

Definition 3.2. *For a trapezoid, we can define a common algebraic measure along the parallel sides and so the ratio of the algebraic measures of the parallel sides. The ratio of the smallest by the largest*[2] *will be called the characteristic invariant of the trapezoid.*

Proposition 3.2. *Two trapezoids are affinely equivalent if and only if they have the same characteristic invariant.*

Proof. We do not look at the special case of a parallelogram because it is a obvious equivalence.

The equality of their characteristic invariant is clearly a necessary condition to their affine equivalence.

Conversely, let $ABCD$ and $A'B'C'D'$ be two trapezoids which are not parallelograms. Let us suppose that the parallel sides are $(AB) \parallel (CD)$ and $(A'B') \parallel (C'D')$. Without loss of generality, we can assume that $A' = A$, $B' = B$ and $D' = D$, even if it means using an affine transformation sending (A, B, D) onto (A', B', D'). Because now $\overline{A'B'} = \overline{AB}$ the equality of the two characteristic invariants gives that $\overline{D'C'} = \overline{DC}$ and so $C' = C$ because $D' = D$.

[2]Even if this ratio is an algebraic quantity that can be positive or negative, these terms of comparison must be understood in the sense of comparing absolute values. For a parallelogram, this definition is unambiguous, since for any pair of sides, this ratio is equal to 1.

3.2. **Non-symmetrical Varignon's theorem.**

The presence of a parallelogram as the output figure in Varignon's original theorem conceals a multitude of possibilities and that the parallelogram is just one of many figures.

Theorem 3.1. *Let $A_1A_2\cdots A_k$ be a fixed non-degenerate k-gon. Then there exists a barycentric map f such that for any non-degenerate k-gon $P_1P_2\cdots P_k$, the k-gon defined by the vertices of the k-tuple $f(P_1, P_2, \cdots, P_k)$ is affinely equivalent to $A_1A_2\cdots A_k$.*

Proof. The non-degeneracy assumption guarantees that (A_1, A_2, A_3) is an affine frame of the plane $\mathcal{P}$. So, for any $i \in [\![4, k]\!]$ there are real numbers $\alpha_i^1, \alpha_i^2, \alpha_i^3$ such that $\alpha_i^1 + \alpha_i^2 + \alpha_i^3 = 1$ and $A_i = \alpha_i^1 A_1 + \alpha_i^2 A_2 + \alpha_i^3 A_3$.

Let us recursively define the barycentric map $f : \mathcal{P}^k \to \mathcal{P}^k$ by the formulae $f(P_1, \cdots, P_k) = (P_1', \cdots, P_k')$ where the P_i' are $P_1' = P_1, P_2' = P_2, P_3' = P_3$ and for $i \in [\![4, k]\!]$, $P_i' = \alpha_i^1 P_1' + \alpha_i^2 P_2' + \alpha_i^3 P_3'$. Because all the considered k-gons are assumed to be non-degenerate, for any k-gon $P_1P_2\cdots P_k$ there is some $g \in GA(\mathcal{P})$ such that $g.(P_1, P_2, P_3) = (A_1, A_2, A_3)$. In this case, because g is an affine transformation, $g.f(P_1, P_2, \cdots, P_k) = (A_1, A_2, \cdots, A_k)$. It remains to verify that using this map $f : \mathcal{P}^k \to \mathcal{P}^k$, we can construct a map from the set of k-gons to the set of classes of k-gons modulo the affine equivalence. Indeed, let us consider the k-cycle $c = (12\cdots k) \in C_k$, where C_k denotes the cyclic group of order k generated by c. Then $f(c.(P_1, P_2, \cdots, P_k))$ is equal to $(P_2, P_3, P_4, \alpha_4^1 P_2 + \alpha_4^2 P_3 + \alpha_4^3 P_4, \cdots, \alpha_k^1 P_2 + \alpha_k^2 P_3 + \alpha_k^3 P_4)$. This last one is not equal to $c.f(P_1, P_2, \cdots, P_k)$ but, because of the non-degeneracy assumption, (P_2, P_3, P_4) is an affine frame. It results that it exists $g' \in GA(\mathcal{P})$ such that $(P_2, P_3, P_4) = g'.(P_1, P_2, P_3)$ and so finally $f(c.(P_1, P_2, \cdots, P_k)) = g'.f(P_1, P_2, \cdots, P_k)$. Thus $f(c.(P_1, P_2, \cdots, P_k))$ and $f(P_1, P_2, \cdots, P_k)$ are affinely equivalent. So, finally, we get that the k-tuple $f(P_1, \cdots, P_k)$ defines the vertices of a k-gon affinely equivalent to $A_1 \cdots A_k$.

We have thus obtained a general method for constructing from any non-degenerate k-gon another that is affinely equivalent to an initially prescribed k-gon with a given level of symmetry. Let us just note that the barycentric map given here does not satisfy any cyclic condition.

3.3. **Symmetrical Varignon's theorem.**

We are now going to look at a geometric object that generalises the parallelogram and is called a p-gram. Among polygons with an even number of vertices, it can be considered highly symmetrical. It was introduced and studied in [15] with the following definition:

Definition 3.3. *A $2k$-gon $A_1\cdots A_{2k}$ is called a p-gram (or a k-p-gram if we want to precise the number of its sides), if its "opposite sides" are equal in the sense that for all $i \in [\![1, k]\!]$, $\overrightarrow{A_iA_{i+1}} = \overrightarrow{A_{k+i+1}A_{k+i}}$, with the convention that $A_{2k+1} = A_1$.*

Remark 3.1.

(1) *For instance a parallelogram is a 2-p-gram*[3].

[3] The notion of a parallelogram and, more generally, of a p-gram is invariant to cyclic permutations of vertices and is therefore well defined on polygons.

(2) *In the definition of* [15], *a* k*-p-gram is defined with* k *distinct vertices. Here, for the convenience of one of our statements, we will also consider the "degenerate" case where several vertices may be identical. For example,* $AABB$ *can be considered a degenerate parallelogram.*
(3) *The conditions of the previous definition are equivalent to say that for all* $i \in [\![1, k]\!]$, $\frac{1}{2}(A_i + A_{i+k}) = \frac{1}{2}(A_{i+1} + A_{i+1+k})$.

In section 2. we have presented a few barycentric maps which give a Varignon-type result. A natural problem is to find *all* the barycentric maps that would associate a parallelogram to a quadrilateral as Varignon did with his particular construction. In fact, here, we will deal with a more general problem by replacing parallelograms by p-grams. Varignon's theorem will therefore be extended in two ways, firstly by considering p-grams instead of simple parallelograms, and secondly by searching exhaustively for all possible sets of weights defining a barycentric map which allows any non-degenerate $2k$-gon to be sent onto a k-p-gram.

Before stating the following theorem, let us point out that the weight w defining a barycentric map $f_w : \mathcal{P}^{2k} \to \mathcal{P}^{2k}$ is given as a $2k$-tuple $w = (\alpha_i)_{1\le i\le 2k}$ where each α_i is itself a $2k$-tuple of real numbers, which we can denote $\alpha_i = (\alpha_i^1, \cdots, \alpha_i^{2k})$. Thus, such a weight w can be seen as a matrix of $2k$ rows and $2k$ columns with the condition that the sum of the coefficients of each column is equal to 1. Moreover, such a weight w will be said *cyclic* if the associated barycentric map f_w verifies that for all $A \in \mathcal{P}^{2k}$, $f_w(c.A) = c.f_w(A)$, where c denotes the $2k$-cycle $(12\cdots 2k)$ and for all $M = (M_1, \cdots, M_{2k})$, $c.M = (M_2, \cdots, M_{2k}, M_1)$.

The following remark will be useful and frequently used throughout this article.

Remark 3.2. *An affine relation satisfied by the image points of a barycentric map* f_w *gives the same relation between the components of* w.

Theorem 3.2. *Let* k *be a positive integer.*

1. A barycentric map f_w *associated to a cyclic-weight* $w = (\alpha_i)_{1\le i\le 2k}$ *transforms the vertices of any* $2k$*-gon into the vertices of a* k*-p-gram if and only if for all* i *in* $[\![1, k-1]\!]$, $\alpha_i + \alpha_{i+k} = \alpha_{i+1} + \alpha_{i+1+k}$.

2. A barycentric map f_w *associated to the cyclic weight* $w = (\alpha_i)_{1\le i\le 2k}$ *transforms the vertices of any* $2k$*-gon into the vertices of a* k*-p-gram if, and only if,* $w = (\alpha, c.\alpha, \cdots, c^{2k-1}.\alpha)$*, where* c *is the* $2k$*-cycle* $(12\cdots(2k))$*, and* α *belongs to the* k*-parameters family* $\mathcal{W}$ *defined as*

$$\mathcal{W} := \left\{ \left(t_1, \cdots, t_k, \frac{1}{k} - t_1, \cdots, \frac{1}{k} - t_k\right),\ (t_1, \cdots, t_k) \in \mathbb{R}^k \right\}.$$

Proof.

1. Let us first prove that these conditions are necessary. Let f_w be a barycentric map with weight $w = (\alpha_i)_{1\le i\le 2k}$. Let us put $\alpha_i = (\alpha_i^1, \cdots, \alpha_i^{2k})$ with $\sum_{j=1}^{2k} \alpha_i^j = 1$. If $A_1 \cdots A_{2k}$ is a non-degenerate $2k$-gon, we will denote $A'_1 \cdots A'_{2k}$ its image by the map f_w. We want $A'_1 \cdots A'_{2k}$ to be a p-gram for any choice of the input $2k$-gon $A_1 \cdots A_{2k}$. Thus, we must satisfy, according the equalities of the third point of remark 3.1, the equalities

$$\forall i \in [\![1, k-1]\!],\ \frac{1}{2}(A'_i + A'_{i+k}) = \frac{1}{2}(A'_{i+1} + A'_{i+1+k}),$$

where $A'_i = \sum_{j=1}^{2k} \alpha_i^j A_j$, and so

$$\forall i \in [\![1, k-1]\!],\ \frac{1}{2}\sum_{j=1}^{2k}(\alpha_i^j + \alpha_{i+k}^j)A_j = \frac{1}{2}\sum_{j=1}^{2k}(\alpha_{i+1}^j + \alpha_{i+1+k}^j)A_j.$$

Thanks to the non-degeneracy assumption made on our polygons, these equalities are in fact valid more generally for any $2k$-tuples. This allows us to identify the coefficients in both members of these equalities using the following easy-to-check uniqueness lemma:

Lemma 3.1. *Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ two n-tuples of real numbers such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. If for any n-tuples $(P_1, \cdots, P_n)$ of points of the plane we have the equality $\displaystyle\sum_{i=1}^n \alpha_i P_i = \sum_{i=1}^n \beta_i P_i$ then $\alpha = \beta$.*

Conversely, if these conditions on the weights are satisfied, we can verify that the image of any $2k$-gon by such a barycentric map is a k-p-gram, possibly degenerate in the sense of the second point of remark 3.1. Note that these algebraic relations are exactly the same as those satisfied by the vertices of the target image, as already pointed out in the remark 3.2.

2. The cyclicity condition on f_w means that the matrix $(\alpha_i^j)_{1\le i,j\le 2k}$ satisfies the following relations:

$$\forall (i,j) \in [\![1, 2k]\!]^2,\ \alpha_i^{j-1} = \alpha_{i+1}^j,$$

with the usual convention of circular identification of indices. These relations imply that if the first row is given, the other ones are obtained by cyclic permution of it. So, if we denote $\alpha_1 = (a_1, \cdots, a_{2k})$ this first row, the property of the first claim of this theorem implies the equalities:

$$a_1 + a_{1+k} = a_2 + a_{2+k} = \cdots = a_k + a_{2k}.$$

Now, because of the normalization conditions of weights, the sum of the elements of each row (and now column) is equal to 1, so when we sum the k previous equal terms we must get 1 and so each of them is equal to $1/k$. Thus, we can take the k first entries of α_1 as free parameters and get the k last ones as indicated in the statement.

Corollary 3.1. ***A family of Varignon's theorems***

(1) *In the case $k = 2$, we obtain all the barycentric maps that transform any quadrilateral into a parallelogram. Precisely, they are the maps f_w with $w = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$.*
(2) *Moreover, if w is supposed cyclic, its components are the 4-tuples $(s, t, 1/2 - s, 1/2 - t)$, $(t, 1/2 - s, 1/2 - t, s)$, $(1/2 - s, 1/2 - t, s, t)$, $(1/2 - t, s, t, 1/2 - s)$, where s and t are arbitrary real numbers.*

Remark 3.3. *Among this family, we can find the three examples of section 2, in particular the original Varignon's theorem. Moreover, as in Varignon's result, initial quadrilateral and associated parallelogram have the same centroid. More generally, this last property is also true for any construction given in claim 2. theorem 3.2.*

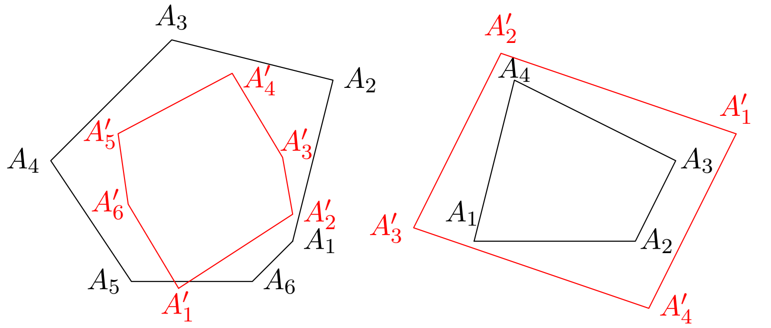

FIGURE 4. Illustration of theorem 3.2 and corollary 3.1 respectively for weights $(1/2, -1/3, 1/4, -1/6, 2/3, 1/12)$ and $(-1/2, 1/3, 1, 1/6)$.

Taking $t_1 = \cdots = t_k = 1/k$, we obtain a corollary which is nothing else than Theorem 3 in [15]:

Corollary 3.2. *The set of centroids of consecutive k-tuples of vertices of a $2k$-gon is a k-p-gram.*

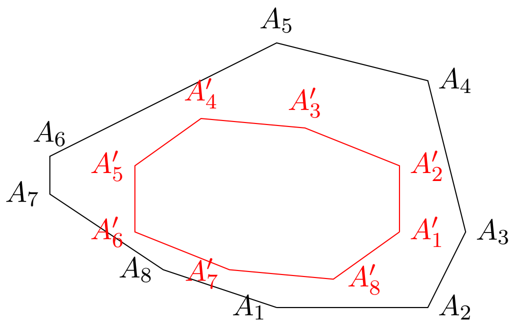

FIGURE 5. Illustration of corollary 3.2 in the case $k = 4$, where the A'_i are cyclically the centroids of four consecutive vertices.

4. Cyclic barycentric maps for affine regular polygons

In this last section, we deal with regular polygons. The standard notion of *regular polygon* is a Euclidean one, since it concerns lengths and angles, and therefore has nothing to do with our subject. Nevertheless, we are now going to define an affine notion of regularity, which we will be working with here.

4.1. What is a regular polygon in affine context?

Theorem and definition 4.1. *Let $A_1 ... A_n$ be a non-degenerate polygon with n vertices. The two following assertions are equivalent:*

(1) *It exists an affine transformation g of the plane such that for all $i \in [\![1, n]\!]$, $g(A_i) = A_{i+1}$, with the convention $A_{n+1} = A_1$. In other*

words, the cyclic action on the vertices is realizable by an affine transformation.

(2) *It exists an affine transformation φ of the plane such that the polygon $\varphi(A_1)\cdots\varphi(A_n)$ is a regular euclidean polygon.*

A polygon verifying one of these equivalent properties will be called an affine regular polygon*.*

Proof.

(1. $\Longrightarrow$ 2.) Let g be such an affine transformation. Then the centroid of the polygon is a fixed point of g. So we can treat our problem as if g were a plane linear transformation of order n. Let us denote M its matrix in fixed vectorial basis. Then M is similar to one of the three types $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, $\begin{pmatrix}\rho\cos\theta & -\rho\sin\theta\\ \rho\sin\theta & \rho\cos\theta\end{pmatrix}$, where λ, μ, θ are real numbers and $\rho > 0$. The conditions $M^n = I_2$ and $n \geq 3$ imply that the only possible case is that M is similar to a matrix of third type with $\rho = 1$ and $\theta = 2p\pi/n$ with p an integer relatively prime to n. This leads to property 2.

(2. $\Longrightarrow$ 1.) Let r be a rotation of angle $2p\pi/n$, with p relatively prime with n. If φ satisfies property 2, then the affine transformation g defined by $g = \varphi^{-1} \circ r \circ \varphi$ satisfies property 1.

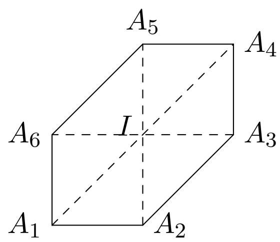

FIGURE 6. A regular 3-p-gram.

Proposition 4.1. *The three following properties are equivalent for any* 3-*p-gram* $A_1A_2A_3A_4A_5A_6$*:*

(1) *It is affinely regular.*
(2) *Each of its diagonals* $[A_1A_4], [A_2A_5], [A_3A_6]$ *is parallel to the two non-adjacent sides.*
(3) $\overrightarrow{A_1A_2} = \overrightarrow{A_5A_4} = \frac{1}{2}\overrightarrow{A_6A_3}$, $\overrightarrow{A_2A_3} = \overrightarrow{A_6A_5} = \frac{1}{2}\overrightarrow{A_1A_4}$, $\overrightarrow{A_3A_4} = \overrightarrow{A_1A_6} = \frac{1}{2}\overrightarrow{A_2A_5}$.

This result is easy to prove. We can remark that, in its claim 2, as soon as the condition is met for two of these diagonals, it is also met for the third.

All regular 3-p-grams are affinely equivalent, but without the regularity condition this is no longer true.

Indeed, let $A_1A_2A_3A_4A_5A_6$ and $B_1B_2B_3B_4B_5B_6$ be two 3-p-grams such that $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are trapezoids. If these two trapezoids are not affinely equivalent[4], then the two 3-p-grams $A_1A_2A_3A_4A_5A_6$ and $B_1B_2B_3B_4B_5B_6$ cannot be affinely equivalent.

[4] We leave it to the reader to construct examples of such situations.

Regular 3-p-grams are to 6-gons what parallelograms are to quadrilaterals: the most *symmetrical* of them.

Corollary 4.1. *The output* 3*-p-gram associated to a cyclic barycentric construction of the type described by claim 2 theorem 3.2 is regular if and only if* $t_1 - t_2 + t_3 = \frac{1}{6}$.

Proof.

Let $A_1A_2A_3A_4A_5A_6$ be any non-degenerate 6-gon and $A'_1A'_2A'_3A'_4A'_5A'_6$ the output 3-p-gram. Following Theorem 3.2 we can write

$$A'_1 = t_1A_1 + t_2A_2 + t_3A_3 + \left(\frac{1}{3} - t_1\right) A_4 + \left(\frac{1}{3} - t_2\right) A_5 + \left(\frac{1}{3} - t_3\right) A_6,$$

and the other points $A'_2, A'_3, A'_4, A'_5, A'_6$ follow the same formulae obtained cyclically. The necessary and sufficient condition for which $A'_1A'_2A'_3A'_4A'_5A'_6$ is regular is given by the equalities $\overrightarrow{A'_1A'_4} = 2\overrightarrow{A'_6A'_5}$ and $\overrightarrow{A'_6A'_3} = 2\overrightarrow{A'_1A'_2}$. The non-degeneracy condition allowing to identify coefficient in the two members, we get easily the announced condition.

Let us consider a regular 3-p-gram as the image by an affine transformation of a Euclidean regular hexagon $A_1 \cdots A_6$. In this case, it is easy to verify that for $k = 4, 5, 6,\ A_k = A_{k-3} - 2A_{k-2} + 2A_{k-1}$. Since an affine transformation preserves the barycentre, these relationships are also true for our initial 3-p-gram. Using the remark 3.2, we obtain the announced relationships.

Among all the solutions of the previous corollary, we can find a geometric construction by choosing $t_1 = 1/6$, $t_2 = t_3 = 1/3$ for which the corresponding weight is $(1/6, 1/3, 1/3, 1/6, 0, 0)$. Geometrically, it means taking the midpoint of diagonal $[A_1A_4]$ as I_1 and the centroid of the triangle $I_1A_2A_3$ as A'_1. This construction is then repeated cyclically to obtain other points A'_2, A'_3, A'_4, A'_5 and A'_6. We emphasize that this construction is the perfect counterpart to Varignon's, here for hexagons. The following figure illustrates this construction.

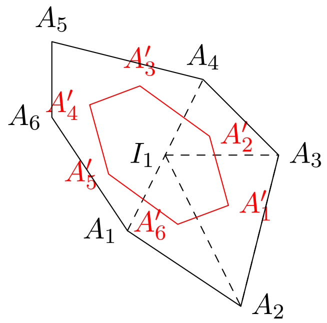

FIGURE 7. Illustration of the geometric construction described above.

4.2. Extension to affine regular {n/p}-star polygons.

In this subsection we will give cyclic barycentric maps that send any n-gon to a regular $\{n/p\}$-star polygon, where p is a fixed integer relatively prime with n.

A regular $\{n/p\}$-star polygon $A_1A_2\cdots A_n$ is defined recursively by

$$\forall k \in [\![4, n]\!],\ A_k = A_{k-3} - wA_{k-2} + wA_{k-1},\ \text{where } w = 1 + 2\cos\left(\frac{2\pi p}{n}\right).$$

This choice of w allows that the polygon closes.

Then if f is a cyclic barycentric map defined by a weight $\alpha = (t_1, \cdots, t_n)$, the lemma 3.1 allows us to obtain that the real numbers t_k satisfy for all $k \in [\![4, n]\!]$, $t_k = t_{k-3} - wt_{k-2} + wt_{k-1}$, which, in accordance with the remark 3.2, are exactly the same as those verified by the vertices.

We now need to solve this linear difference equation of ordre three. By introducing its characteristic equation $r^3 - wr^2 + wr - 1 = 0$, whose roots are $1, e^{\imath\theta}, e^{-\imath\theta}$ where $\theta = 2\pi p/n$, we know that t_k is a complex linear combination of the k-th powers of these roots, or, because here the t_k are real numbers, a real linear combination of $1, \cos(k\theta), \sin(k\theta)$. There are therefore real numbers a, b, c such that for all k, $t_k = a + b\cos(k\theta) + c\sin(k\theta)$. But let us not forget that the sum of t_k is equal to 1, which leads to the condition $a = 1/n$ because the sum of the cosines (and also the sum of the sines) is zero. Thus, we have a 2-parameters family of solutions given by the previous equality. Note that by adding t_k and t_{k+2} and using the trigonometric formulae for the sum of two cosines and two sines, we obtain:

$$t_k + t_{k+2} = \frac{2}{n} + 2b\cos\left((k+1)\theta\right)\cos\theta + 2c\sin\left((k+1)\theta\right)\cos\theta.$$

and so, $t_k + t_{k+2} = \frac{2}{n} + 2\left(t_{k+1} - \frac{1}{n}\right)\cos\theta = \frac{2}{n}(1 - \cos\theta) + 2t_{k+1}\cos\theta$.

So we get the recursion relation $t_{k+2} - 2t_{k+1}\cos\theta + t_k = \frac{2}{n}(1 - \cos\theta)$ which contains special cases that we have already found, such as for parallelograms ($t_1 + t_3 = 1/2$) or, in the Corollary 4.1, for 3-p-regular grams ($t_1 - t_2 + t_3 = \frac{1}{6}$).

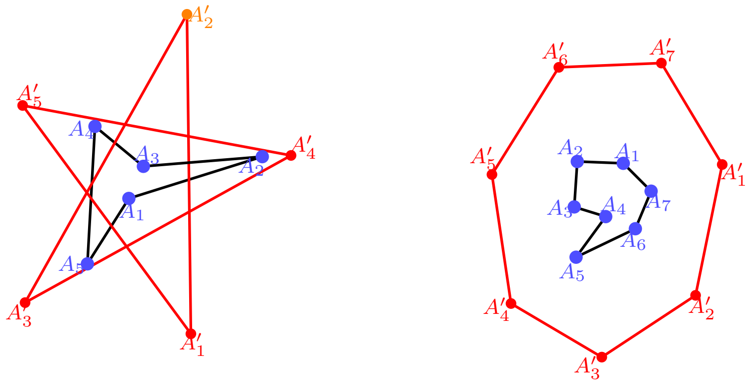

Figure 8. A pentagon and an heptagon and their images (respectively a $\{5/2\}$-star regular pentagon and a regular heptagon) under a cyclic barycentric map.

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