INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 14 (2025), No. 4, 51 - 67

SIX DIVISIONS OF THE SIDES AND DIAGONALS OF A QUADRILATERAL BY CIRCLES THAT FORM PASCAL POINTS, AND THEIR REPRESENTATION BY CIRCLE CENTERS

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Abstract. Each circle that forms Pascal points intersects one pair of opposite sides of a quadrilateral at two points and forms Pascal points on the other pair of sides. In addition, it intersects the extensions of the diagonals at two points. These six points divide the four sides of the quadrilateral (internally) and its diagonals (externally). In this paper, we show that these six divisions are equivalent to the divisions of the six fixed segments (which depend only on the quadrilateral) using the center of the circle that forms the Pascal points. To this end, we extend the notion of a "circle that forms Pascal points on the sides of a quadrilateral" to include also circles that do not intersect a pair of opposite sides at interior points. We then establish an equality among the seven ratios defined using three circles that form Pascal points.

1. Introduction

The theory of a convex quadrilateral and a circle that forms Pascal points on its sides is a relatively new topic in Euclidean geometry (see [1], [2], [3], [4], [5], [6] , [7]). Therefore, we begin by recalling the definitions of Pascal points and a circle that forms Pascal points on the sides of a quadrilateral. All definitions and properties are illustrated using dynamic GeoGebra applets.

A circle that forms Pascal points (see [1]).
For a convex quadrilateral $ABCD$ in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD. A circle that forms Pascal points is defined as any circle passing through points E and F, as well as through interior points of sides BC and AD (see Figure 1)

Pascal points on the sides of the quadrilateral (see [1]).
Let ω_i be a circle that forms Pascal points, and let $M_i = \omega_i \cap [BC]$,

Keywords and phrases: Divisions, Ratio, Pascal points, Circle that forms Pascal points, GeoGebra.
(2010)Mathematics Subject Classification: 51M04, 51M05, 51M99, 51-04
Received: 13.06.2025. In revised form: 26.07.2025. Accepted: 19.08.2025.

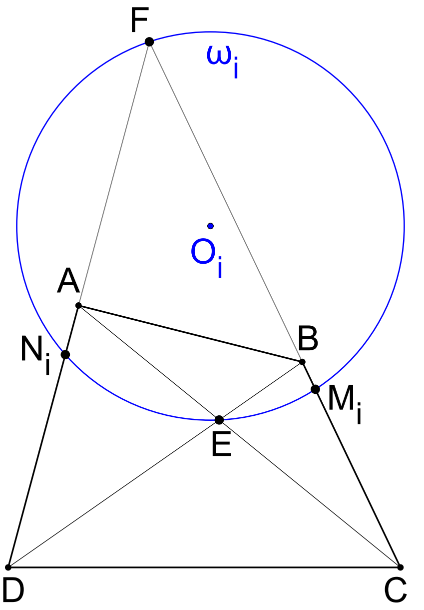


Figure 1: The circle ω_i passes through points E and F and intersects sides BC and AD at interior points M_i and N_i, respectively.

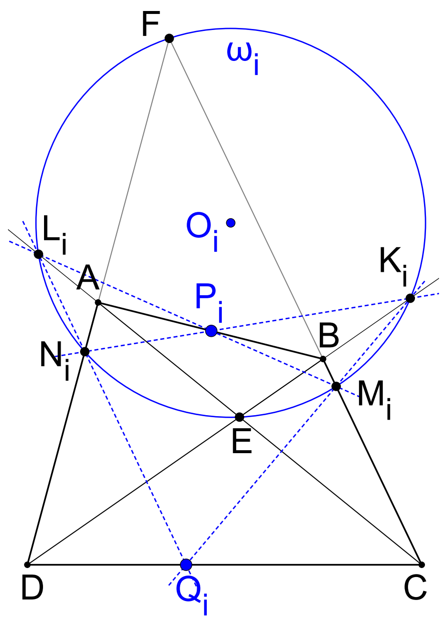


Figure 2: P_i and Q_i are Pascal points formed by the circle ω_i on the sides AB and CD.

$N_i = \omega_i \cap [AD]$. Let K_i and L_i be the points where ω_i intersects the extensions of diagonals BD and AC, respectively. We denote $P_i = K_iN_i \cap L_iM_i$ and $Q_i = K_iM_i \cap L_iN_i$ (see Figure 2).

In this case, the following property holds:

Property 1. $P_i \in [AB]$, $Q_i \in [CD]$.

The proof of this property is based on the application of the general Pascal theorem (see [1, Theorem 1]). Therefore, the points P_i and Q_i are called Pascal points formed by the circle ω_i on the sides AB and CD.

The following applet allows you to move the circle ω_i (by dragging its center O_i or by clicking the *Move Circle* ω_i button) and to observe that Property 1 holds for every valid position of ω_i.
https://www.geogebra.org/m/xwvrs9wp

2. Extension of the Notion of a "Circle that Forms Pascal Points"

Until now, we have considered circles that pass through interior points of the sides BC and AD of the quadrilateral. As a result, the corresponding Pascal points formed by these circles lie on the sides AB and CD.
If we extend the definition of Pascal points to include points lying on the lines AB and CD (and not necessarily on the sides themselves), the notion of a circle that forms Pascal points can be extended as follows:

Definition 2.1. *Let $ABCD$ be a quadrilateral in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD. A circle that forms Pascal points is any circle passing through points E and F.*

In this case, the Pascal points are defined as follows:

Definition 2.2. *Let ω_i be a circle that forms Pascal points, and let M_i and N_i be the points of intersection of ω_i with the lines BC and AD, respectively (distinct from the point F). Let K_i and L_i be the intersection points of ω_i with the lines BD and AC, respectively (distinct from the point E). We denote $P_i = K_iN_i \cap L_iM_i$ and $Q_i = K_iM_i \cap L_iN_i$* (see Figure 3).
The points P_i and Q_i are called the Pascal points formed by the circle ω_i.

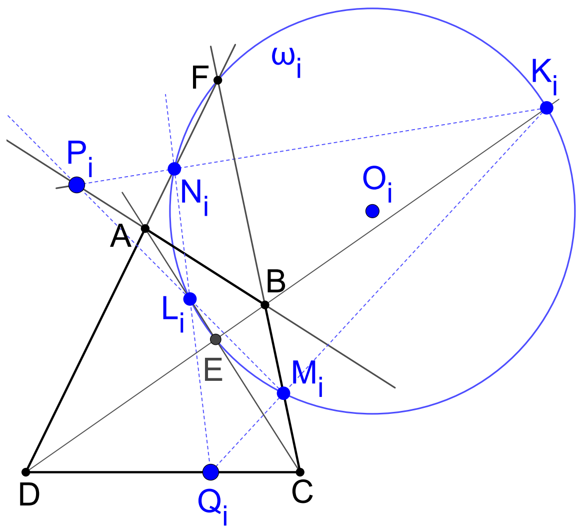


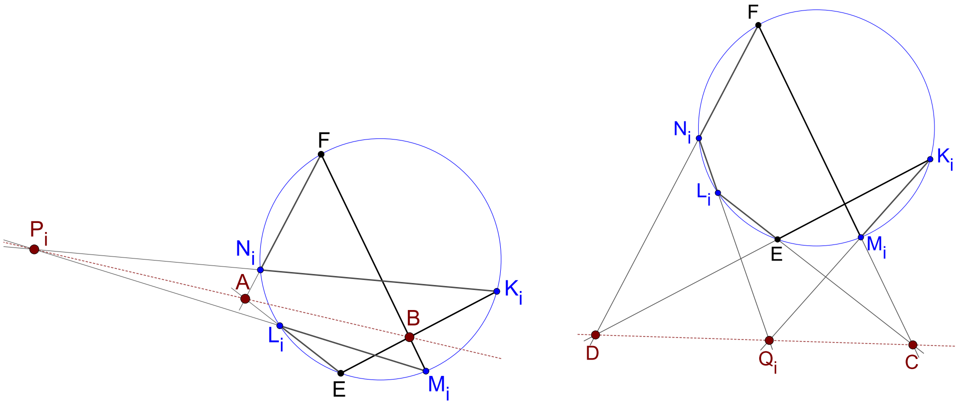
Figure 3: The points M_i, N_i, K_i and L_i lie on the lines BC, AD, BD and AC, respectively.

For these extended definitions, a property similar to Property 1 holds.

Property 2. *Let P_i and Q_i be Pascal points formed by the circle ω_i. Then $P_i \in AB$, $Q_i \in CD$* (see Figure 3).

This theorem states that the points P_i and Q_i lie on the lines AB and CD (not necessarily on the sides themselves). Its proof is simpler than that of Property 1 and follows directly from Pascal's theorem.

Proof. The points E, F, M_i, N_i, K_i, and L_i lie on the same circle. We will consider them as the vertices of the hexagon $EK_iN_iFM_iL_i$. Each of the three pairs of lines: EL_i and FN_i, L_iM_i and K_iN_i, FM_i and EK_i, passes through a pair of opposite sides of the hexagon (see Figure 4(a)).
Therefore, according to Pascal's theorem, the three points of intersection of these pairs (the points A, P, and B respectively) lie on a straight line. Hence, point P lies on the line AB.
Similarly, if we consider the same six points as the vertices of the hexagon



(a) The points $A = EL_i \cap FN_i$, $B = FM_i \cap EK_i$, and $P_i = L_iM_i \cap K_iN_i$ lie on the same straight line.

(b) The points $C = EL_i \cap FM_i$, $D = EK_i \cap FN_i$, and $Q_i = K_iM_i \cap N_iL_i$ lie on the same straight line.

Figure 4

$EK_iM_iFN_iL_i$, then by Pascal's theorem, the three points of intersection of the lines EL_i and FM_i, L_iN_i and K_iM_i, FN_i and EK_i lie on a single straight line.
The three points of intersection of these pairs of lines are C, Q, and D, respectively. Therefore, the point Q lies on the line CD (see Figure 4(b)) $\square$.

For any given convex quadrilateral (that is not a parallelogram) and three circles forming Pascal points, the following theorem holds:

Theorem 2.1. *Let $ABCD$ be a convex quadrilateral in which E is the intersection point of the diagonals, and F is the intersection point of the extensions of sides AD and BC. Let ω_1, ω_2, and ω_3 be three circles with centers O_1, O_2, and O_3, respectively, passing through the points E and F. Let M_1, M_2, M_3 and N_1, N_2, N_3 be the points where these circles intersect the lines BC and AD, respectively, and let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be the Pascal points formed by these circles on the lines AB and CD, respectively* (see Figure 5). *Then the following equality of segment ratios holds:*

$$(1) \qquad \frac{O_1O_2}{O_2O_3} = \frac{P_1P_2}{P_2P_3} = \frac{Q_1Q_2}{Q_2Q_3} = \frac{M_1M_2}{M_2M_3} = \frac{N_1N_2}{N_2N_3} = \frac{K_1K_2}{K_2K_3} = \frac{L_1L_2}{L_2L_3}$$

The following applet allows you to move the circles ω_1, ω_2, and ω_3 (either by dragging their centers O_1, O_2, and O_3, or by clicking the appropriate buttons), and to observe that Theorem 2.1 holds for every valid position of the circles. https://www.geogebra.org/m/qkgdpw4p

Proof. We make use of the following lemma.

Lemma 2.1. *Let Ω_1 and Ω_2 be two circles that intersect at points X and Y. Let line x pass through point X and intersect Ω_1 at point R and Ω_2 at point S. Similarly, let line y pass through point Y and intersect Ω_1 at point T and Ω_2 at point U. Then the lines RT and SU are parallel.*

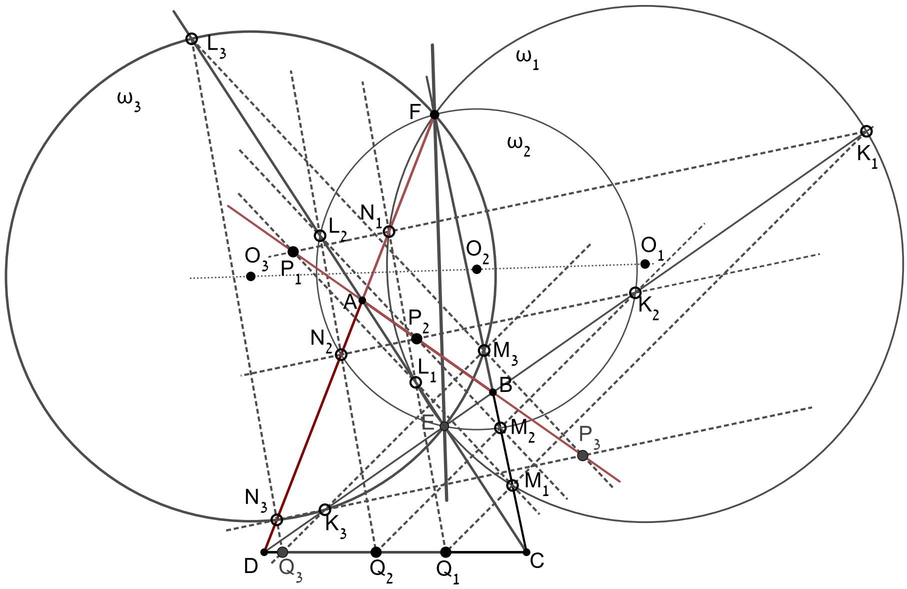


Figure 5: Three circles that form Pascal points, their centers, and six points (M, N, K, L, P, Q) determined using each of the circles.

Proof of Lemma. We consider three possible configurations of the points R, S, T, and U with respect to the line XY:

(i) Points R and T lie on one side of the line XY, and points S and U lie on the other side (see Figure 6(a)).
(ii) All four points lie on the same side of the line XY (see Figure 6(b)).
(iii) Points R and S lie on one side of the line XY, and points T and U lie on the other side (see Figure 6(c)).

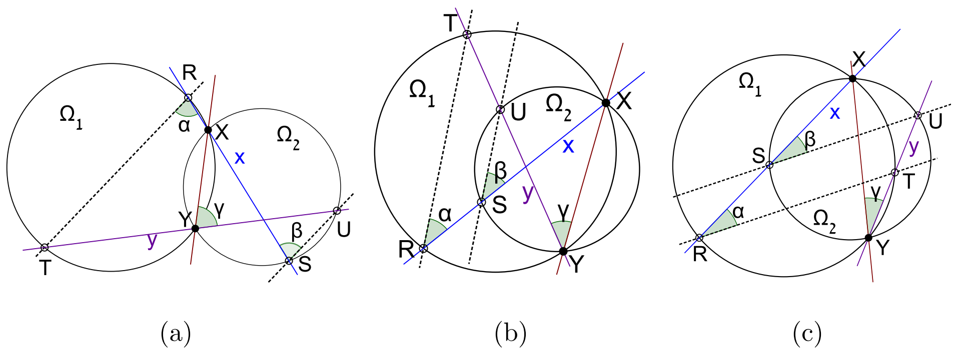


Figure 6: In all three cases, $RT \parallel US$ holds

In all three cases, the angles $\angle XRT = \alpha$ and $\angle XSU = \beta$ are equal to the angle $\angle XTU = \gamma$, hence $\alpha = \beta$. Therefore, $RT \parallel SU$. □

We proceed to the proof of Theorem 2.1, making use of Lemma 2.1.

We begin by considering the circles ω_1 and ω_2, which intersect at points E and F. The lines EA and FA pass through these points (see Figure 7). The line EA intersects ω_1 at L_1 and ω_2 at L_2, while the line FA intersects ω_1 at N_1 and ω_2 at N_2. Therefore, by Lemma 2.1 (specifically, case (ii) in its proof, where the four points L_1, L_2, N_1, and N_2 lie on the same side of the line EF), we have:

$$L_1N_1 \parallel L_2N_2 \tag{2}$$

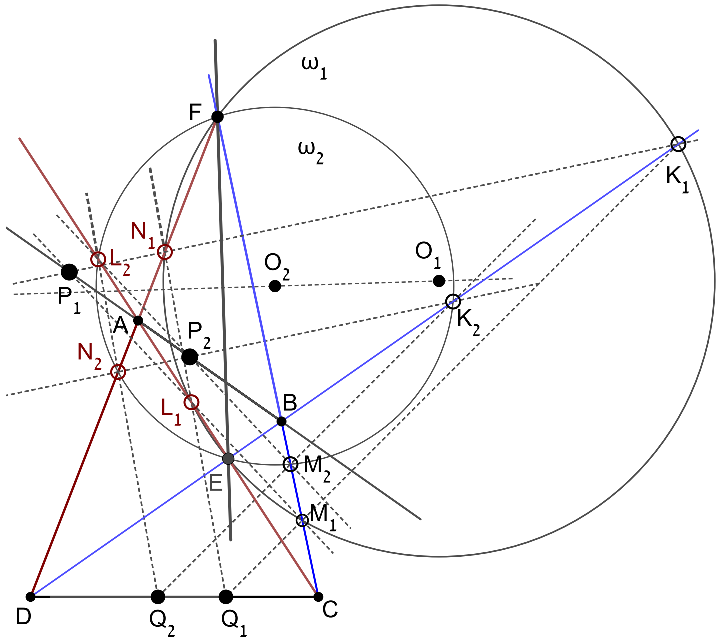


Figure 7: $L_1N_1 \parallel L_2N_2$, $K_1M_1 \parallel K_2M_2$, $K_1N_1 \parallel K_2N_2$, and $L_1M_1 \parallel L_2M_2$.

Similarly, the lines EB and FB also pass through the points E and F. The line EB intersects ω_1 at point K_1 and ω_2 at point K_2. The line FB intersects ω_1 at point M_1 and ω_2 at point M_2. In addition, all four points K_1, K_2, M_1, and M_2 lie on the same side of the line EF. Therefore, we have:

$$K_1M_1 \parallel K_2M_2 \tag{3}$$

For the lines EB and FA, the following holds: the line EB intersects ω_1 at point K_1 and ω_2 at point K_2, and the line FA intersects ω_1 at point N_1 and ω_2 at point N_2. The points K_1 and K_2 lie on one side of the line EF, and the points N_1 and N_2 lie on the other side. Therefore, according to case (iii) of Lemma 2.1, we have:

$$K_1N_1 \parallel K_2N_2 \tag{4}$$

Finally, for the lines FB and EA, the following holds: the line FB intersects ω_1 at point M_1 and ω_2 at point M_2, and the line EA intersects ω_1 at point L_1 and ω_2 at point L_2. The points M_1 and M_2 lie on one side of the line EF, and the points L_1 and L_2 lie on the other side. Therefore, by case (iii) in the proof of Lemma 2.1, we have:

(5) $$L_1M_1 \parallel L_2M_2$$

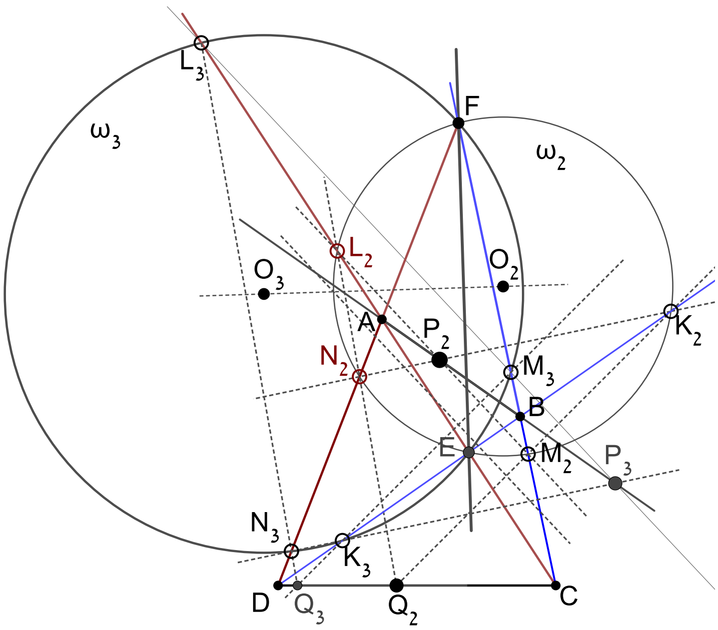


Figure 8: $L_2N_2 \parallel L_3N_3$, $K_2M_2 \parallel K_3M_3$, $K_2N_2 \parallel K_3N_3$, and $L_2M_2 \parallel L_3M_3$.

We now turn to the circles ω_2 and ω_3. These circles also intersect at points E and F. The line EA intersects ω_2 at L_2 and ω_3 at L_3, while the line FA intersects ω_2 at N_2 and ω_3 at N_3 (see Figure 8). All four of these points lie on the same side of the line EF. Therefore,

(6) $$L_2N_2 \parallel L_3N_3$$

Similarly, for the lines EB and FB intersecting the circles ω_2 and ω_3 at points K_2 and K_3, M_2 and M_3, respectively, we have:

(7) $$K_2M_2 \parallel K_3M_3$$

For the lines EB and FA intersecting the circles ω_2 and ω_3 at points K_2 and K_3, and N_2 and N_3, respectively, we have (see Figure 8):

(8) $$K_2N_2 \parallel K_3N_3$$

Finally, for the lines FB and EA intersecting the circles ω_2 and ω_3 at points M_2 and M_3, L_2 and L_3, respectively, we have:

(9) $$L_2M_2 \parallel L_3M_3$$

From statements (2) and (6), it follows that the three lines L_1N_1, L_2N_2, and L_3N_3 are parallel to each other. In addition, they intersect the sides DF and DC of the angle FDC at the points N_1, N_2, N_3 and Q_1, Q_2, Q_3, respectively. Therefore, according to an extension of Thales's theorem, we have:

$$\frac{N_1N_2}{N_2N_3} = \frac{Q_1Q_2}{Q_2Q_3} \tag{10}$$

From statements (3) and (7), it follows that the three lines K_1M_1, K_2M_2, and K_3M_3 are parallel to each other. These lines intersect the sides CD and CA of the angle DCA at the points Q_1, Q_2, Q_3 and M_1, M_2, M_3, respectively. Therefore, we have:

$$\frac{Q_1Q_2}{Q_2Q_3} = \frac{M_1M_2}{M_2M_3} \tag{11}$$

From statements (4) and (8), it follows that the three lines K_1N_1, K_2N_2, and K_3N_3 are parallel to each other. These lines intersect the lines AB and AD at the points P_1, P_2, P_3 and N_1, N_2, N_3, respectively (see Figure 5). Therefore, according to an extension of Thales's theorem, we have:

$$\frac{N_1N_2}{N_2N_3} = \frac{P_1P_2}{P_2P_3} \tag{12}$$

In addition, these three parallel lines intersect the sides of the angle ADB at the points N_1, N_2, N_3 (on side DA) and K_1, K_2, K_3 (on side DB). Therefore, we have:

$$\frac{N_1N_2}{N_2N_3} = \frac{K_1K_2}{K_2K_3} \tag{13}$$

From statements (5) and (9), it follows that the three lines L_1M_1, L_2M_2, and L_3M_3 are parallel to each other. These lines intersect the sides CA and CF of the angle ACF at the points L_1, L_2, L_3 and M_1, M_2, M_3, respectively. Therefore, we have:

$$\frac{L_1L_2}{L_2L_3} = \frac{M_1M_2}{M_2M_3} \tag{14}$$

From the equalities (10)–(14), it follows that:

$$\frac{P_1P_2}{P_2P_3} = \frac{Q_1Q_2}{Q_2Q_3} = \frac{M_1M_2}{M_2M_3} = \frac{N_1N_2}{N_2N_3} = \frac{K_1K_2}{K_2K_3} = \frac{L_1L_2}{L_2L_3} \tag{15}$$

It remains to prove that the ratio $\frac{O_1O_2}{O_2O_3}$ is also equal to the ratios from (15). For this, it is sufficient to prove that this ratio is equal to one of the ratios from (15). We will show that: $\frac{O_1O_2}{O_2O_3} = \frac{Q_1Q_2}{Q_2Q_3}$
Let us first prove the following lemma:

Lemma 2.2. *Let ω_i be an arbitrary circle forming the Pascal points P_i and Q_i on the lines AB and CD, respectively, and let O_{ω_C} and O_{ω_D} be the centers of the circles passing through the points C, E, F and D, E, F, respectively* (see Figure 9). *Then the center O_i of the circle ω_i divides the segment $O_{\omega_C}O_{\omega_D}$ in the same ratio as the Pascal point Q_i divides the segment CD; that is, the proportion $\frac{O_{\omega_C}O_i}{O_iO_{\omega_D}} = \frac{CQ_i}{Q_iD}$ holds.*

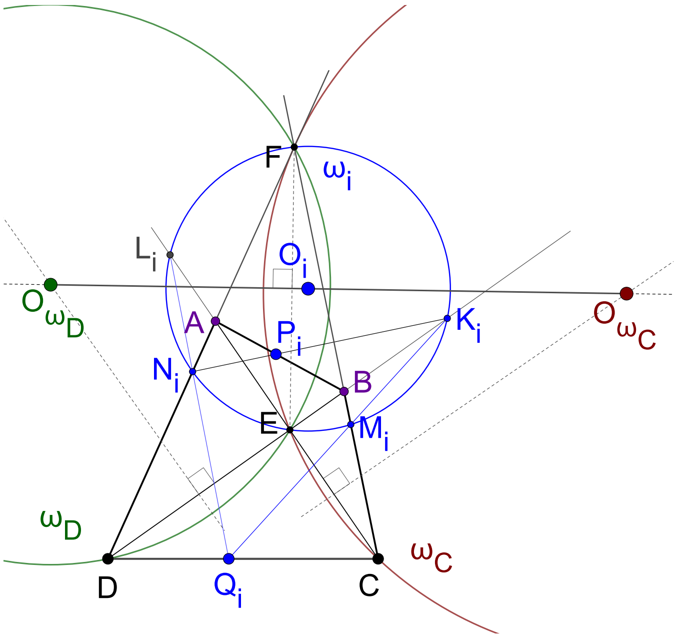


Figure 9: The six points E, M_i, K_i, F, L_i, and N_i lie on the unit circle ω_i. The circle ω_C (with center O_{ω_C}) passes through the points E, F, and C. The circle ω_D (with center O_{ω_D}) passes through the points E, F, and D.

Proof of Lemma. We will use the method of complex numbers in plane geometry.

We choose a system of coordinates so that circle ω_i is the unit circle (the center O_i of circle ω_i is located at the origin, and the radius is $O_iE = 1$). In this system, the equation of the unit circle is $z \cdot \overline{z} = 1$, where z is the complex coordinate of an arbitrary point Z located on circle ω_i, and $\overline{z}$ is the complex conjugate of z.

We denote the complex coordinates of points E, F, K_i, L_i, M_i, and N_i as e, f, k, l, m, and n, respectively. These points are located on the unit circle ω_i (see Figure 9), and therefore there holds: $\overline{e} = \frac{1}{e}$, $\overline{f} = \frac{1}{f}$, $\overline{k} = \frac{1}{k}$, $\overline{l} = \frac{1}{l}$, $\overline{m} = \frac{1}{m}$, $\overline{n} = \frac{1}{n}$.

We use the following property (see [8, pp. 157–158]): Let $T(t)$, $R(r)$, $V(v)$, and $W(w)$ be four points on the unit circle, and let $U(u)$ be the point of intersection of straight lines TR and VW (see Figure 10). Then for the coordinate u and its conjugate $\overline{u}$, there holds:

$$\overline{u} = \frac{t + r - v - w}{tr - vw} \tag{16}$$

and

$$u = \frac{rvw + tvw - trw - trv}{vw - tr} \tag{17}$$

The points C, D, and Q are the points of intersection of the lines FM_i and EL_i, FN_i and EK_i, and K_iM_i and L_iN_i, respectively. Therefore, according

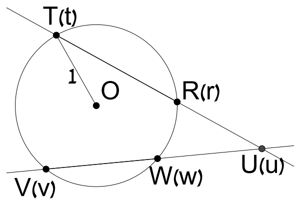


Figure 10: The points T, R, V, and W lie on the unit circle. The lines TR and VW intersect at point U.

to formulas (16) and (17), these points have the following complex coordinates (and their complex conjugate numbers):

$\overline{c} = \dfrac{f+m-e-l}{fm-el}$ and $c = \dfrac{eml+efl-fml-efm}{el-fm}$,

$\overline{d} = \dfrac{f+n-e-k}{fn-ek}$ and $d = \dfrac{enk+efk-fnk-efn}{ek-fn}$,

$\overline{q} = \dfrac{m+k-n-l}{mk-nl}$ and $q = \dfrac{kln+lmn-klm-kmn}{nl-mk}$.

Let UZ be the perpendicular bisector of the segment VW, where U is the midpoint of the segment and Z is an arbitrary point on the perpendicular bisector. Then the equation of the perpendicular bisector is:

$$\overline{z} = \frac{\overline{v}-\overline{w}}{v-w} \cdot z + \frac{1}{2}\left((\overline{w}+\overline{v}) + (w+v) \cdot \frac{\overline{w}-\overline{v}}{w-v}\right).$$

For the perpendicular bisector of the segment EF, we obtain the following equation: $\overline{z} = \dfrac{\overline{e}-\overline{f}}{f-e} \cdot z + \dfrac{1}{2}\left((\overline{f}+\overline{e}) + (f+e) \cdot \dfrac{\overline{f}-\overline{e}}{f-e}\right)$.

From this, $\overline{z} = \dfrac{\frac{1}{e}-\frac{1}{f}}{f-e} \cdot z + \dfrac{1}{2}\left(\underbrace{(\frac{1}{f}+\frac{1}{e}) + (f+e) \cdot \frac{\frac{1}{f}-\frac{1}{e}}{f-e}}_{=0}\right)$.

And finally,

$$\overline{z} = \frac{1}{ef} \cdot z \tag{18}$$

For the perpendicular bisector of the segment EC, we obtain the following equation:

$$\overline{z} = -\frac{\overline{c}-\overline{e}}{c-e} \cdot z + \frac{1}{2}\left((\overline{c}+\overline{e}) + (c+e) \cdot \frac{\overline{c}-\overline{e}}{c-e}\right) \tag{19}$$

We substitute into this equation (step by step) the expressions for the variables $\overline{e}$, c, and $\overline{c}$, and obtain:

$\overline{c}+\overline{e} = \dfrac{f+m-e-l}{fm-el} + \dfrac{1}{e} = \dfrac{ef+em-e^2-2el+fm}{e(fm-el)}$,

$c+e = \dfrac{eml+efl-fml-efm}{el-fm} + e = \dfrac{eml+efl-fml-2efm+e^2l}{el-fm}$,

$$\frac{\overline{c}-\overline{e}}{c-e} = \frac{\dfrac{f+m-e-l}{fm-el} - \dfrac{1}{e}}{\dfrac{eml+efl-fml-efm}{el-fm} - e} = \frac{\dfrac{ef+em-e^2-fm}{e(fm-el)}}{\dfrac{eml+efl-fml-e^2l}{el-fm}}$$

$$= \frac{-(ef+em-e^2-fm)}{el(em+ef-fm-e^2)} = -\frac{1}{el}.$$

Therefore, the equation of the perpendicular bisector of the segment EC takes the following form:

$$\overline{z} = \frac{1}{el}\cdot z + \frac{1}{2}\left(\frac{ef+en-e^2-2el+fn}{e(fn-el)} - \frac{enl+efl-fnl-2efn+e^2l}{el-fn}\cdot\frac{1}{el}\right).$$

Or, after simplifications

$$\overline{z} = \frac{1}{el}\cdot z + \frac{(l-m)(f-l)}{l(fm-el)} \tag{20}$$

The solution of the system of (18) and (20) gives us the complex coordinate of the point O_{ω_C}: $\frac{1}{ef}\cdot z = \frac{1}{el}\cdot z + \frac{(l-m)(f-l)}{l(fm-el)}$. Therefore, for the complex coordinate o_{ω_C} we obtain $z = o_{\omega_C} = \dfrac{\frac{(l-m)(f-l)}{l(fm-el)}}{\frac{1}{ef}-\frac{1}{el}} = \frac{(m-l)ef}{fm-el}$.

Hence, the number $\overline{o_{\omega_C}}$, which is the conjugate of the coordinate o_{ω_C}, is: $\overline{o_{\omega_C}} = \frac{m-l}{fm-el}$.

Similarly, for the point O_{ω_D} (the intersection point of the perpendicular bisectors of the segments EF and ED), we obtain $o_{\omega_D} = \frac{(n-k)ef}{fn-ek}$.

Hence, for the conjugate number $\overline{o_{\omega_D}}$, we get: $\overline{o_{\omega_D}} = \frac{n-k}{fn-ek}$.

Let λ_{O_i} denote the ratio $\frac{O_{\omega_C}O_i}{O_iO_{\omega_D}}$ between the lengths of the segments connecting the centers of the circles ω_i and ω_C, and ω_i and ω_D. Using complex coordinates, this ratio is expressed as:

$$\lambda_{O_i} = \frac{0-o_{\omega_C}}{o_{\omega_D}-0} = -\frac{o_{\omega_C}}{o_{\omega_D}} = -\frac{\frac{(m-l)ef}{fm-el}}{\frac{(n-k)ef}{fn-ek}} = \frac{(m-l)(fn-ek)}{(k-n)(fm-el)},$$

where $\lambda_{O_i} = \overline{\lambda_{O_i}}$.

That is, $\lambda_{O_i} = \frac{O_{\omega_C}O_i}{O_iO_{\omega_D}} = \frac{(m-l)(fn-ek)}{(k-n)(fm-el)}$.

Let λ_{Q_i} denote the ratio of the lengths of the segments $\frac{CQ_i}{Q_iD}$. Using complex coordinates, this ratio can be expressed as: $\lambda_{Q_i} = \frac{q-c}{p-q}$, where $\lambda_{Q_i} = \overline{\lambda_{Q_i}}$ is a real number. Hence:

$$\lambda_{Q_i} = \overline{\lambda_{Q_i}} = \frac{\overline{q}-\overline{c}}{\overline{d}-\overline{q}} = \frac{\frac{m+k-n-l}{mk-nl} - \frac{f+m-e-l}{fm-el}}{\frac{f+n-e-k}{fn-ek} - \frac{m+k-n-l}{mk-nl}}$$

$$= \frac{((m+k-n-l)(fm-el)-(f+m-e-l)(mk-nl))\,(fn-ek)}{((f+n-e-k)(mk-nl)-(m+k-n-l)(fn-ek))\,(fm-el)}$$

$$= \frac{(fn-fm+mk-ek-nl+el)(m-l)(fn-ek)}{(fn-fm+mk-ek-nl+el)(k-n)(fm-el)} = \frac{(m-l)(fn-ek)}{(k-n)(fm-el)}.$$

We obtain $\lambda_{Q_i} = \dfrac{CQ_i}{Q_iD} = \dfrac{(m-l)(fn-ek)}{(k-n)(fm-el)}$.

By comparing λ_{O_i} and λ_{Q_i}, we get: $\lambda_{O_i} = \lambda_{Q_i}$, and therefore: $\dfrac{CQ_i}{Q_iD} = \dfrac{O_{\omega_C}O_i}{O_iO_{\omega_D}}$. Thus, we have proved Lemma 2.2.

According to Lemma 2.2, for the three circles ω_1, ω_2, and ω_3 (see Figure 11), the following three proportions hold:
$\dfrac{CQ_1}{Q_1D} = \dfrac{O_{\omega_C}O_1}{O_1O_{\omega_D}}$, $\dfrac{CQ_2}{Q_2D} = \dfrac{O_{\omega_C}O_2}{O_2O_{\omega_D}}$ and $\dfrac{CQ_3}{Q_3D} = \dfrac{O_{\omega_C}O_3}{O_3O_{\omega_D}}$.
We add the following auxiliary constructions (see Figure 11):

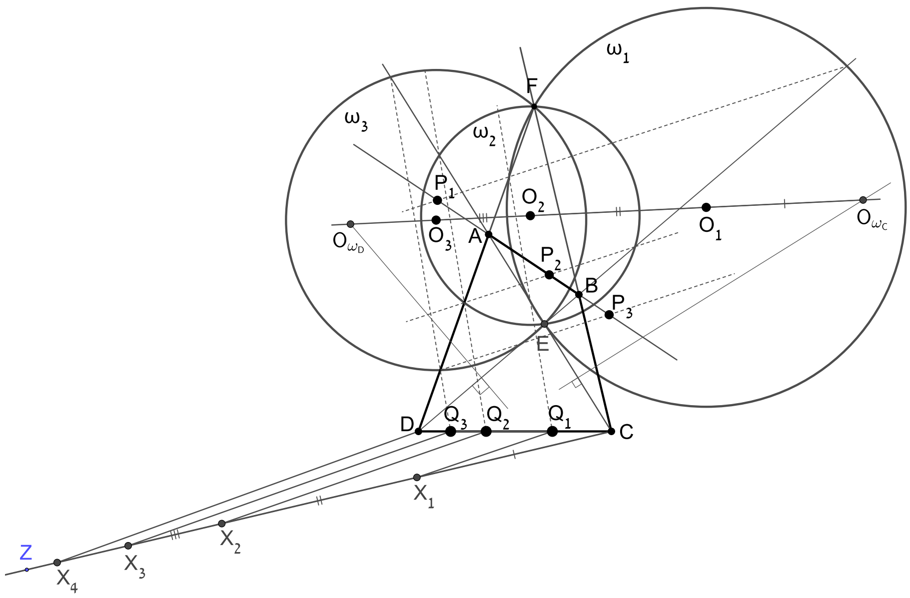


Figure 11: $CX_1 = O_{\omega_C}O_1$, $CX_2 = O_{\omega_C}O_2$, $CX_3 = O_{\omega_C}O_3$, $CX_4 = O_{\omega_C}O_{\omega_D}$ and $\dfrac{Q_1Q_2}{Q_2Q_3} = \dfrac{O_1O_2}{O_2O_3}$.

We draw an arbitrary ray CZ and mark four points on it, X_1, X_2, X_3, and X_4, such that:
$CX_1 = O_{\omega_C}O_1$, $CX_2 = O_{\omega_C}O_2$, $CX_3 = O_{\omega_C}O_3$, and $CX_4 = O_{\omega_C}O_{\omega_D}$.
Therefore, the following proportions also hold:
$\dfrac{CQ_1}{Q_1D} = \dfrac{CX_1}{X_1X_4}$, $\dfrac{CQ_2}{Q_2D} = \dfrac{CX_2}{X_2X_4}$ and $\dfrac{CQ_3}{Q_3D} = \dfrac{CX_3}{X_3X_4}$.
Through the points Q_1 and X_1, Q_2 and X_2, Q_3 and X_3, and D and X_4, we draw four lines. According to the converse of Thales' theorem, from the previous proportions it follows that each of the lines Q_1X_1, Q_2X_2, and Q_3X_3 is parallel to the line DX_4.
Therefore, we have: $Q_1X_1 \parallel Q_2X_2 \parallel Q_3X_3$.
Hence, by Thales' theorem, it follows that: $\dfrac{Q_1Q_2}{Q_2Q_3} = \dfrac{X_1X_2}{X_2X_3}$. From the last proportion and from the fact that $X_1X_2 = O_1O_2$ and $X_2X_3 = O_2O_3$, we get: $\dfrac{Q_1Q_2}{Q_2Q_3} = \dfrac{O_1O_2}{O_2O_3}$. □

3. Divisions of the sides and diagonals of the quadrilateral and the representation of these divisions using the centers of the circles

In the following theorem, we consider again the circle ω_i, which forms the Pascal points on the sides of the quadrilateral, and establish an equality between the partition of six fixed segments by the center O_i of the circle ω_i and the partition of the sides and diagonals of the quadrilateral by six points determined via ω_i.

Theorem 3.1. *Let $ABCD$ be a convex quadrilateral in which E is the intersection point of the diagonals, and F is the intersection point of the extensions of sides AD and BC. Let ω_i be an arbitrary circle with center O_i that forms the Pascal points P_i and Q_i on the sides AB and CD (rather than on their extensions). Let M_i, N_i, K_i, and L_i be the points where ω_i intersects the sides BC and AD, and the extensions of the diagonals BD and AC, respectively. Let us denote:*
O_{ω_A} – the center of the circle ω_A passing through the points E, F, and A;
O_{ω_B} – the center of the circle ω_B passing through the points E, F, and B;
O_{ω_C} – the center of the circle ω_C passing through the points E, F, and C;
O_{ω_D} – the center of the circle ω_D passing through the points E, F, and D.
Then the following equalities hold:

3.1.1. $\dfrac{AP_i}{P_iB} = \dfrac{O_{\omega_A}O_i}{O_iO_{\omega_B}}$

3.1.2. $\dfrac{BM_i}{M_iC} = \dfrac{O_{\omega_B}O_i}{O_iO_{\omega_C}}$

3.1.3. $\dfrac{CQ_i}{Q_iD} = \dfrac{O_{\omega_C}O_i}{O_iO_{\omega_D}}$

3.1.4. $\dfrac{DN_i}{N_iA} = \dfrac{O_{\omega_D}O_i}{O_iO_{\omega_A}}$

3.1.5. $\dfrac{BK_i}{K_iD} = \dfrac{O_{\omega_B}O_i}{O_iO_{\omega_D}}$

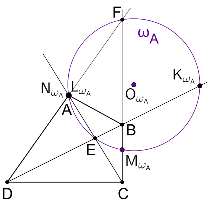
3.1.6. $\dfrac{AL_i}{L_iC} = \dfrac{O_{\omega_A}O_i}{O_iO_{\omega_C}}$

The following applet allows you to move the circle ω_i and to observe that Theorem 3.1 holds for every valid position of ω_i.
https://www.geogebra.org/m/qprb7gsr

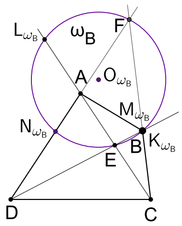
Note: We formulated Theorem 3.1 for a circle that forms Pascal points on the sides of the quadrilateral, since for such circles, equalities (1)–(4) of Theorem 3.1 describe the internal divisions of the sides of the quadrilateral. However, an analogous theorem can also be formulated for a circle that forms Pascal points according to the more general Definition 2.1. In this case, each of the six divisions can be either internal or external. The proof of the more general theorem is identical to the proof presented below.

Proof. According to the statement of the theorem, all four circles ω_A, ω_B, ω_C, and ω_D pass through the points E and F, and therefore, by Definition

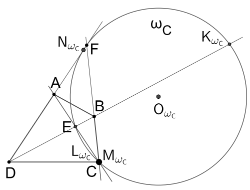
2.1, they are circles that form Pascal points (not necessarily on the sides of the quadrilateral).



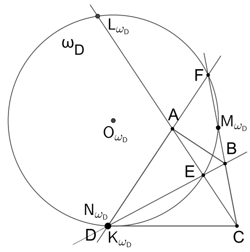
(a) The circle ω_A passes through the points E, F, and A. Therefore, $N_{\omega_A} = L_{\omega_A} = A$.



(b) The circle ω_B passes through the points E, F, and B. Therefore, $M_{\omega_B} = K_{\omega_B} = B$.



(c) The circle ω_C passes through the points E, F, and C. Therefore, $M_{\omega_C} = L_{\omega_C} = C$.



(d) The circle ω_D passes through the points E, F, and D. Therefore, $N_{\omega_D} = K_{\omega_D} = D$.

Figure 12

Let M_{ω_A} and N_{ω_A} be the intersection points of the circle ω_A with the lines BC and AD, respectively (distinct from point F). Let K_{ω_A} and L_{ω_A} be the intersection points of the circle ω_A with the lines BD and AC, respectively (distinct from point E). Let P_{ω_A} and Q_{ω_A} be the Pascal points formed by ω_A on the lines AB and CD.

The circle ω_A intersects the line AD at the points N_{ω_A}, A, and F. The point F lies on the extension of the side AD, and therefore $F \neq A$. Additionally, by the definition of N_{ω_A}, we have $F \neq N_{\omega_A}$. Therefore, it necessarily follows that (see Figure 12(a))

$$N_{\omega_A} = A \tag{21}$$

The circle ω_A intersects the line AC at the points L_{ω_A}, A, and E. The point E is the intersection point of the diagonals of the quadrilateral, and therefore $E \neq A$. Additionally, by the definition of L_{ω_A}, we have $E \neq L_{\omega_A}$. It necessarily follows that

$$L_{\omega_A} = A \tag{22}$$

According to Definition 2.2, for the Pascal point P_{ω_A}, the following holds: $P_{\omega_A} = K_{\omega_A}N_{\omega_A} \cap L_{\omega_A}M_{\omega_A} = K_{\omega_A}A \cap AM_{\omega_A} = A$. That is,

$$P_{\omega_A} = A \tag{23}$$

Let M_{ω_B} and N_{ω_B} be the intersection points of the circle ω_B with the lines BC and AD, respectively (distinct from point F). Let K_{ω_B} and L_{ω_B} be the intersection points of the circle ω_B with the lines BD and AC, respectively (distinct from point E). Let P_{ω_B} and Q_{ω_B} be the Pascal points formed by ω_B on the lines AB and CD.

The circle ω_B intersects the line BC at the points M_{ω_B}, B, and F. Therefore, it necessarily follows that (see Figure 12(b))

$$M_{\omega_B} = B \tag{24}$$

The circle ω_B intersects the line BD at the points K_{ω_B}, B, and E. Therefore, it necessarily follows that

$$K_{\omega_B} = B \tag{25}$$

For the Pascal point P_{ω_B}, the following holds:
$P_{\omega_B} = K_{\omega_B}N_{\omega_B} \cap L_{\omega_B}M_{\omega_B} = BN_{\omega_B} \cap L_{\omega_B}B = B$. That is,

$$P_{\omega_B} = B \tag{26}$$

Let M_{ω_C} and N_{ω_C} be the intersection points of the circle ω_C with the lines BC and AD, respectively (distinct from point F). Let K_{ω_C} and L_{ω_C} be the intersection points of the circle ω_C with the lines BD and AC, respectively (distinct from point E).
The circle ω_C intersects the line BC at the points M_{ω_C}, C, and F. Therefore, it necessarily follows that (see Figure 12(c))

$$M_{\omega_C} = C \tag{27}$$

The circle ω_C intersects the line AC at the points L_{ω_C}, C, and E. Therefore, it necessarily follows that

$$L_{\omega_C} = C \tag{28}$$

Finally, let M_{ω_D} and N_{ω_D} be the intersection points of the circle ω_D with the lines BC and AD, respectively (distinct from point F). Let K_{ω_D} and L_{ω_D} be the intersection points of the circle ω_D with the lines BD and AC, respectively (distinct from point E).
The circle ω_D intersects the line AD at the points N_{ω_D}, D, and F. Therefore, it necessarily follows that (see Figure 12(d))

$$N_{\omega_D} = D \tag{29}$$

The circle ω_D intersects the line BD at the points K_{ω_D}, D, and E. Therefore, it necessarily follows that

$$K_{\omega_D} = D \tag{30}$$

We now proceed to the proof of the six equalities stated in the theorem. According to equality (1) in Theorem 2.1, the following holds:
$\frac{P_1P_2}{P_2P_3} = \frac{O_1O_2}{O_2O_3}$. We choose the three circles as follows: $\omega_1 = \omega_A$, $\omega_2 = \omega_i$, $\omega_3 = \omega_B$, and we obtain: $\frac{P_{\omega_A}P_i}{P_iP_{\omega_B}} = \frac{O_{\omega_A}O_i}{O_iO_{\omega_B}}$.
It follows from equalities (23) and (26) that: $\frac{AP_i}{P_iB} = \frac{O_{\omega_A}O_i}{O_iO_{\omega_B}}$.
We have proved part 3.1.1 of Theorem 3.1.

To prove part 3.1.2, we use the equality $\frac{M_1M_2}{M_2M_3} = \frac{O_1O_2}{O_2O_3}$ (see equality (1)).
We choose $\omega_1 = \omega_B$, $\omega_2 = \omega_i$, and $\omega_3 = \omega_C$, and obtain: $\frac{M_{\omega_B}M_i}{M_iM_{\omega_C}} = \frac{O_{\omega_B}O_i}{O_iO_{\omega_C}}$.
It then follows from equalities (24) and (27) that: $\frac{BM_i}{M_iC} = \frac{O_{\omega_B}O_i}{O_iO_{\omega_C}}$.

Part 3.1.3 holds by Lemma 2.2.

To prove part 3.1.4, we use the equality $\frac{N_1N_2}{N_2N_3} = \frac{O_1O_2}{O_2O_3}$ (see equality (1)).
We choose $\omega_1 = \omega_D$, $\omega_2 = \omega_i$, and $\omega_3 = \omega_A$, and obtain: $\frac{N_{\omega_D}N_i}{N_iN_{\omega_A}} = \frac{O_{\omega_D}O_i}{O_iO_{\omega_A}}$.
It then follows from equalities (29) and (21) that: $\frac{DN_i}{N_iA} = \frac{O_{\omega_D}O_i}{O_iO_{\omega_A}}$.

To prove part 3.1.5, we use the equality $\frac{K_1K_2}{K_2K_3} = \frac{O_1O_2}{O_2O_3}$ (see equality (1)).
We choose $\omega_1 = \omega_B$, $\omega_2 = \omega_i$, and $\omega_3 = \omega_D$, and obtain: $\frac{K_{\omega_B}K_i}{K_iK_{\omega_D}} = \frac{O_{\omega_B}O_i}{O_iO_{\omega_D}}$.
It then follows from equalities (25) and (30) that: $\frac{BK_i}{K_iD} = \frac{O_{\omega_B}O_i}{O_iO_{\omega_D}}$.

To prove part 3.1.6, we use the equality $\frac{L_1L_2}{L_2L_3} = \frac{O_1O_2}{O_2O_3}$ (see equality (1)).
We choose $\omega_1 = \omega_A$, $\omega_2 = \omega_i$, and $\omega_3 = \omega_C$, and obtain: $\frac{L_{\omega_A}L_i}{L_iL_{\omega_C}} = \frac{O_{\omega_A}O_i}{O_iO_{\omega_C}}$.
It then follows from equalities (22) and (28) that: $\frac{AL_i}{L_iC} = \frac{O_{\omega_A}O_i}{O_iO_{\omega_C}}$.

4. Conclusion

In this paper, we presented new properties of the centers of circles that form Pascal points on the sides of a quadrilateral.
For any quadrilateral (that is not a parallelogram), there exist four fixed points—namely, the centers of four circles passing through the points E and F, and one of the vertices of the quadrilateral.
The center of each circle that forms Pascal points divides the six segments, determined by the four centers of the fixed circles, in the same ratio as the six points determined by the circle that forms Pascal points divide the four sides and the two diagonals of the quadrilateral.

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