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ON CERTAIN RELATIONS IN THE ARBELOS

ARNE ERIKSON

Abstract. In this article we consider certain relations in the arbelos connecting its twin circles and incircle with two other circles, each internally tangent to the outer semicircle and each externally tangent respectively to the left and the right internal semicircle of the arbelos. The main result is that the sum of the two circles' radi equals the incircle radius.

1. Introduction

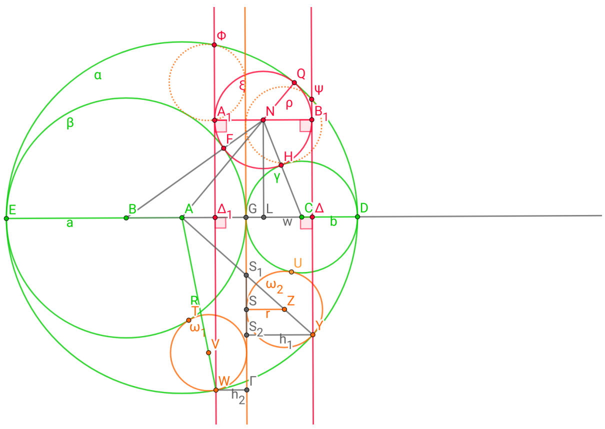

Figure 1. The chords $W\Phi$ and $Y\Psi$ are tangent to the arbelos incircle ξ at A_1 and B_1.

For convenience, in all figures we reflect the arbelos semicircles about the baseline DE to obtain full circles (see Figure 1). We will use the term arbelos for this full circle configuration although strictly speaking an arbelos is only defined by semicircles ("Archimedes' Book of Lemmas" in [3], prop 4, p.562). The twin circles of an arbelos (dotted in the figure) and their reflections ω_1 and ω_2 in the baseline, touch the outer circle α at the points $\{W,\Phi\}$ and $\{Y,\Psi\}$ respectively.

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Proposition 1.1. *The chords $W\Phi$ and $Y\Psi$ are tangent to the arbelos incircle ξ at A_1 and B_1, respectively.*

Proof. This will be shown in two ways, first (i) not assuming the hypothesis, and second (ii) assuming its truth.

(i) We have the following circles (see Figure 1): $\alpha(A,R)$, $\beta(B,a)$, $\gamma(C,b)$, $\xi(N,\rho)$, $\omega_1(V,r)$ and $\omega_2(Z,r)$, where ([2] p.210,[6])

$$R = a + b, \quad \rho = \frac{ab(a+b)}{a^2+ab+b^2}, \quad r = \frac{ab}{a+b}.$$

Also

$AY = R, \quad AZ = R-r, \quad SZ = r, \quad AG = R-2b, \quad YS_2 = h_1, \quad S_1Z = y.$

The triangles $\{SZS_1, AGS_1\}$ are similar giving

$$\frac{SZ}{AG} = \frac{S_1Z}{AS_1} \Leftrightarrow \frac{r}{R-2b} = \frac{y}{(R-r)-y},$$

$$\Rightarrow y = \frac{r(R-r)}{R-2b+r}. \tag{1}$$

Also, the triangles $\{YS_1S_2, SZS_1\}$ are similar giving

$$\frac{YS_2}{SZ} = \frac{YS_1}{S_1Z} \Leftrightarrow \frac{h_1}{r} = \frac{y+r}{y},$$

$$\Rightarrow h_1 = r(1+\frac{r}{y}). \tag{2}$$

Substituting for y from (1) we find

$$h_1 = \frac{2r}{R-r}(R-b), \tag{3}$$

and similarly we find

$$h_2 = \frac{2r}{R-r}(R-a). \tag{4}$$

Using expressions from above for R, ρ, and r we have

$$h_1 = \frac{2a^2b}{a^2+ab+b^2} \tag{5}$$

$$h_2 = \frac{2ab^2}{a^2+ab+b^2} \tag{6}$$

$$\Rightarrow h_1 + h_2 = 2\rho.$$

(ii) By hypothesis: $\Delta_1L=L\Delta=\rho$ (see Figure 1). In the triangle $\{NCL\}$ we have $NC=b+\rho$, $LC=w$, $NL=v$, and in the triangle $\{NAL\}$ we have $NL=v$, $AL=a-w$, $AN=(a+b)-\rho$. Using the Pythagorean theorem we find

$$LC = w = \frac{b^2(b+2a)}{a^2+ab+b^2}. \tag{7}$$

$$GL = b - w = \frac{ab(a-b)}{a^2+ab+b^2}. \tag{8}$$

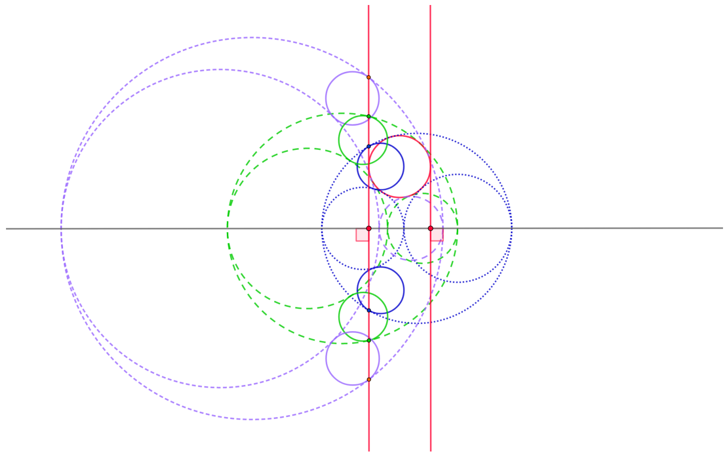

FIGURE 2. Three generations of arbeli (dashed) by theorem 5.4 of [7], all sharing the parallel lines shown. Representative twin circles (whole) closely connected with the parallel lines are shown on the left line. The shared incircle is shown in red.

$$\Rightarrow \Delta_1 L - GL = \rho - GL = \frac{2ab^2}{a^2+ab+b^2} = h_2.$$

And similarly

$$L\Delta + GL = \rho + GL = h_1.$$

Hereafter we refer to a member of the circles ω_i as a "twin circle". We note that the extended lines of the parallel chords in proposition 1.1 are shared by all arbeli in theorem 5.4 of [7], meaning that the twin circles in each arbelos all have their tangent points with the outer semicircle on the same line (either left or right) of the two parallels. In figure 2, we show three generations of arbeli reflected in their common baseline (green, blue, purple full circles), and their respective twin circles on the left parallel line. The shared incircle (red) is also shown.

2. Parallel and perpendicular lines.

We quote a result (adjusting the notation to our figures) from [5], (cor.4, p.4, fig.4): "If EJ_1 cuts β at P and DJ_1 cuts γ at I, then PI is a common tangent to β and γ" (see Figure 3). From this result a corollary can be made:

Corollary 2.1. *J_1J_2, BP and CI are parallel $\Rightarrow$ $J_2\Sigma$ is perpendicular to PI.*

Proof. E is the external similarity center of the circles β and α, so the points P and J_1 are homologous. Thus the radi BP and AJ_1 are parallel implying the perpendicularity of the lines $J_2\Sigma$ and PI. Also, the points P and I are homologous with respect to the external similarity center K (see Figure 4) of β and γ, so their radi BP and CI are parallel.

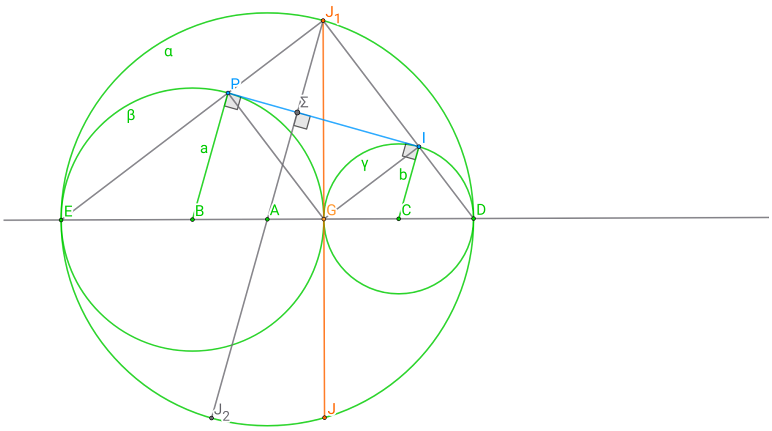

FIGURE 3. PI is a common tangent to the circles β and γ. $J_2\Sigma$ and PI are perpendicular to each other.

3. SUM OF RADI.

Construct a common tangent $K\Lambda$ to the circles β and γ from their external similarity center K (see Figure 4). Construct at each of the two tangent points P and I a circle externally tangent to the circles β and γ respectively and also internally tangent to the circle α. The two circles, δ_1 and δ_2, thus constructed have certain relations with the twin circles ω_1, ω_2 and the incircle ξ.

Theorem 3.1. *The sum of the radi r_L and r_R of the circles δ_1 and δ_2 respectively, equals the the radius ρ of the incircle ξ.*

Proof. [1] We have (see Figure 4):

$$KD = x, \quad KB = Q = a + 2b + x, \quad KP = T, \quad KC = x + b.$$

[1]A different proof has been proposed by Paris Pamfilos (private communication).

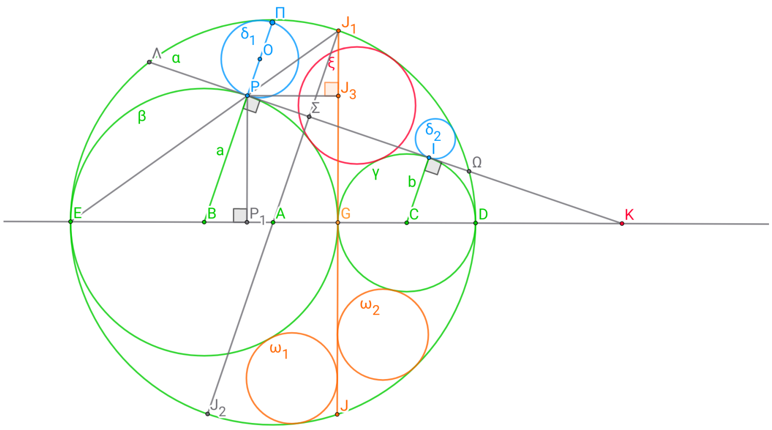

FIGURE 4. Tangent circles δ_1 and δ_2.

$$\frac{a}{KB} = \frac{b}{KC} \Rightarrow x = \frac{2b^2}{a-b}. \tag{9}$$

$$Q = \frac{a(a+b)}{a-b}. \tag{10}$$

$$T^2 = Q^2 - a^2 \Rightarrow T = \frac{2a\sqrt{ab}}{a-b}. \tag{11}$$

Also we have $PP_1{=}h$ and $BP_1{=}s$

$$\frac{h}{a} = \frac{T}{Q} \Rightarrow h = \frac{2a\sqrt{ab}}{a+b}. \tag{12}$$

$$s^2 = a^2 - h^2 \Rightarrow s = \frac{a(a-b)}{a+b}. \tag{13}$$

We find an interesting result that P_1G equals the diameter of the twin circle:

$$P_1G = a - s = \frac{2ab}{a+b} = 2r.$$

Similarly, if we drop a perpendicular (not shown in Figure 4) from I to the baseline DE, the distance from its intersection with DE to the point G will also be equal to the diameter of the twin circle. This will be used later.
We have the parallels $(BP,\ A\Sigma,\ CI)$ (see Section 2) and the lines

$$BP = a, \quad CI = b, \quad A\Sigma = c, \quad KA = x + a + b.$$

Thus,

$$\frac{a}{KB} = \frac{c}{KA} \Rightarrow c = \frac{a^2+b^2}{a+b}. \tag{14}$$

We may confirm a well known result, also known as the "Bankoff quadruplet" circle [2], p. 203 : $J_1\Sigma = R - c = (a+b) - c = \frac{2ab}{a+b} = 2r$, i.e. the diameter of the twin circle.
We have the collinearity (see Section 2) EPJ_1, thus finding:

$$P\Lambda \cdot P\Omega = PJ_1 \cdot PE = P\Pi \cdot J_2 \Rightarrow P\Pi = 2r_L = \frac{PJ_1 \cdot PE}{J_2\Sigma} \tag{15}$$

We used prop.6 p.31 from [1] in the above. We have (see Figure 4):

$$J_1G = 2\sqrt{ab}, \quad P_1G = 2r, \quad PP_1 = h = \frac{2a\sqrt{ab}}{a+b}, \quad J_1J_3 = J_1G - PP_1 = \frac{2\sqrt{ab}(R-a)}{R},$$

and using these we find:

$$PJ_1 = 2b\sqrt{\frac{a}{R}}, \tag{16}$$

$$PE = 2a\sqrt{\frac{a}{R}}, \tag{17}$$

and

$$J_2\Sigma = 2(R-r). \tag{18}$$

Substituting (16), (17) and (18) into (15) gives the radius r_L of the circle δ_1:

$$r_L = \frac{ra}{R-r} = \frac{a\rho}{R}. \tag{19}$$

Similarly for radius r_R of the circle δ_2:

$$r_R = \frac{b\rho}{R}. \tag{20}$$

Summing (19) and (20) we get:

$$r_L + r_R = \frac{(a+b)}{R}\rho = \rho. \tag{21}$$

4. Common tangents.

(See Figure 5).

Proposition 4.1. *The lines $\{W\Phi, Y\Psi\}$ are common tangents to the circle pairs $\{\delta_1, \xi\}$ and $\{\delta_2, \xi\}$, respectively.*

Proof. We use inversion of the lines $\{W\Phi, Y\Psi\}$ relative to the circles $\{\lambda(E), \lambda_1(D)\}$ respectively as our tool (see Figure 6). Both lines are perpendicular to the baseline ED. We focus on the line $W\Phi$ as the same method and result applies to the line $Y\Psi$. An inversion of the line $W\Phi$ relative to $\lambda(E)$ maps it to the circle α ([8], p.361). We know that the points $\{E, P, P_2, J_1\}$ are collinear: $\{E, P, J_1\}$ by ([5], cor.4, p.4, fig.4), and (as $P_2P_3 \| EG$), $\{E, P, P_2\}$ by "Archimedes' Book of Lemmas" in ([3], prop. 1, p. 561). We also know that J_1 is a point on the circumference of the circle α. If P_2 is on the line $W\Phi$, then the points $\{P_2, J_1\}$ are inverses of each other relative to the circle $\lambda(E)$. This then implies that the circle δ_1 is tangent to the line $W\Phi$. We want to show that:

$$EJ_1 \cdot EP_2 = E\Phi^2.$$

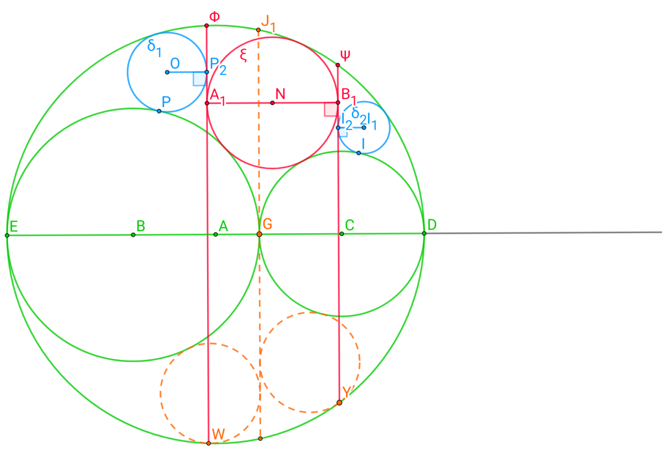

Figure 5. Common tangents $W\phi$ and $Y\Psi$ to circle pairs $\{\delta_1, \xi\}$ and $\{\delta_2, \xi\}$, respectively.

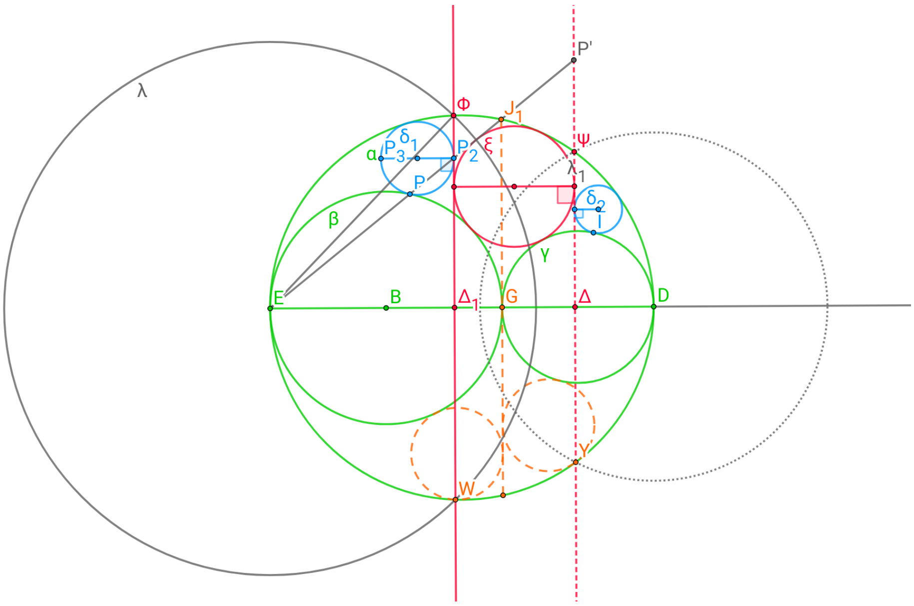

Figure 6. Inverting lines $W\Phi, Y\Psi$ relative to the circles $\lambda(E), \lambda_1(D)$ respectively, maps them each to the circle α.

The calculations proceed as follows (see Figure 7):

$$E\Delta_1 \cdot D\Delta_1 = (\Phi\Delta_1)^2. \tag{22}$$

$$(E\Delta_1)^2 + (\Phi\Delta_1)^2 = (E\Phi)^2. \tag{23}$$

First we find the segments $\{EP_2, EJ_1\}$. From the similar triangles $\{PJ_3J_1, PE_1P_2\}$ we have:

$$\frac{J_1J_3}{PJ_3} = \frac{P_2E_1}{PE_1} \Rightarrow P_2E_1 = \frac{PE_1 \cdot J_1J_3}{PJ_3}, \tag{24}$$

where the following formulas are found in the introduction (1):

$$h_1 = \frac{2a\rho}{R}, \quad h_2 = \frac{2b\rho}{R}, \quad PE_1 = 2r - h_2 = \frac{2b}{R}(a-\rho), \quad J_1J_3 = \frac{2b\sqrt{ab}}{R}, \quad PJ_3 = P_1G = 2r.$$

This gives

$$P_2E_1 = \frac{2b\sqrt{ab}(a-\rho)}{Ra}, \tag{25}$$

and we find:

$$(PE_1)^2 + (P_2E_1)^2 = (PP_2)^2 \Rightarrow PP_2 = \frac{r}{R-r}2a\sqrt{\frac{a}{R}}. \tag{26}$$

Using EP from equation (17) we have:

$$EP_2 = EP + PP_2 = 2a\sqrt{\frac{a}{R}} + \frac{r}{R-r}2a\sqrt{\frac{a}{R}} = 2a\sqrt{\frac{a}{R}}\left(\frac{R}{R-r}\right). \tag{27}$$

We can easily find $EJ_1 = 2\sqrt{aR}$, thus giving:

$$EP_2 \cdot EJ_1 = 4a^2\left(\frac{R}{R-r}\right) = \frac{4a^2R^2}{N}, \tag{28}$$

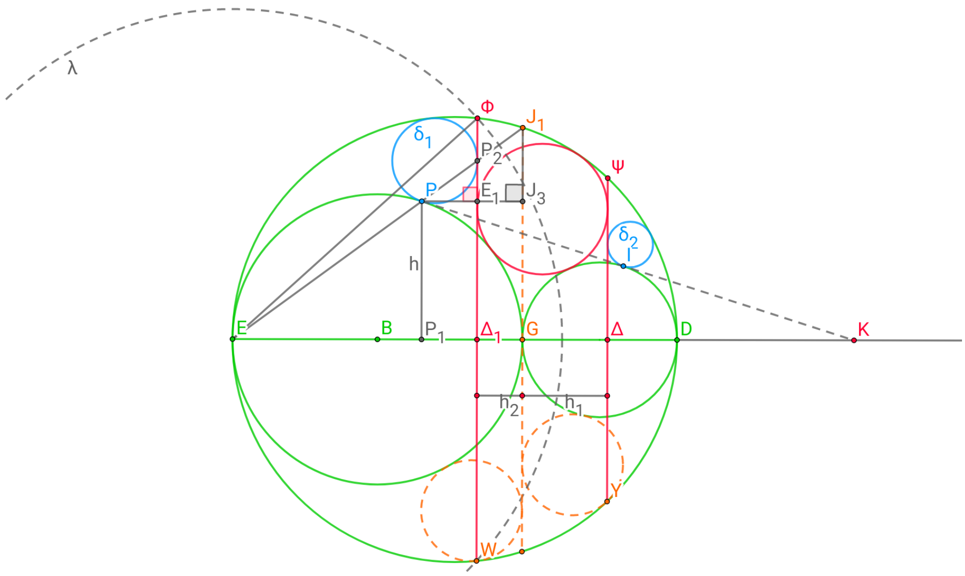

FIGURE 7. Segments used in the proof of Proposition 4.1.

where $N = a^2 + ab + b^2$ i.e. the denominator of the arbelos incircle radius, ρ. Now,

$$\Delta D = DG - \Delta G = 2b - h_1 = \frac{2}{R}(Rb - a\rho) = \frac{2}{R}\left(\frac{b^2R^2}{N}\right), \tag{29}$$

and

$$\Delta_1 D = \Delta D + 2\rho = \frac{2b}{R}(R+\rho) = \frac{2bR^2}{N}. \tag{30}$$

Using $E\Delta_1 = 2R - \Delta_1$ we find (see equation (22)):

$$(\Phi\Delta_1)^2 = (2R - \Delta_1 D)\Delta_1 D. \tag{31}$$

and using equation (23),

$$(E\Phi)^2 = (E\Delta_1)^2 + (\Phi\Delta_1)^2 = (2R - \Delta_1 D)2R = \frac{4R^2a^2}{N}, \tag{32}$$

which agrees with equation (28), thus confirming that the points P_2 and J_1 are indeed inverses relative to the circle $\lambda(E)$. The same will be found for analogous points related to an inversion of the line $Y\Psi$ relative to the circle $\lambda_1(D)$ (see Figure 6).

As a consequence of the above we have the following corollary, which can easily be verified using expressions previously found, where we also indicate the twin circles (dotted) found in section 3 (see Figure 8):

Corollary 4.1.

$$h_1 + 2r_R = 2\rho \tag{33}$$

$$h_2 + 2r_L = 2\rho. \tag{34}$$

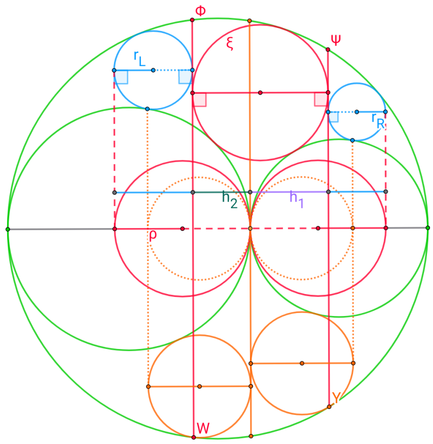

FIGURE 8. Showing $h_1 + 2r_R = 2\rho$ and $h_2 + 2r_L = 2\rho$.

5. Equal circles.

Construct a new arbelos within each of the reference arbelos' internal circles β (left) and γ (right), i.e. with the chords EG and GD as respective baselines. In each new arbelos use a twin circle ω_3 and ω_4 as one of the internal circles (see Figure 9). We will use the subscripts $\{l, r\}$ associated with the left and right new arbeli, respectively.

Proposition 5.1. *The incircles δ_3 and δ_4 of the arbelos $\{\beta,\ \beta_1,\ \omega_3\}$ and $\{\gamma,\ \gamma_1,\ \omega_4\}$, respectively, are congruent to the circles δ_1 and δ_2. Also, the extended line of the segment OO_1 joining the pair of circle centers $\{\delta_1(O),\delta_3(O_1)\}$ is perpendicular to the baseline EG. The same is true for the other pair of congruent circles $\{\delta_2,\ \delta_4\}$.*

Note: we have not shown the analogous segment of OO_1 relating to the circle centers of $\{\delta_2,\ \delta_4\}$ in the figure 9.

Proof. We prove this for the arbelos $\{\beta,\ \beta_1,\ \omega_3\}$ with the circles $\beta(B, R_l)$, $\beta_1(T_1, u)$, $\omega_3(C_1, r)$ having incircle $\delta_3(O_1, \rho_l)$ and a twin circle with radius r_l (not shown in Figure 9), where

$$R_l = u + r = a, \quad \rho_l = \frac{R_l r_l}{R_l - r_l}, \quad r_l = \frac{ur}{u+r} = \frac{r(a-r)}{a}.$$

Then

$$\rho_l = \frac{R_l r_l}{R_l - r_l} = \frac{a\left(\frac{r(a-r)}{a}\right)}{a - \frac{r(a-r)}{a}} = \frac{a^2 b}{(a+b)^2 - ab} = \frac{a\rho}{R}. \tag{35}$$

This last expression is equal to the equation (19), i.e. to the radius r_L of the circle δ_1. In exactly the same way we find that the radius r_R of the circle δ_2 equals the radius ρ_r of the circle δ_4.
We now show that the segments OG' and O_2G are equal, thus proving the perpendicularity of OO_2 and EG. As both circle centers $\{O,\ O_1\}$ are on OO_2, this will prove the last part of proposition 5.1. First, consider the

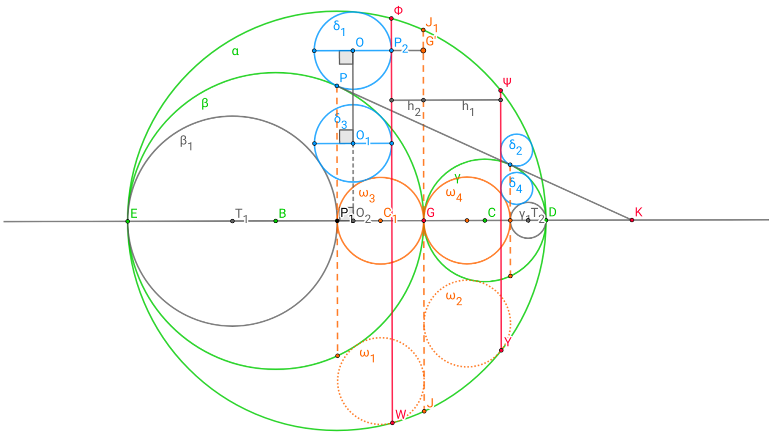

Figure 9. Equal pairs of circles: $\delta_1 = \delta_3$ and $\delta_2 = \delta_4$.
Equal circles: $\omega_1 = \omega_2 = \omega_3 = \omega_4$ (twin circles).

reference arbelos $\{\alpha,\ \beta,\ \gamma\}$ where we found in equation (7) an expression for the segment $w = LC$ (see Figure 1). Adding the radius b to this segment, we find:

$$w + b = \rho\frac{R+b}{a}. \tag{36}$$

This segment is analogous to the segment O_2G, and using the appropriate analogue expressions for the arbelos $\{\beta,\ \beta_1,\ \omega_3\}$ we find:

$$O_2G = \rho_l\frac{R_l + r}{u} = \frac{\rho}{R}(R+b). \tag{37}$$

From figure 9 we deduce $OG' = r_L + h_2$:

$$OG' = r_L + h_2 = \frac{a\rho}{R} + \frac{2b\rho}{R} = \frac{\rho}{R}(R+b) = O_2G. \tag{38}$$

6. Further relations.

(See Figure 10).

Proposition 6.1. *The circle pairs $\{\beta,\ \gamma\}$ and $\{\delta_1,\ \delta_2\}$ have a common external similarity center at the point K.*

Proof. Use E as a center of similarity for the circle $\delta_1(O)$ to make a homothetic circle thus: draw lines $\{EP_3P_4, EPP_2\}$ to intersect the extended chords of $W\Phi$ and $Y\Psi$ at the points A_2 and B_2 respectively. Also draw $EO \cap A_2B_2 = N_1$. We have $A_2B_2 \| P_3P_2$ and $\angle A_2J_1B_2 = 90°$, so $A_2J_1B_2$ is a circle $\xi_1(N_1)$ homothetic to $\delta_1(O)$ and with diameter equal to that of the arbelos incircle $\xi(N)$. We note that J_1 is a tangent point for the cirles $\{\xi_1, \alpha\}$. Likewise we make the same circle $\xi_1(N_1)$ homothetic to the circle $\delta_2(I_1)$ from the similarity center D. We now have two external similarity centers, E and D, one for each of the circle pairs $\{\delta_1,\xi_1\}$ and $\{\delta_2,\xi_1\}$ respectively. It is known ([8] p.507, [4], p.151) that these two external similarity centers will

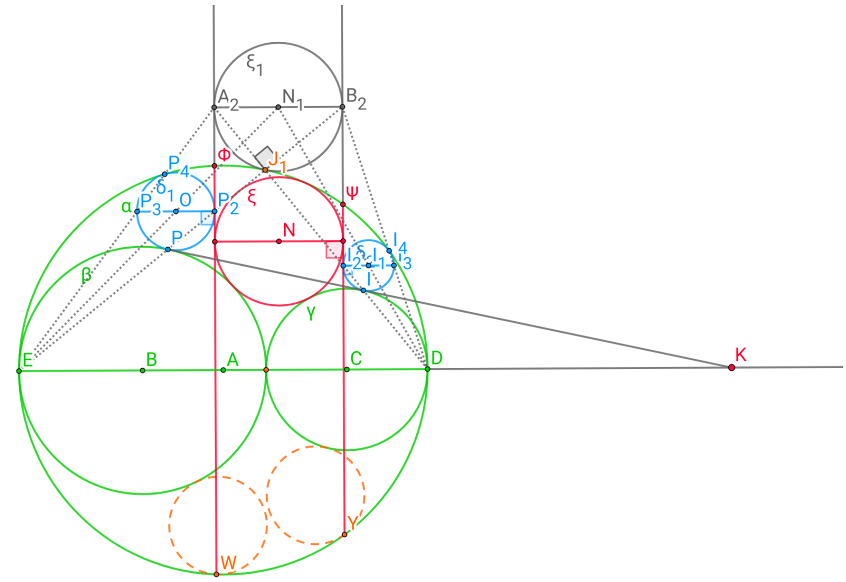

FIGURE 10. K is a common external similarity center of the circle pairs $\{\beta,\, \gamma\}$ and $\{\delta_1,\, \delta_2\}$

be collinear with that of the circle pairs $\{\delta_1,\, \delta_2\}$. From the point K on the line ED we have the common tangent PI to the circle pairs $\{\delta_1,\, \delta_2\}$ and $\{\beta,\, \gamma\}$. Thus K is a common external similarity center for these two circle pairs.

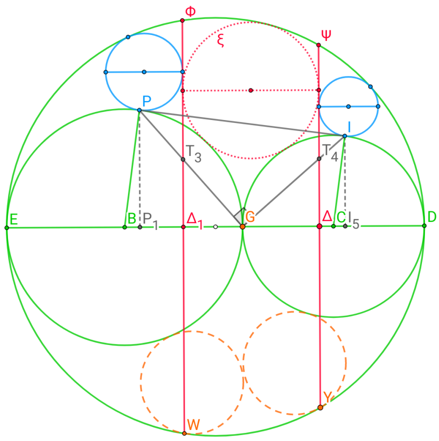

FIGURE 11. $T_3\Delta_1 = T_4\Delta$.

Join GP and GI to intersect $W\Phi$ and $Y\Psi$ in T_3 and T_4 respectively (see Figure 11).

Proposition 6.2. $T_3\Delta_1 = T_4\Delta$.

Proof. We have the similar triangles $\{PP_1G, T_3\Delta_1G\}$ and know the distances $\{PP_1, PG, \Delta_1G\}$ from previously, hence:

$$T_3\Delta_1 = \frac{\Delta_1G \cdot PP_1}{P_1G} = \frac{2\rho}{R}\sqrt{ab}. \tag{39}$$

And likewise we have the similar triangles $II_5G, T_4\Delta G\}$ and know the distances $\{II_5, I_5G, \Delta G\}$ giving:

$$T_4\Delta = \frac{II_5 \cdot \Delta G}{GI_5} = \frac{2\rho}{R}\sqrt{ab} = T_3\Delta_1. \tag{40}$$

We know that $\angle T_3GT_4 = 90°$ [5] (cor.4, p.4, fig.4) so now we have a circle ξ_2 through points $\{T_3, G, T_4\}$ congruent to the arbelos incircle. Its' center T_5, is on the extended line NN_1, and $NN_1 \perp ED$ (see Figure 12). We also know that the points $\{P_4, P_2, D\}$ and $\{P_4, P_3, E\}$ are collinear by [3] prop.1, p.561, as the circles $\{\alpha, \delta_1,\}$ are tangent at the point P_4 and $P_3P_2 \| ED$. The same applies to analogous points related to the circle δ_2.

Proposition 6.3. *Related to the circle $\delta_1(O)$: Points $\{K_2, P_4, P_2, T_4, D\}$ are collinear, where K_2 is the external similarity center for the circles $\{\delta_1, \xi_2, \gamma\}$. Similarly related to the circle $\delta_2(I_1)$: Points $\{K_1, I_4, I_2, T_3, E\}$ are collinear, where K_1 is the external similarity center for the circles $\{\delta_2, \xi_2, \beta\}$. Also $K_2E \perp ED$ and $K_1D \perp ED$.*

See Figure 10 for points I_i.

Proof. We consider only the case related to the circle δ_1 as the same method will apply to that of the circle δ_2. Let the line P_4D intersect the chord $Y\Psi$ in T_4'. We want to show that $T_4' = T_4$ (see Figures 12 and 13). We have the similar triangles $\triangle P_2\Delta_1 D$ and $\triangle T_4'\Delta D$, hence:

$$\frac{P_2\Delta_1}{\Delta_1 D} = \frac{T_4'\Delta}{\Delta D} \Rightarrow T_4' = \frac{\Delta D \cdot P_2\Delta_1}{\Delta_1 D}. \tag{41}$$

Now $P_2\Delta_1 = P_2E_1 + PP_1$, where $PP_1 = h = \frac{2a\sqrt{ab}}{R}$, and $P_2E_1 = \frac{2b\sqrt{ab}(a-\rho)}{Ra}$ (equations (12),(25)), so:

$$P_2\Delta_1 = P_2E_1 + PP_1 = \frac{2\sqrt{ab}}{R}\left[\frac{b}{a}(a-\rho) + a\right] = \frac{2aR\sqrt{ab}}{N}, \tag{42}$$

where as before $N = a^2 + ab + b^2$ i.e. the denominator of the arbelos incircle radius. From equations (29),(30) we have $\Delta D = \frac{2}{R}(\frac{b^2R^2}{N})$ and $\Delta_1 D = \frac{2bR^2}{N}$. Thus

$$T_4'\Delta = \frac{\Delta D \cdot P_2\Delta_1}{\Delta_1 D} = \frac{(\frac{2Rb^2}{N})(\frac{2Ra\sqrt{ab}}{N})}{\frac{2R^2b}{N}} = \frac{2abR\sqrt{ab}}{NR} = \frac{2\rho}{R}\sqrt{ab} = T_4\Delta. \tag{43}$$

(See equation (40)). We have established the collinearity of the points $\{K_2, P_4, P_2, D\}$. The collinearity of the points $\{G, T_3, P\}$ from proposition 6.2 and the collinearity of the points $\{G, P, P_3\}$ (again making use of [3] prop.1, p.561) here related to the circles $\{\beta, \delta_1\}$ and their tangent point P) combined give the collinearity of the points $\{K_2, P_3, P, T_3, G\}$. We then see that the three diameters $\{P_3P_2, T_3T_4, GD\}$ of the circles $\{\delta_1, \xi_2, \gamma\}$ respectively are all parallell, thus K_2 is their common external similarity center. The same line of argument applies for showing that K_1 is the external similarity center for the circles $\{\delta_2, \xi_2, \beta\}$. Furthermore, $\{K_2, E\}$ are external

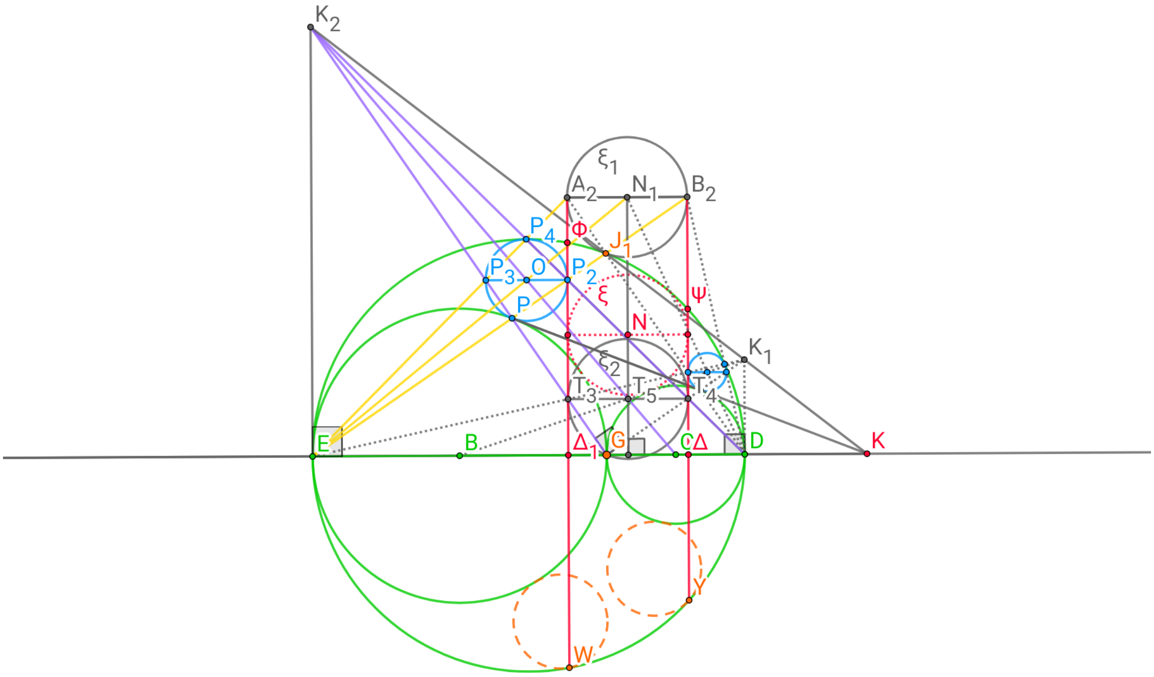

FIGURE 12. Collinear points $\{K_2, P_4, P_2, T_4, D,\}$ and $\{K_1, I_4, I_2, T_3, E\}$ (see Figure 10 for points I_i). $K_2E \perp ED$ and $K_1D \perp DE$.

similarity centers for circles $\{\delta_1, \xi_2\}$ and$\{\delta_1, \xi_1\}$ respectively. Invoking definition 9 in [4] p.6:"...Two or more parallel lines shall be said to have a common point at infinity or to intersect at infinity...", we may state that the external similarity center for the congruent circles $\{\xi_1, \xi_2\}$ is a common point at infinity $\{K_\infty\}$ where the two parallel lines $\{T_3A_2, T_5N_1\}$ intersect. These two lines are perpendicular to the line ED. In order that the three centers of similarity $\{K_2, E, K_\infty\}$ be collinear ([8] p.507, [4] p.151) we must have $\{K_2E \| T_3A_2 \| T_5N_1\}$, i.e. $K_2E \perp ED$. An identical argument holds for $K_1D \perp ED$.

By the reasoning above, we also see that the external similarity centers (not shown in the figures) for the circles $\{\delta_1, \xi\}$ and $\{\delta_2, \xi\}$ will lie on the lines K_2E and K_1D respectively. We note that the point J_3 is also on the line K_2D which is true if left and right side of $\frac{J_3G}{GD} = \frac{T_4\Delta}{\Delta D}$ are equal. This is found true and may easily be verified using expressions from above.

Proposition 6.4. *The points* $\{K_2, J_1, K_1, K\}$ *are collinear.*

Proof. The points $\{K_2, K_1, K\}$ are collinear by ([8] p.507, [4] p.151) so we need to show that J_1 is also on the line K_2K. We consider first the similar triangles $\{K_1DG, T_4, \Delta G\}$ (see Figure 13). The points $\{K_1, T_4, G\}$ are collinear since the points $\{G, T_4\}$ are homologous relative to the circles $\{\beta, \xi_2\}$ with K_1 as external similarity center (see Figure 12). We use formulas

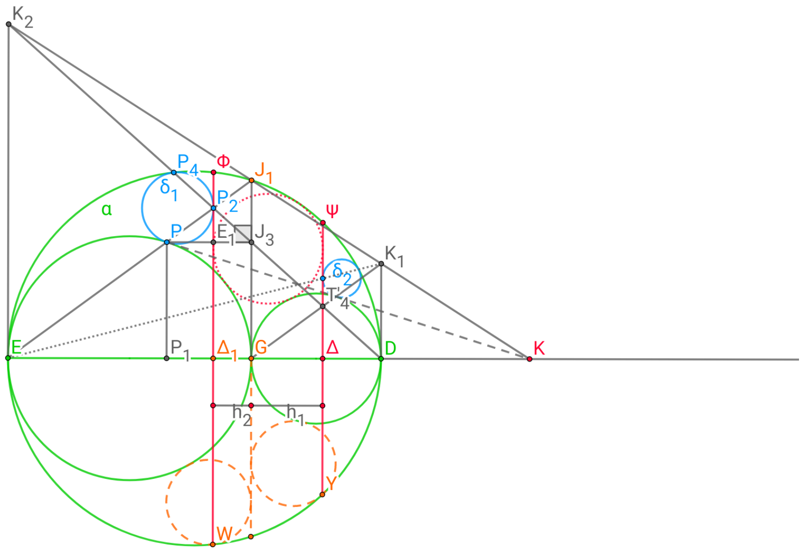

FIGURE 13. Collinear points $\{K_2, P_4, P_2, T_4', D\}$ and $\{K_2, J_1, K_1, K\}$. See propositions 6.3 and 6.4 respectively.

found above

$$GD = 2b, \quad \Delta G = h_1 = \frac{2a\rho}{R}, \quad T_4\Delta = \frac{2a\rho}{R}$$

to find:

$$\frac{K_1D}{GD} = \frac{T_4\Delta}{\Delta G} \Rightarrow K_1D = \frac{GD \cdot T_4\Delta}{\Delta G} = \frac{(2b)\left(\frac{2\rho\sqrt{ab}}{R}\right)}{\left(\frac{2a\rho}{R}\right)} = \frac{2b\sqrt{ab}}{a}. \tag{44}$$

Next we consider the similar triangles $\{J_1GK, K_1DK\}$ and use formulas found previously

$$DK = x = \frac{2b^2}{a-b}, \quad GK = x + 2b = \frac{2ab}{a-b}.$$

to find:

$$\frac{J_1G}{K_1D} = \frac{GK}{DK} \Rightarrow J_1G = \frac{GK \cdot K_1D}{DK} = \frac{\left(\frac{2ab}{a-b}\right)\left(\frac{2b\sqrt{ab}}{a}\right)}{\left(\frac{2b^2}{a-b}\right)} = 2\sqrt{ab}. \tag{45}$$

By the power of the point G with regard to the circle α we have:

$$(J_1G)^2 = GE \cdot GE = (2a)(2b) \Rightarrow J_1G = 2\sqrt{ab}, \tag{46}$$

which agrees with equation (45) and the point J_1 is on the line K_2K.

Referring back to figure 6 and also considering figure 12, we see that an inversion of the circles $\{\alpha, \beta\}$ w.r.t. the circle λ maps them to the lines $\{W\Phi, Y\Psi\}$ respectively. Considering the circle δ_1 and its' tangent points $\{P_4, P, P_2\}$ on $\{\alpha, W\Phi, \beta\}$ respectively, we see that an inversion of δ_1 w.r.t. the circle λ maps δ_1 to ξ_1 i.e. $P_4 \mapsto A_2, P \mapsto B_2, P_2 \mapsto J_1$. By the same token, an inversion of the circle δ_2 w.r.t. the circle λ_1 maps δ_2 to the circle ξ_1 as well, i.e. $I_4 \mapsto B_2, I \mapsto A_2, I_2 \mapsto J_1$.

7. Scaling factors.

We can see in figure 12 the beginnings of a dilation, which is followed up in figure 14. From the dilation centers E and D, left and right sets of arbeli are made. In the figure, considering only outer (full) circles, we have made a right set of four {green, pink, turquoise, yellow} and a left set of two {green, pink} arbeli (the green one with baseline ED is the reference arbelos). The whole and dashed lines connect circle centers common to each arbelos. The purple parallell lines are the equivalent of the extended chords $\{W\Phi, Y\Psi\}$ described in the introduction (1). We want to establish scaling factors for the right and left dilations relative to the dilation centers E and D, respectively. We use the following formulas previously found:

$$r_L = \frac{a\rho}{R} = \frac{r\rho}{b}, \quad r_R = \frac{b\rho}{R} = \frac{r\rho}{a},$$

and the notations

$$\rho'_R, \quad r', \quad r'_{R_R}, \quad r'_{L_R}, \quad R'_R, \quad a', \quad b'$$

for the radi of the first dilated arbelos on the right i.e. with dilation center E. These are the equivalent of the radi

$$\rho, \quad r, \quad r_R, \quad r_L, \quad R, \quad a, \quad b$$

associated with the reference arbelos. We use some relations previously found:

$$r'_{L_R} = \rho, \quad r' = b, \quad a' = R.$$

Thus we have for the first right dilated arbelos,

$$r'_{L_R} + r'_{R_R} = \rho'_R = \rho + \frac{b\rho'_R}{R} \Rightarrow \rho'_R = \rho\left(\frac{R}{a}\right) = \rho\left(\frac{b}{r}\right), \tag{47}$$

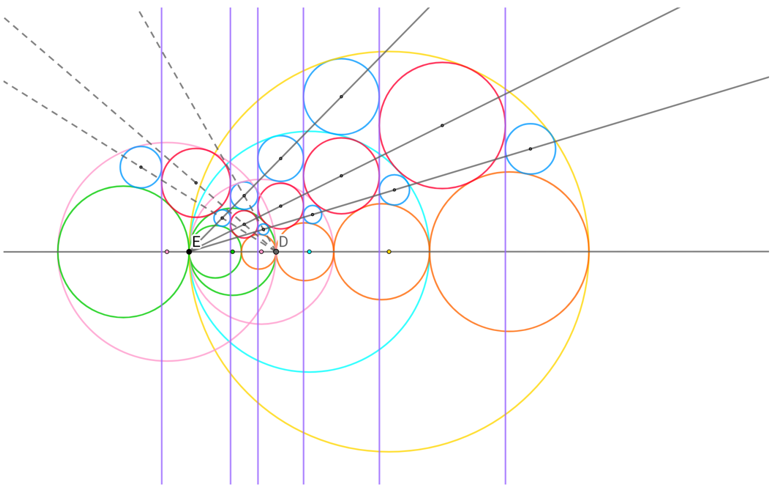

Figure 14. Dilation of the reference arbelos from centers E and D.

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i.e. a scaling factor of $\frac{R}{a} = \frac{b}{r}$. An equivalent calculation for the left side gives:

$$\rho'_L = \rho\left(\frac{R}{b}\right) = \rho\left(\frac{a}{r}\right), \tag{48}$$

i.e. a scaling factor of $\frac{R}{b} = \frac{a}{r}$. We may then find some relations (sum, ratio and product of ρ'_R and ρ'_L):

$$\rho'_R + \rho'_L = \frac{\rho R}{r}, \qquad \left(\frac{\rho'_R}{\rho'_L}\right) = \frac{b}{a}, \qquad \rho'_R\rho'_L = \left(\frac{\rho}{r}\right)^2. \tag{49}$$

Also:

$$\rho'_R r_L = \rho^2 = \rho'_L r_R. \tag{50}$$

Using the scaling factors from above we confirm that $r_{R_L} = r_{L_R} = \rho$, i.e. the radius of the circle ξ_1 (see Figure 12):

(51)

$$r_{R_L} = r_L\left(\frac{R}{a}\right) = \left(\frac{\rho a}{R}\right)\left(\frac{R}{a}\right) = \rho, \quad r_{L_R} = r_R\left(\frac{R}{b}\right) = \left(\frac{\rho b}{R}\right)\left(\frac{R}{b}\right) = \rho.$$

8. Six polar circles and their radi.

We consider polar circles of six triangles, three left (L) and three right (R) "oriented" triangles, all having bases (i.e. two vertices) alligned with the arbelos baseline ED. For the meaning of "oriented", we refer to a comparison of figures 16 and 17, where it can be seen that the left oriented triangles have their third vertex $\{P_5, P_2, T_4\}$ to the left relative to that $\{T_3, I_2, I_5\}$ of the right oriented triangles and vice versa. The polar circles have centers on

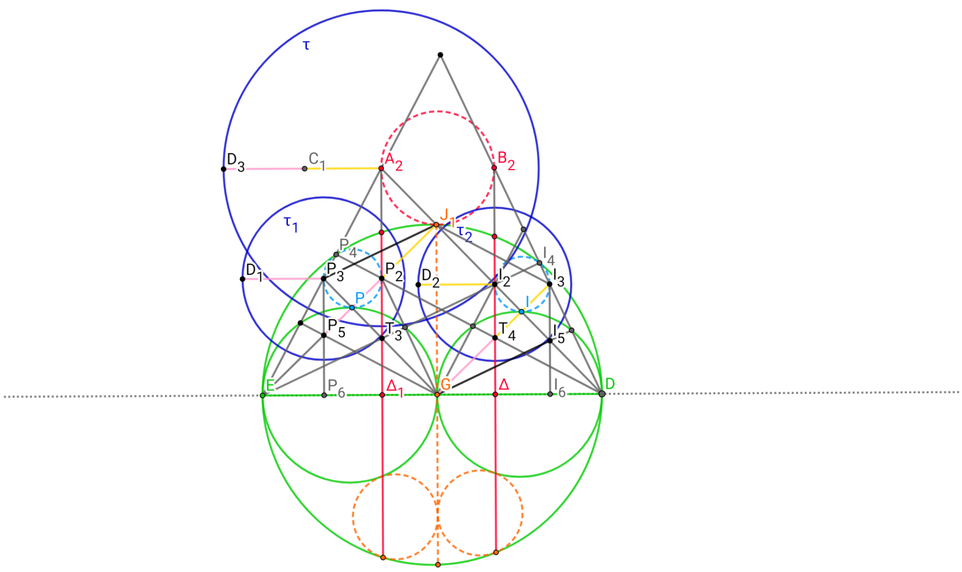

FIGURE 15. Polar circles $\tau(A_2, r_{p_{L_\alpha}})$, $\tau_1(P_3, r_{p_{L_\beta}})$ and $\tau_2(I_2, r_{p_{L_\gamma}})$.

the circles we have discussed in previous sections $\{\delta_1, \delta_2, \xi_1\}$. In figure 15 we have made three polar circles $\tau(A_2, r_{p_{L_\alpha}})$, $\tau_1(P_3, r_{p_{L_\beta}})$ and $\tau_2(I_2, r_{p_{L_\gamma}})$ of the left oriented triangles$\{EP_2D\}$, $\{EP_5G\}$ and $\{GT_4D\}$ respectively. The notation of the radi is as follows: p stands for polar, $L_\alpha, L_\beta, L_\gamma$ stands for the left oriented triangle associated with the arbelos circle α, β, γ. The right oriented triangles are $\{EI_2D, ET_3G, GI_5D\}$ with respective polar circles (not shown) $\tau_3(B_2, r_{p_{R_\alpha}})$, $\tau_4(P_2, r_{p_{R_\beta}})$, $\tau_5(I_3, r_{p_{R_\gamma}})$.

Proposition 8.1. *For the left oriented triangles* $\{EP_2D, EP_5G, GT_4D\}$ *the following is valid:* $I_3G = J_1P_5 = r_{p_{L_\alpha}} = A_2D_3 = r_{p_{L_\beta}} + r_{p_{L_\gamma}}$, $P_2P_5 = T_4G = r_{p_{L_\beta}} = P_3D_1$, *(pink colour code) and* $J_1P_2 = I_3T_4 = r_{p_{L_\gamma}} = I_2D_2$ *(yellow colour code). The same relations will be true for the radi of the polar circles associated with the right oriented triangles viz.* $P_3G = J_1I_5 = r_{p_{R_\alpha}} = r_{p_{R_\beta}} + r_{p_{R_\gamma}}$, $P_3T_3 = J_1I_2 = r_{p_{R_\beta}}$ *and* $T_3G = I_2I_5 = r_{p_{R_\gamma}}$.

Segments referred to above and their associated triangles are shown more explicitly in figures 16 and 17. We find it remarkable that these segments and the radi $\{r_{p_{L_\alpha}}, r_{p_{L_\beta}}, ..., r_{p_{R_\gamma}}\}$ are equal.

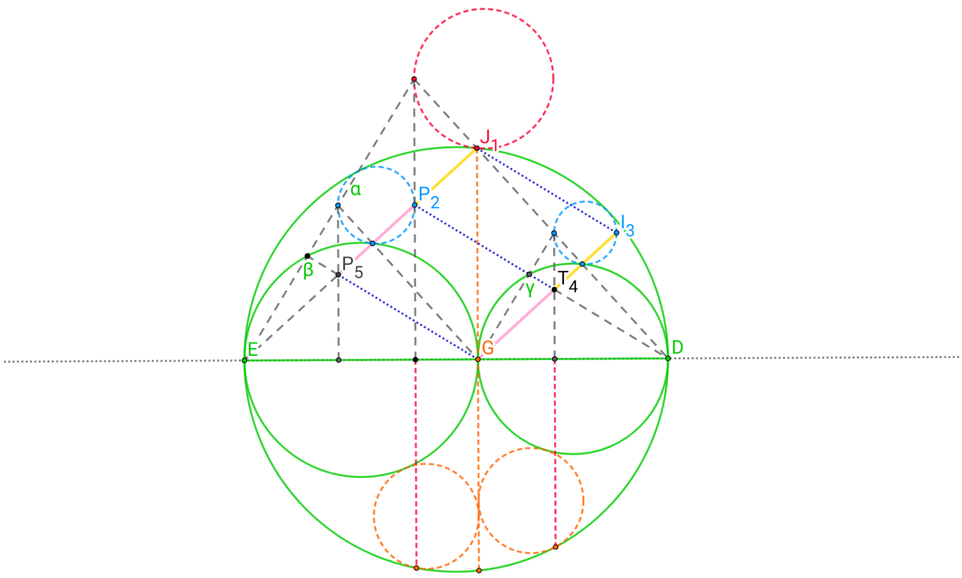

FIGURE 16. Radi $r_{p_{L_\beta}} = P_5P_2 = GT_4$ (pink), $r_{p_{L_\gamma}} = P_2J_1 = T_4I_3$ (yellow) and $r_{p_{L_\alpha}} = r_{p_{L_\beta}} + r_{p_{L_\gamma}} = P_5J_1 = GI_3$ of polar circles of left oriented triangles $\{EP_5G, GT_4D, EP_2D\}$ respectively.

Proof. We will first employ the definition of the radius of a polar circle ([4] p. 176) to find the radius $r_{p_{L_\beta}}$:

$$\left(r_{p_{L_\beta}}\right)^2 = (P_3P_5) \cdot (P_3P_6). \tag{52}$$

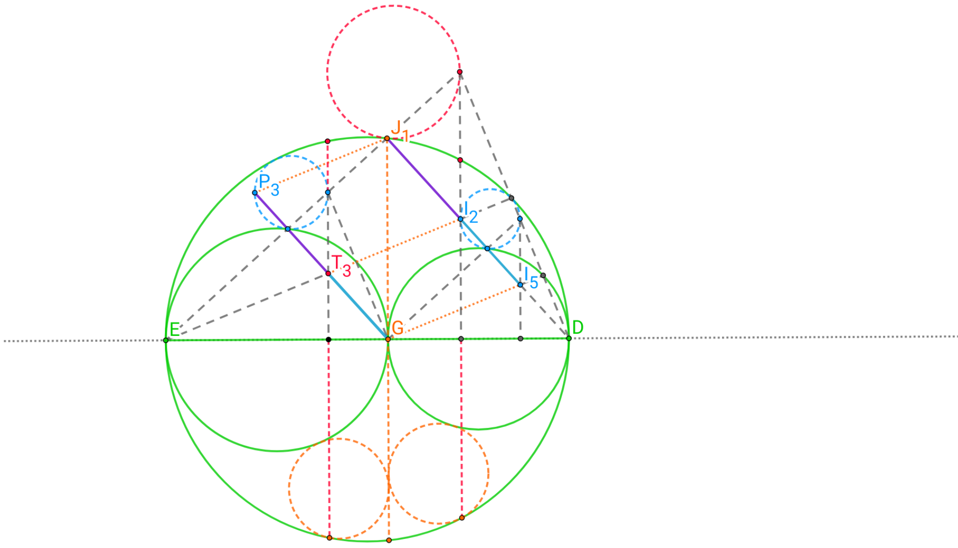

FIGURE 17. Radi $r_{p_{R_\beta}} = P_3T_3 = J_1I_2$ (purple), $r_{p_{R_\gamma}} = T_3G = I_2I_5$ (blue) and $r_{p_{R_\alpha}} = r_{p_{R_\beta}} + r_{p_{R_\gamma}} = P_3G = J_1I_5$ of polar circles of right oriented triangles $\{ET_3G, GI_5D, EI_2D\}$ respectively.

We only need to find P_3P_5 as we have $P_3P_6 = P_2\Delta_1 = \frac{2aR\sqrt{ab}}{N} = 2\rho\sqrt{\frac{a}{b}}$, from equation (42). We have the similar triangles $\{P_3P_5P_2, J_1GE\}$ where

$$P_3P_2 = 2r_L = h_1 = \frac{2\rho a}{R}, \quad J_1G = 2\sqrt{ab}, \quad EG = 2a,$$

thus:

$$\frac{P_3P_5}{P_3P_2} = \frac{J_1G}{EG} \Rightarrow P_3P_5 = \frac{(P_3P_2)\cdot(J_1G)}{EG} = \frac{2\rho\sqrt{ab}}{R}. \tag{53}$$

Substituting for P_3P_6 and P_3P_5 we find from equation (52):

$$\left(r_{p_{L_\beta}}\right)^2 = \left(2\rho\sqrt{\frac{a}{b}}\right)\cdot\left(\frac{2\rho\sqrt{ab}}{R}\right) \Rightarrow r_{p_{L_\beta}} = 2\rho\sqrt{\frac{a}{R}}.$$

Considering the right triangle $\{GT_4\Delta\}$, where we previously found

$$T_4\Delta = \frac{2\rho\sqrt{ab}}{R}, \quad \Delta G = h_1 = \frac{2\rho a}{R},$$

we have:

$$(T_4G)^2 = (\Delta G)^2 + (T_4\Delta)^2 = \frac{4\rho^2 a}{R} \Rightarrow T_4G = 2\rho\sqrt{\frac{a}{R}} = r_{p_{L_\beta}}. \tag{54}$$

This can easily be seen to equal P_5P_2 by the parallel lines $GP_5 \| DP_4$ and $EJ_1 \| GI$. It is also trivial to see the sum $r_{p_{L_\alpha}} = r_{p_{L_\beta}} + r_{p_{L_\gamma}}$ from the similar left oriented triangles defined above: the bases of the two smaller triangles sum to that of the larger. We may now calculate $r_{p_{L_\alpha}}$ by using the scaling factor (see Section 7) $\frac{R}{a}$ on $r_{p_{L_\beta}}$ and find:

$$r_{p_{L_\alpha}} = 2\rho\sqrt{\frac{R}{a}},$$

which agrees with the segments $\{I_3G, J_1P_5\}$ as can easily be verified by similar triangles and previously found expressions. We also find from the sum of radi:

$$r_{p_{L_\gamma}} = r_{p_{L_\alpha}} - r_{p_{L_\beta}} = 2\rho\frac{b}{\sqrt{Ra}}.$$

In exactly the same way we verify the relations quoted in proposition 8.1 for the right oriented triangles and their associated polar radi. We find:

$$r_{p_{R_\alpha}} = 2\rho\sqrt{\frac{R}{b}}, \quad r_{p_{R_\beta}} = 2\rho\frac{a}{\sqrt{Rb}}, \quad r_{p_{R_\gamma}} = 2\rho\sqrt{\frac{b}{R}}.$$

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BRATTBAKKEN 124
9018 TROMSØ, NORWAY.
E-mail address:arerikso@gmail.com