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ON A CONSTRUCTION PROBLEM

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Abstract. Given the points O, I and I_a, we set ourselves the problem of constructing with an ungraduated ruler and a compass the triangle ABC which has the three given points as the centers of the circles: O circumscribed, I inscribed, respectively I_a escribe of the triangle ABC. This construction problem is also found in [2], problem 614, pages 446–447. In this article, we approach the problem differently, giving both the necessary and sufficient condition for the existence of the construction, that is, what relationship exists between OI, OI_a and II_a.

1. Introduction

In this section we will recall some known results which we will use in the following.
For a triangle ABC, we will use the established notations: A, B, C the measures of the angles; a, b, c the lengths of the sides; $\mathcal{C}(O, R)$ the circumscribed circle with center O and radius R; $\mathcal{C}(I, r)$ the inscribed circle with center I and radius r; $\mathcal{C}(I_a, r_a)$ the escribe circle with center I_a and radius r_a, corresponding to the side BC; T the area of the triangle ABC and s the semiperimeter.

Lemma 1. *In a triangle ABC, the following relationships hold*

$$r_a > r, \tag{1.1}$$

$$\sin\frac{A}{2} = \frac{1}{2}\sqrt{\frac{r_a - r}{R}}\,, \tag{1.2}$$

$$r_a + r = \frac{(b+c)T}{s(s-a)} \tag{1.3}$$

and

$$r_a - r = \frac{aT}{s(s-a)}\,. \tag{1.4}$$

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Theorem 1. *In triangle ABC, the following relationships hold*

$$OI = \sqrt{R^2 - 2Rr}\,, \tag{1.5}$$

$$OI_a = \sqrt{R^2 + 2Rr_a}\,, \tag{1.6}$$

$$OI < OI_a \tag{1.7}$$

and

$$II_a = 2\sqrt{R(r_a - r)}. \tag{1.8}$$

Proof. Since relation (1.8) is not very known, we will prove it.

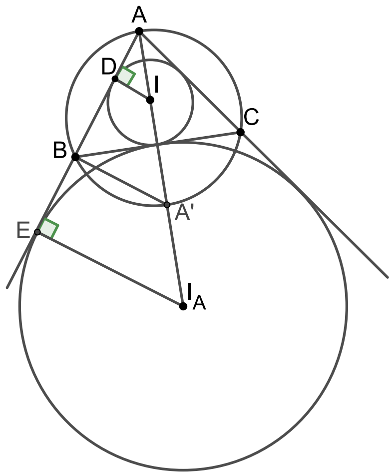

Figure 1.1

Let the points D and E such that $ID \perp AB$, $I_aE \perp AB$, $D, E \in AB$ (see Fig. 1.1).

In the triangles AID and AI_aE we have $AI = \dfrac{r}{\sin\dfrac{A}{2}}$ and $AI_a = \dfrac{r_a}{\sin\dfrac{A}{2}}$,

from where $II_a = AI_a - AI = \dfrac{r_a - r}{\sin\dfrac{A}{2}}$.

Taking (1.2) into account, the relation (1.8) follows.

Remark 1. The relation from (1.5) is named Euler's Theorem. Leonard Euler published this result in 1765. However, the same result was published earlier by William Chapple in 1746.

Lemma 2. *If in triangle ABC, $AI \cap \mathcal{C}(O, R) = \{A, A'\}$, then $A'B = A'I$.*

Proof. In triangle $BA'I$ (see Fig. 1.1), we have that $\widehat{IBA'} = \dfrac{B}{2} + \dfrac{A}{2} = \dfrac{\pi - C}{2}$ and $\widehat{BA'I} = C$, from where $\widehat{BIA'} = \pi - \widehat{IBA'} - \widehat{BA'I} = \dfrac{\pi - C}{2}$. Then $\widehat{IBA'} = \widehat{BIA'}$, so triangle $A'BI$ is isosceles. □

In what follows, we will remind you how to make two constructions, using only ungraduated ruler and a compass.

Construction 1. The perpendicular from a given point M to a gven line d (see Fig. 1.2).

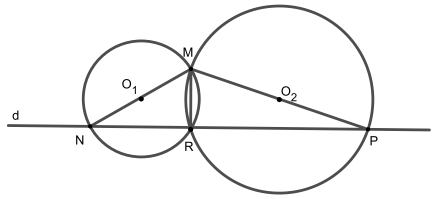

Figure 1.2

Let the points $N, P \in d$. Using the compass, we construct the means O_1, O_2 of the segments MN and MP. Using the compass, we construct the circles with centers O_1, O_2 and diameters MN and respectively MP. These two circles intersect at a point R which is on the line d and we have as $MR \perp d$.

Construction 2. The tangent from a given point M to a given circle $\mathcal{C}(O)$ (see Fig. 1.3).

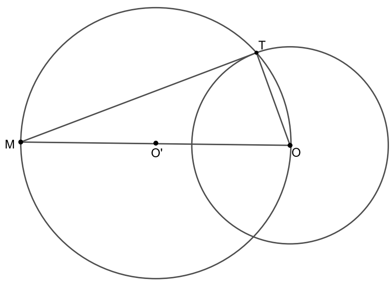

Figure 1.3

Using the compass, we construct the middle O' of the segment MO. With the compass, we construct a circle with center O' and diameter OM, which intersects the circle $\mathcal{C}(O)$ in T and T'. It is immediately seen that $\widehat{MTO} = \dfrac{\pi}{2}$, so MT is a tangent to the circle $\mathcal{C}(O)$.

2. Main results and construction

In this section we will give some results that will help us in the construction of triangle ABC.

Lemma 3. *In a triangle ABC, the following inequality*

$$\frac{(r_a + r)^2}{r_a - r} \le 4R \tag{2.1}$$

holds.

Proof. Taking (1.3) and (1.4) into account, the inequality (2.1) becomes

$$\frac{(b+c)^2T^2}{(s(s-a))^2} \cdot \frac{s(s-a)}{aT} \le 4\frac{abc}{4T},$$

equivalent to $(b+c)^2(s-b)(s-c) \le a^2bc$, equivalent to $(b+c)^2(a-b+c)(a+b-c) \le 4a^2bc$, equivalent to $(b+c)^2(a^2-(b-c)^2) \le 4a^2bc$, equivalent to $0 \le 4a^2bc - a^2(b+c)^2 + (b+c)^2(b-c)^2$, equivalent to $0 \le -a^2(b-c)^2 + (b+c)^2(b-c)^2$, from where we obtain the true inequality $0 \le (b-c)^2(b+c-a)(b+c+a)$. From the inequality it results that the equality in (2.1) holds if and only if $b = c$, so the triangle ABC is an isosceles triangle. □

Theorem 2. *The numbers* $d = \sqrt{R^2 - 2Rr}$, $d_a = \sqrt{R^2 + 2Rr_a}$ *and* $d_A = 2\sqrt{R(r_a - r)}$ *can be the lengths of the sides of a triangle, which can also be degenerate.*

Proof. We must show that $d < d_a + d_A$, $d_a \le d_A + d$ and $d_A \le d + d_a$. The first inequality is true. The second inequality is equivalent to $\sqrt{R + 2r_a} - \sqrt{R - 2r} \le 2\sqrt{r_a - r}$, equivalent to $(R + 2r_a) + (R - 2r) - 2\sqrt{(R + 2r_a)(R - 2r)} \le 4r_a - 4r$, equivalent to

$$R - r_a + r \le \sqrt{(R + 2r_a)(R - 2r)}. \tag{2.2}$$

The third inequality is equivalent to $2\sqrt{r_a - r} \le \sqrt{R - 2r} + \sqrt{R + 2r_a}$, equivalent to $4r_a - 4r \le (R - 2r) + (R + 2r_a) + 2\sqrt{(R + 2r_a)(R - 2r)}$, equivalent to

$$-(R - r_a + r) \le \sqrt{(R + 2r_a)(R - 2r)}. \tag{2.3}$$

From (2.2) and (2.3) it follows that we must prove that $|R - r_a + r| \le \sqrt{(R + 2r_a)(R - 2r)}$. This inequality is equivalent to $R^2 + r_a^2 + r^2 - 2Rr_a + 2Rr - 2r_a r \le R^2 - 2Rr + 2Rr_a - 4r_a r$, equivalent to $(r_a + r)^2 \le 4R(r_a - r)$. Taking relation (2.1) into account, it follows that the above inequality is true, so the inequalities (2.2) and (2.3) are true.

Remark 2. According to Theorem 2, there exists triangle OII_a, which can also be degenerate.

Theorem 3. *Let the points* O, I, I_a *and the distances between them be* $OI = d$, $OI_a = d_a$, $II_a = d_A$. *The points* O, I, I_a *are the centers of the circumscribed, inscribed, respectively escribe circles of a triangle* ABC *if and only if*

$$2(d_a^2 - d^2) > d_A^2 > 0 \tag{2.4}$$

and we have that

$$\begin{cases} R = \dfrac{\sqrt{2(d_a^2 + d^2) - d_A^2}}{2} \\ r = \dfrac{2(d_a^2 - d^2) - d_A^2}{8R} \\ r_a = \dfrac{2(d_a^2 - d^2) + d_A^2}{8R}, \end{cases} \tag{2.5}$$

where R, r, r_a *are the radius of these circles.*

Proof. If points O, I, I_a are the centers of the circumscribed, inscribed, respectively escribe circles of triangle ABC, then relations (1.5), (1.6) and (1.8) hold, so $d^2 = R^2 - 2Rr$, $d_a^2 = R^2 + 2Rr_a$ and $d_A^2 = 4R(r_a - r)$. From the first two relations we have $d_a^2 + d^2 = 2R^2 + R(r_a - r)$ and using the last relation it results that $R^2 = \dfrac{2(d_a^2 + d^2) - d_A^2}{4}$.
Since $2(d_a^2 + d^2) > 2(d_a^2 - d^2)$ and taking into acoount the second in (2.4), it results that $2(d_a^2 + d^2) - d_A^2 > 0$.

From the above, the first relation in (2.5) follows.
From the first two relations above we obtain that $r = \dfrac{R^2 - d^2}{2R}$ and $r_a = \dfrac{d_a^2 - R^2}{2R}$. Substituting R, after calculus we obtain the last two relations from (2.5). Taking into account (2.4), if follows that $r > 0$. From (2.4) it results that $d_a > d$ and then it follows that $r_a > 0$.
Conversely, if given triangle ABC, then $d_A^2 = 4R(r_a - r)$ and taking (1.1) into account, it results that $d_A^2 > 0$. Replacing d, d_a, d_A we have that $2(d_a^2 - d^2) - d_A^2 = 2((R^2 + 2Rr_a) - (R^2 - 2Rr)) - 4R(r_a - r) = 8Rr > 0$, so $2(d_a^2 - d^2) > d_A^2$. From the above it follows that the inequalities in (2.4) hold.

Theorem 4. *In triangle OII_a, the length of the median from vertex O is equal to R.*

Proof. Let m_0 the length of the median from the vertex O (see Fig. 2.1).

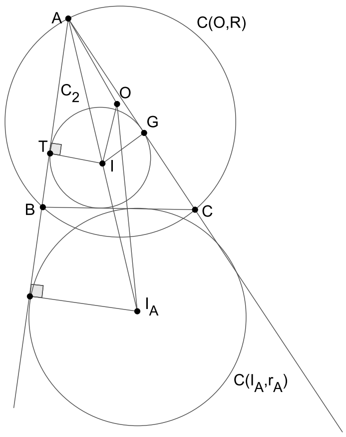

Figure 2.1

We have $m_0^2 = \dfrac{2(OI^2 + OI_a^2) - II_a^2}{4}$ and taking (1.5), (1.6), (1.8) into account it results that $m_0^2 = $
$= \dfrac{2(R^2 - 2Rr + r^2 + 2Rr_a) - (4Rr_a - 4Rr)}{4} = R^2$, from where $m_0 = R$.

Remark 3. If A' is the midpoint of segment II_a, then $OA = m_0 = R$, so $A' \in \mathcal{C}(O, R)$.

Three points are given. The task is to construct with an ungraduated ruler and compass the triangle ABC for which O, I, I_a are respectively the center of the circumscribed circle, the center of the inscribed circle, and the center of the escribe circle of the triangle ABC. Also to the construct the three circles.
We do the construction through following steps.

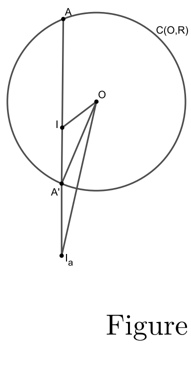

Figure 2.2

Step 1. Using the compass, we construct the midpoint A' ofn the segment II_a. According to Theorem 4, the length of the median OA' is equal to R. We take the compass opening OA' and draw the circle $\mathcal{C}(O, R)$ (see Fig. 2.2). We have that $II_a \cap \mathcal{C}(O, R) = \{A, A'\}$.

Step 2. Considering that AA' is the bisector of angle A in triangle ABC, it follows that $A'B = A'C$ and according to Lemma 2 we have that $A'B = A'I$. We take the compass opening $A'I$ and with the opening with the compass vertex at the point A', we intersect the circle $\mathcal{C}(O, R)$ in B and C (see Fig. 2.3).

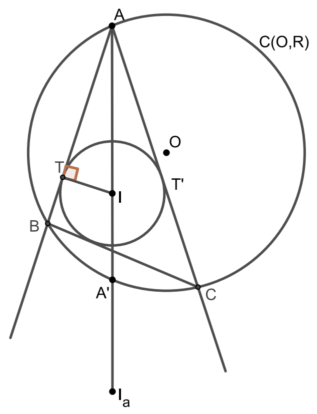

Figure 2.3

Using the idea from Construction 1, we construct the point $T \in AB$, such that $IT \perp AB$. The length of the IT is equal to r. We take the compass opening IT, with tha compass vertex at the point I we draw the circle $\mathcal{C}(I, r)$.

Step 3. Using the idea from Construction 1, we construct the point $U \in BC$ succh that $I_aU \perp BC$. Then I_aU is equal to r_a and we draw the circle $\mathcal{C}(I_a, r_a)$.
With this the construction is finished.

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