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CONICS WITH THREE EQUAL CHORDS THROUGH A POINT

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Abstract. In this note we study the existence of a conic passing through the endpoints of three segments of equal lengths, their supporting lines intersecting at a point. We show that, in general, there are always two solutions of the problem.

1. Introduction

Given a conic κ, one can easily construct several chords of equal length. However, if we impose the additional restriction that these equal-length chords must have supporting lines passing through the same fixed point O, then the existence of more than two such chords becomes non-trivial.

For example, in the case of a circle, there are generally two equal-length chords passing through a given point O. If the chord length equals the diameter of the circle and O is distinct from the center, there is only one such chord. On the other hand, if O coincides with the center, there are infinitely many diameters (which are chords of maximal length). In this note we deal with the somewhat converse problem. We are given three segments $\{AB, CD, EF\}$ of equal length k, whose supporting lines pass through the same point O (see Figure 1). We investigate whether there exists a conic passing through their endpoints.

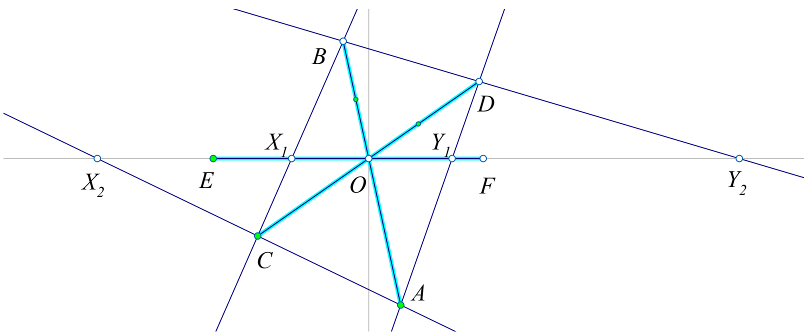

Figure 1. Three equal segments through the origin

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Without loss of generality, we may assume: The common point is the origin $O(0,0)$. The last segment EF lies along the x-axis, with endpoints $E(e,0)$ and $F(e+k,0)$. The other two segments AB and CD can be described using unit vectors $\{e_1(c_1,s_1), e_2(c_2,s_2)\}$ along with scalar parameters a and c:

$$(1)\qquad A = ae_1 = (ac_1, as_1)\ ,\quad B = (a+k)e_1 = ((a+k)c_1, (a+k)s_1)\ ,$$

$$(2)\qquad C = ce_2 = (cc_2, cs_2)\ ,\quad D = (c+k)e_2 = ((c+k)c_2, (c+k)s_2)\ .$$

The search for a conic passing through the six endpoints of these segments relies on *"Desargues' involution theorem"* ([1, I,p.128]), which states:
"The member-conics of a pencil intersect, on an arbitrary line in pairs of points that form an involution."
Here we consider the pencil $\mathcal{P}$ of conics passing through the four points $\{A,B,C,D\}$ and seek the conic in $\mathcal{P}$ that also passes through E and F. The involution induced on the x-axis by its intersections with the conics of $\mathcal{P}$ is uniquely determined by the intersections with two particular members of $\mathcal{P}$ (for background on homographies and involutions, see [2]). By selecting two such members and computing the involution they define, we can test whether the points E and F satisfy this involution.

A convenient choice is to use the *"singular members"* of $\mathcal{P}$, represented by the degenerate conics (BC, AD) and (AC, BD). These intersect the x-axis at points (X_1, Y_1) and (X_2, Y_2) respectively (see Figure 1).

2. The involution

The involution, induced on the x-axis by the intersections of the conics in the pencil $\mathcal{P}$ has the general form

$$(3)\quad y = \frac{ux+v}{wx-u} \quad\Leftrightarrow\quad wxy - u(x+y) - v = 0 \quad \text{with} \quad u^2 + vw \neq 0\ .$$

The coefficients (u,v,w) are uniquely determined up to a non-zero multiplicative constant by specifying two corresponding pairs $\{(x_1,y_1)\ ,\ (x_2,y_2)\}$. Given such pairs and denoting a general pair of corresponding points by (x,y), the coefficients must satisfy the following system derived from (3):

$$\begin{cases} wx_1y_1 - u(x_1+y_1) - v = 0,\\ wx_2y_2 - u(x_2+y_2) - v = 0,\\ wxy - u(x+y) - v = 0. \end{cases}$$

This homogeneous linear system in (u,v,w) has a non-trivial solution only if its determinant vanishes:

$$(4)\qquad \begin{vmatrix} x_1y_1 & x_2y_2 & xy \\ x_1+y_1 & x_2+y_2 & x+y \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

which simplifies to the involution's equation:

$$(5)\qquad Lxy + M(x+y) + N = 0,$$

where

$$L = (x_1 + y_1) - (x_2 + y_2),$$
$$M = x_2y_2 - x_1y_1,$$
$$N = x_1y_1(x_2 + y_2) - x_2y_2(x_1 + y_1).$$

The coordinates of the corresponding points are explicitly given by:

$$x_1 = \frac{c(k+a)(c_1s_2 - c_2s_1)}{cs_2 - (k+a)s_1}, \qquad y_1 = \frac{a(k+c)(c_1s_2 - c_2s_1)}{(k+c)s_2 - as_1}, \tag{6}$$

$$x_2 = \frac{ac(c_1s_2 - c_2s_1)}{cs_2 - as_1}, \qquad y_2 = \frac{(k+a)(k+c)(c_1s_2 - c_2s_1)}{(k+c)s_2 - (k+a)s_1}. \tag{7}$$

Substituting these into the determinant condition (4) yields the explicit formula for the involution (5).

3. The condition

The necessary and sufficient condition for the existence of a conic passing through the six points $\{A, B, C, D, E, F\}$ results by replacing the preceding values of (x_1, y_1), (x_2, y_2) and $x = e$, $f = e+k$ in the determinant of equation (4). This leads to a quadratic equation for e in terms of the other constants. In fact, setting

$$U = s_2c(k + 2a)(k + c), \quad V = s_1a(k + 2c)(k + a) \ ,$$
$$W = (s_1c_2 - s_2c_1)ac(k + a)(k + c),$$

and evaluating the determinant in equation (4), we get, after some factorization, the equation:

$$e^2(U - V) + e(2W - k(V - W)) + kW = 0. \tag{8}$$

The discriminant of the quadratic is seen to be equal to

$$q = k^2(U - V)^2 + 4W^2,$$

showing that the problem, in general, always has a solution. Thus, for any given values of the other constants, we can, in general, find two solutions for e producing two conics solving the problem at hand:

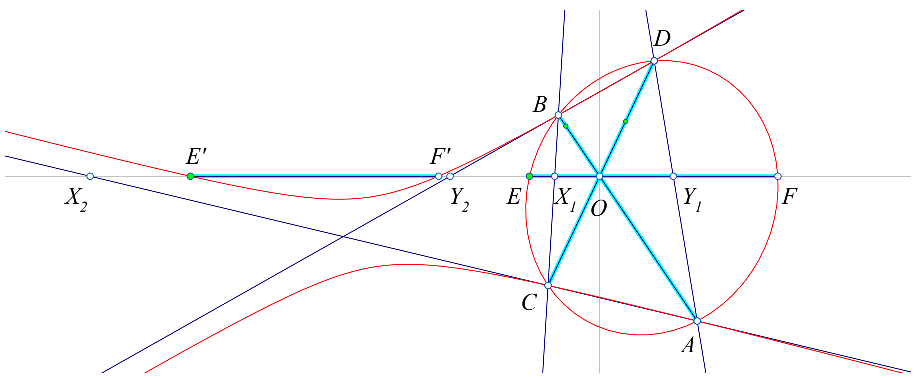

Figure 2. Two conics-solutions of the problem

Theorem 3.1. *Given three segments $\{AB, CD, EF\}$ of length equal to $k > 0$, whose supporting lines pass through the same point O, we can, varying the segment EF on its support line while preserving its length, find in general two positions for it, so that through the six points $\{A, B, C, D, E, F\}$ passes a conic.*

Figure 2 shows the two solution-conics for the particular configuration of $\{AB, CD\}$ and the two appropriate positions of the segment EF and $E'F'$, all segments having the same length. The first triple $\{AB, CD, EF\}$ are chords of an ellipse. The second triple $\{AB, CD, E'F'\}$ are chords of a hyperbola.

The investigation of particular cases, in which $M^2 - LN = 0$ or the discriminant or some of the coefficients of the quadratic equation vanish, and we have only one, or infinitely many solutions, as in the case of a circle and its diameters, is left as exercise.

References

[1] Veblen, O. , Young, J. *Projective Geometry vol. I, II.* Ginn and Company, New York, (1910).

[2] Pamfilos, P., *Homographic relation*, http://users.math.uoc.gr/~pamfilos/eGallery/problems/HomographicRelation.pdf, 2024.

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