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ANOTHER REFINEMENT OF THE GARFUNKEL–BANKOFF INEQUALITY

TRAN QUANG HUNG

Abstract. In this article, we present a new refinement of the classical Garfunkel–Bankoff inequality, which also yields strengthened versions of the Finsler–Hadwiger and Kooi inequalities. We compare our result with previous refinements by Wei-Dong Jiang and the author, emphasizing the independence and originality of our approach. A conjecture on the sharpness of the correction term's constant is also proposed.

1. Introduction

In triangle ABC, we adopt the standard notations: A, B, and C denote the interior angles, while a, b, and c represent the lengths of the sides opposite these angles, respectively. Furthermore, R, r, and s denote the circumradius, inradius, and semiperimeter of triangle ABC.

In [3], J. Garfunkel established the following inequality:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{1}$$

The first proof of inequality (1) was given by L. Bankoff in [1], and thus inequality (1) is also referred to as the Garfunkel–Bankoff inequality [1, 2, 3, 5, 10, 14]. Notably, inequality (1) serves as an enhancement of both the Finsler–Hadwiger and Weitzenböck inequalities (see [10, 11, 12]):

$$a^2+b^2+c^2\geq 4\sqrt{3}S+(b-c)^2+(c-a)^2+(a-b)^2, \tag{2}$$

and is equivalent to Kooi's inequality (see [8]):

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}. \tag{3}$$

In [4], Wei-Dong Jiang proposed a refinement of the Garfunkel–Bankoff inequality:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\frac{r^2(R-2r)}{4R^2(R-r)}. \tag{4}$$

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In the same vein, the author in [13] presented an even stronger result:

(5)
$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}.$$

Clearly, inequality (5) constitutes a strengthened version of inequality (1). Consequently, inequality (1) may be viewed as a refinement of both the Finsler–Hadwiger and Kooi inequalities. Additionally, M. Lukarevski and D. S. Marinescu introduced a related refinement of Kooi's inequality (see [8]):

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}-\frac{r^2(R-2r)}{4R}. \tag{6}$$

However, inequality (5) remains stronger than (6) (see [13]).

In this paper, we propose a new refinement of the Garfunkel–Bankoff inequality in the following form:

(7)
$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\frac{4(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}.$$

By applying the Law of Sines and some basic trigonometric transformations, we also obtain the trigonometric form of inequality (7) as follows:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+ \\ +4\left(\frac{\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\right)^2. \tag{8}$$

It is worth noting that a notable trigonometric refinement of the Garfunkel–Bankoff inequality also appeared in [9], stated as follows:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+ \\ +\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}. \tag{9}$$

However, we also note that the remainders of inequalities (8) and (9) cannot be compared.

Evidently, inequality (7) constitutes a direct extension of the Garfunkel–Bankoff inequality (1). From inequality (7), we derive the following new refinements of the Finsler–Hadwiger and Lukarevski–Marinescu inequalities:

- A new refinement of the Finsler–Hadwiger inequality:

(10)
$$a^2+b^2+c^2\geq 4\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{4R}+\frac{4(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}}S+ \\ +(b-c)^2+(c-a)^2+(a-b)^2,$$

- A new refinement of the Lukarevski–Marinescu inequality (which includes Kooi's inequality as a special case):

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R} - \frac{4(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}. \tag{11}$$

2. Proof of the inequality (7)

In this proof, we employ the following sharppened of Gerretsen's inequality [7]

$$16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r}. \tag{12}$$

Proof. Using the well-known identities

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} = \frac{(4R+r)^2}{s^2} - 2,$$

and

$$\frac{4(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2} = \frac{-s^4 + (4R^2+20Rr-r^2)s^2 - r^4 - 12Rr^3 - 48R^2r^2 - 64R^3r}{R^2s^2}.$$

The inequality (7) is equivalent to

$$\frac{s^4 - 2(4R^2+9Rr-r^2)s^2 + 16R^4 + r^4 + 12Rr^3 + 49R^2r^2 + 72R^3r}{R^2s^2} \geq 0. \tag{13}$$

Now we write

$$s^4 - 2(4R^2+9Rr-r^2)s^2 + 16R^4 + r^4 + 12Rr^3 + 49R^2r^2 + 72R^3r$$
$$= (4R^2+4Rr+3r^2-s^2)^2 + (10R-8r)\left(4R^2+4Rr+3r^2+\frac{r^2(R-2r)}{R-r} - s^2\right) \tag{14}$$
$$+ \frac{Rr(R+r)(R-2r)}{R-r} \geq 0$$

by the sharppened of Gerretsen's inequality (12). This completes our proof. □

3. Conclusion

First, we observe that the refinement given in inequality (7) is fundamentally different from that in inequality (5), since the two remainder terms

$$\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)} \quad \text{and} \quad \frac{4(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

cannot be directly compared—that is, there exists no known inequality relating them in general. As a result, the two refinements are independent and neither is a consequence of the other.

Another observation, which we currently state only as a conjecture, is that in the generalized inequality

(15)
$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+k\cdot\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2},$$

the constant $k=4$ appears to be the best possible. We welcome any insights or formal proof from interested readers.

In conclusion, inequality (7) represents a novel and independent refinement of the classical Garfunkel–Bankoff inequality. Furthermore, it naturally leads to new strengthened versions of both the Finsler–Hadwiger inequality and the Lukarevski–Marinescu inequality. These results contribute to the broader effort of systematically improving classical triangle inequalities and may inspire further research into sharp bounds involving elementary triangle functions.

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High School for Gifted Students, Hanoi University of Science, Vietnam National University at Hanoi, 182 Luong The Vinh, Thanh Xuan, Hanoi, Vietnam

Email address: tranquanghung@hus.edu.vn