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VERSIONS OF PYTHAGOREAN THEOREM IN TO AND TH PLANES

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Abstract. In this paper we first define the Tetrakis Hexahedron and Truncated Octahedron planes, which are 2-dimensional analytical planes equipped with metrics induced by two dual semi-regular convex polyhedra in space and reduced to plane by projection. Then, we present the analogues of Pythagorean Theorem in these planes and show that, even though the Pythagorean Theorem holds, it is not a prerequisite for a triangle in mentioned planes to posses a rigth angle. Finally, we provide the necessary and sufficient conditions for a triangle in TO and TH planes to have a right angle.

1. Introduction

Development of convex set theory is particularly based on polyhedra. A Polyhedron, in Euclidean geometry, is defined as a three-dimensional object with flat, polygonal faces and straight edges. Technically, a polyhedron is the boundary between the interior and exterior of a solid. In general, polyhedrons are named according to number of faces. A tetrahedron has four faces, a pentahedron five, and so on; a cube is a six-sided regular polyhedron (hexahedron) whose faces are squares. A polyhedron is called uniform if its faces are all regular polygons and its vertices are all congruent.A polyhedron is called regular if its faces are all regular polygons, and each face is congruent to the other faces. Regular polyhedra are uniform, but not vice versa. A polyhedron is said to be convex if its surface (comprising its faces, edges and corners) does not intersect itself and the line segment joining any two points of the polyhedron is contained in the interior or surface of polyhedron.

As mentioned in [33] and [20], Minkowski geometry is a finite-dimensional, non-Euclidean geometry with a similar linear structure to that of Euclidean geometry, that is, points, lines, and planes are the same, and angles are

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measured in the same way. The most significant difference between Minkowski and Euclidean geometries is the distance function used. The distance is not uniform in all directions in Minkowski geometry contrary to Euclidean geometry. This primarily leads to the unit ball being general, symmetric, convex set. Through the studies on metric geometry it has seen that convex polyhedra and some metrics are closely related, for some studies see [25], [24], [22], [10], [27], [5], [9], [30], [14], [17], [18], [6], [28], [3] and [7]. For example in [13] tetrakis hexahedron metric in 3−dimensional analytical space is defined as

$$d_{TH}(P_1,P_2) = \left(\sqrt{3}-1\right)\max\left\{\begin{array}{c} |x_1-x_2|+|y_1-y_2|, \\ |x_1-x_2|+|z_1-z_2|, \\ |y_1-y_2|+|z_1-z_2| \end{array}\right\} + \left(2-\sqrt{3}\right)\max\left\{\begin{array}{c} |x_1-x_2|, \\ |y_1-y_2|, \\ |z_1-z_2| \end{array}\right\}$$

and in [12] truncated octahedron metric in 3−dimensional analytical space is defined as

$$d_{TO}(P_1,P_2) = \max\left\{\tfrac{2}{3}\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right), |x_1-x_2|, |y_1-y_2|, |z_1-z_2|\right\}$$

for the points $P_1=(x_1,y_1,z_1), P_2=(x_2,y_2,z_2)\in\mathbb{R}^3$ and in these studies it has been shown that unit spheres of the spaces furnished by these metrics are a tetrakis hexahedron and a truncated octahedron as in the Figure 1, respectively.

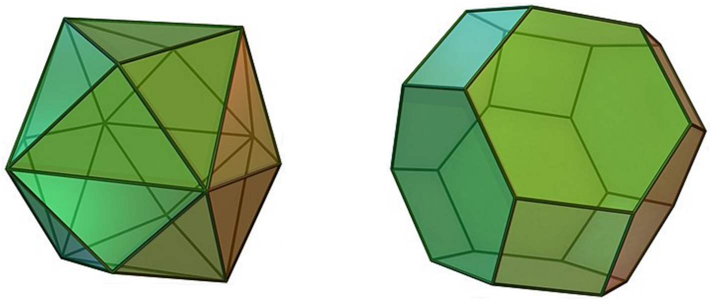


Figure 1: Tetrakis Hexahedron and Truncated Octahedron

Since the main difference between Euclidean and Minkowski geometries is the definition of distance, it becomes intriguing to examine how Euclidean problems involving distance translate into the framework of Minkowski geometry, for some studies on Euclidean problems in Minkowski geometries see [1], [4], [8], [11], [15], [16], [19], [21], [23], [26], [29], [31] and [32].

In this paper, we introduce tetrakis hexahedron and truncated octahedron metrics in 2-dimensional analytical plane. We explore Pythagorean theorem and demonstrate that the converse of the Pythagorean theorem does not hold in these geometries. Finally, we provide a necessary and sufficient condition for triangle in the $\mathbb{R}^2_{TH}$ and $\mathbb{R}^2_{TO}$ planes to be a right triangle.

2. TETRAKIS HEXAHEDRON AND TRUNCATED OCTAHEDRON PLANE GEOMETRIES

In this section we introduce tetrakis hexahedron and truncated octahedron planes by using the projection of d_{TH} and d_{TO} in $\mathbb{R}^3$ which were given in [2] and [13] and give basic concepts and properties of these geometries.

Definition 2.1. *For the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ given in $\mathbb{R}^2$, the tetrakis hexahedron distance function $d_{TH} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0, \infty)$ is defined as*

(1)

$$d_{TH}(P_1, P_2) = \left(\sqrt{3} - 1\right)(|x_1 - x_2| + |y_1 - y_2|) + \left(2 - \sqrt{3}\right)\max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Proposition 2.1. *The distance function d_{TH} in $\mathbb{R}^2$ is a metric.*

Proof. To show that d_{TH} is a metric it must be shown that d_{TH} is non-negative defined, symmetric, $d_{TH}(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$ and triangle inequality is hold. By using the definition of d_{TH} non-negativity, symmetry property and condition $d_{TH}(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$ would easily be obtained. For $d_{TH}(P_1, P_2) \le d_{TH}(P_1, P_3) + d_{TH}(P_3, P_2)$ where $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$, eight cases must be considered. As an example if $d_{TH}(P_1, P_2) = \left(\sqrt{3} - 1\right)|x_1 - x_2| + |y_1 - y_2|$, $d_{TH}(P_1, P_3) = |x_1 - x_3| + \left(\sqrt{3} - 1\right)|y_1 - y_3|$ and $d_{TH}(P_3, P_2) = \left(\sqrt{3} - 1\right)|x_2 - x_3| + |y_2 - y_3|$, then

$$\begin{aligned} d_{TH}(P_1, P_2) &= \left(\sqrt{3} - 1\right)|x_1 - x_2| + |y_1 - y_2| \\ &= \left(\sqrt{3} - 1\right)|x_1 + x_3 - x_3 - x_2| + |y_1 + y_3 - y_3 - y_2| \\ &\le \left(\sqrt{3} - 1\right)|x_1 - x_3| + |y_1 - y_3| + \left(\sqrt{3} - 1\right)|x_3 - x_2| + |y_3 - y_2| \\ &\le |x_1 - x_3| + \left(\sqrt{3} - 1\right)|y_1 - y_3| + \left(\sqrt{3} - 1\right)|x_3 - x_2| + |y_3 - y_2| \\ &\le d_{TH}(P_1, P_3) + d_{TH}(P_3, P_2) \end{aligned}$$

Thus d_{TH} in $\mathbb{R}^2$ is a metric.

Tetrakis hexahedron circle with center $O = (x_0, y_0)$ and the radius r is the set of the points

$$\mathcal{C}_1 = \left\{(x, y) : \left(\sqrt{3} - 1\right)(|x - x_0| + |y - y_0|) + \left(2 - \sqrt{3}\right)\max\{|x - x_0|, |y - y_0|\} = r\right\}$$

and locus of this set is as in the Figure 2.

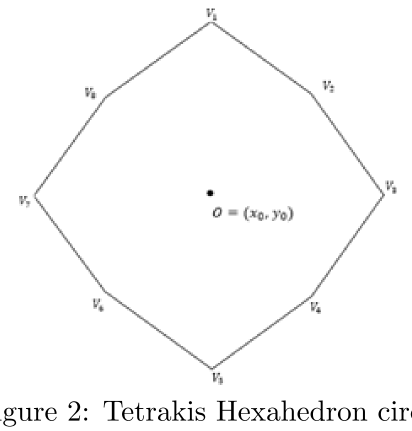


Figure 2: Tetrakis Hexahedron circle

Coordinates of the vertices of this circle are $V_1 : (x_0, y_0 + r)$, $V_2 : \left(x_0 + \frac{\sqrt{3}}{3}r, y_0 + \frac{\sqrt{3}}{3}r\right)$, $V_3 : (x_0 + r, y_0)$, $V_4 : \left(x_0 + \frac{\sqrt{3}}{3}r, y_0 - \frac{\sqrt{3}}{3}r\right)$, $V_5 : (x_0, y_0 - r)$, $V_6 : \left(x_0 - \frac{\sqrt{3}}{3}r, y_0 - \frac{\sqrt{3}}{3}r\right)$, $V_7 : (x_0 - r, y_0)$ and $V_8 : \left(x_0 - \frac{\sqrt{3}}{3}r, y_0 + \frac{\sqrt{3}}{3}r\right)$

Definition 2.2. *For the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ given in $\mathbb{R}^2$, the truncated octahedron distance function $d_{TO} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0, \infty)$ is* *defined as*

$$(2) \qquad d_{TO}(P_1, P_2) = \max\left\{ |x_1 - x_2|, |y_1 - y_2|, \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2|) \right\}$$

Proof. Proof is similar to the d_{TH} case. Similarly by using the definition of d_{TO} non-negativity, symmetry property and condition $d_{TO}(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$ would easily be obtained. For $d_{TO}(P_1, P_2) \leq d_{TO}(P_1, P_3) + d_{TO}(P_3, P_2)$ where $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$, twenty seven cases must be considered. As an example if $d_{TO}(P_1, P_2) = \frac{2}{3}(|x_1 - x_2| + |y_1 - y_2|)$, $d_{TO}(P_1, P_3) = |x_1 - x_3|$ and $d_{TO}(P_3, P_2) = |y_2 - y_3|$, then

$$\begin{aligned} d_{TO}(P_1, P_2) &= \tfrac{2}{3}(|x_1 - x_2| + |y_1 - y_2|) \\ &= \tfrac{2}{3}(|x_1 + x_3 - x_3 - x_2| + |y_1 + y_3 - y_3 - y_2|) \\ &\leq \tfrac{2}{3}(|x_1 - x_3| + |y_1 - y_3|) + \tfrac{2}{3}(|x_3 - x_2| + |y_3 - y_2|) \\ &\leq |x_1 - x_3| + |y_2 - y_3| \\ &\leq d_{TO}(P_1, P_3) + d_{TO}(P_3, P_2) \end{aligned}$$

Thus d_{TO} in $\mathbb{R}^2$ is a metric.

Truncated octahedron circle with center $O = (x_0, y_0)$ and the radius r is the set of the points

$$\mathcal{C}_2 = \left\{ (x, y) : \max\left\{ |x - x_0|, |y - y_0|, \frac{2}{3}(|x - x_0| + |y - y_0|) \right\} = r \right\}$$

and locus of this set is as in the Figure 3.

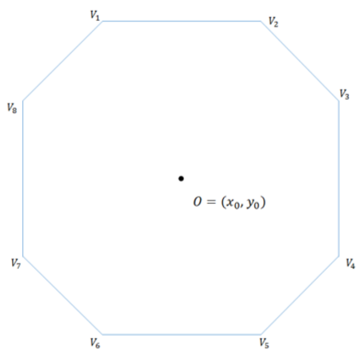


Figure 3: Truncated Octahedron circle

Coordinates of the vertices of this circle are $V_1 : \left(x_0 - \frac{r}{2}, y_0 + r\right)$, $V_2 : \left(x_0 + \frac{r}{2}, y_0 + r\right)$, $V_3 : \left(x_0 + r, y_0 + \frac{r}{2}\right)$, $V_4 : \left(x_0 + r, y_0 - \frac{r}{2}\right)$, $V_5 : \left(x_0 + \frac{r}{2}, y_0 - r\right)$, $V_6 : \left(x_0 - \frac{r}{2}, y_0 - r\right)$, $V_7 : \left(x_0 - r, y_0 - \frac{r}{2}\right)$ and $V_8 : \left(x_0 - r, y_0 + \frac{r}{2}\right)$.

3. TETRAKIS HEXAHEDRON AND TRUNCATED OCTAHEDRON VERSIONS OF THE PYTHAGOREAN THEOREM

Let ABC be a triangle in analytical plane, and a,b,c denote the lenghts of the sides BC, AC, AB respectively. It is well known in Euclidean plane geometry that if ABC is a right triangle with right angle at A, then $a^2 = b^2 + c^2$. Its converse is also true in Euclidean plane. In this section we consider the tetrakis hexahedron and truncated octahedron versions of the Pythagorean theorem. We assume ABC is a triangle in the analytical plane with vertices labeled in counterclockwise order with a right angle at A, tetrakis hexahedron lenghts of sides BC, AC, AB of the triangle ABC are denoted by a_{TH}, b_{TH} and c_{TH} respectively and truncated octahedron enghts of sides BC, AC, AB of the triangle ABC are denoted by a_{TO}, b_{TO} and c_{TO} respectively.

The following proposition which states relation between Euclidean distance and tetrakis hexahedron or truncated octahedron distance of two points in analytical plane plays important role in our study.

Proposition 3.1. *If A and B are two points in $\mathbb{R}^2$, that do not lie on a vertical line and m is the slope of the line passing through these points, then*

$$d_E(A,B) = \rho_{\Delta_i}(m) d_{\Delta_i}(A,B) \tag{3}$$

where $i = 1, 2$, $\Delta_1 = TH$, $\Delta_2 = TO$,

$$\rho_{\Delta_1}(m) = \frac{\sqrt{1+m^2}}{\left(\sqrt{3}-1\right)\{1+|m|\} + \left(2-\sqrt{3}\right)\max\{1,|m|\}}$$

and

$$\rho_{\Delta_2}(m) = \frac{\sqrt{1+m^2}}{\max\left\{1,|m|,\frac{2}{3}(1+|m|)\right\}}.$$

In addition if the points A and B are on a vertical line, then $d_E(A,B) = d_{TH}(A,B) = d_{TO}(A,B)$.

Proof. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ two points where $x_1 \neq x_2$, and the slope of the line passing through these two points is m. By using the definitions of d_E, d_{TH} and d_{TO} in the plane and the equation $m = \frac{y_2-y_1}{x_2-x_1}$, the equation (3) is easily obtained. If also A and B are on a vertical line, then again by the definitions of d_E, d_{TH} and d_{TO}, it is clear that $d_E(A,B) = d_{TH}(A,B) = d_{TO}(A,B)$. This consequence also would be obtained by limiting $m \to \infty$, which means $\rho_{\Delta_i}(m) = 1$ where $i = 1, 2$.

The next proposition gives a very useful property for our arguments of the function $\rho(m)$.

Proposition 3.2. *If $m \in \mathbb{R} - \{0\}$ and $m' = -\frac{1}{m}$, then*

$$\rho_{\Delta_i}(m) = \rho_{\Delta_i}(-m) = \rho_{\Delta_i}(m') = \rho_{\Delta_i}(-m') \tag{4}$$

Proof. If $m \in \mathbb{R} - \{0\}$ then the equations $\rho_{\Delta_i}(m) = \rho_{\Delta_i}(-m)$, $\rho_{\Delta_i}(m) = \rho_{\Delta_i}(m')$ and $\rho_{\Delta_i}(m') = \rho_{\Delta_i}(-m')$ are derived by direct calculations by using definiton of the function $\rho_{\Delta_i}(m)$ for m, $-m$, $m'$and $i = 1, 2$.

The following proposition states that although the Euclidean lengths and the tetrakis hexahedron or truncated octahedron lengths of sides AC and AB of triangle ABC differ from each other, their mutual proportions are equal.

Proposition 3.3. *If ABC is a right triangle with the right angle at A and $d_E(A,C) = b, d_E(A,B) = c, d_{\Delta_i}(A,C) = b_{\Delta_i}, d_{\Delta_i}(A,B) = c_{\Delta_i}$ where $i = 1, 2$, $\Delta_1 = TH$ and $\Delta_2 = TO$, then*

$$\frac{b}{c} = \frac{b_{\Delta_i}}{c_{\Delta_i}} \tag{5}$$

Proof. When the sides AB and AC are parallel to the coordinate axes, it follows that $b = b_{\Delta_i}$ and $c = c_{\Delta_i}$, for $i = 1, 2$. Therefore, the equation (5) holds. However, if one side of the triangle is not parallel to the coordinate axes, the other side will also not be parallel to any axis. Also, since the legs are perpendicular to each other, if the slope of the AB line is m, then the slope of the AC line $m' = -\frac{1}{m}$, thus $b=\rho_{\Delta_i}(m')b_{\Delta_i}$ and $c=\rho_{\Delta_i}(m)c_{\Delta_i}$ are obtained from the equation (3), where $i = 1, 2$. In this case clearly $\frac{b}{c} = \frac{b_{\Delta_i}}{c_{\Delta_i}}$.

The theorem below shows how the a_{Δ_i}, b_{Δ_i}, c_{Δ_i} the tetrakis hexahedron or truncated octahedron lenghts of sides are related to and the slope of one of the legs or the hypotenuse in the right triangle ABC. That is, the tetrakis hexahedron or truncated octahedron version of Pythagorean theoerem includes one more parameter other than the lenghts of sides. If the hypotenus is paralell to a coordinate axis, then an additional parameter is unncessary. That is, the relation depends on only the lenghts of legs.

Theorem 3.1. *Consider the right triangle ABC, where right angle is at A.*

i) If the legs of the triangle ABC are parallel to the coordinate axes, then

$$a_{\Delta_1} = (\sqrt{3}-1)(b_{\Delta_1} + c_{\Delta_1}) + (2-\sqrt{3}) \max\{b_{\Delta_1}, c_{\Delta_1}\} \tag{6}$$

and

$$a_{\Delta_2} = \max\left\{b_{\Delta_2}, c_{\Delta_2}, \frac{2}{3}(b_{\Delta_2} + c_{\Delta_2})\right\} \tag{7}$$

ii) If the legs of the triangle ABC are not parallel to the coordinate axes, and the hypotenuse (ie BC side) is not vertical and slope of one of the legs is m, then

$$a_{\Delta_1} = \frac{(\sqrt{3}-1)\left(|b_{\Delta_1}m+c_{\Delta_1}|+|c_{\Delta_1}m-b_{\Delta_1}|\right)+(2-\sqrt{3})\max\left\{|b_{\Delta_1}m+c_{\Delta_1}|,|c_{\Delta_1}m-b_{\Delta_1}|\right\}}{\left[(\sqrt{3}-1)(1+|m|)+(2-\sqrt{3})\max\{1,|m|\}\right]} \tag{8}$$

and

$$a_{\Delta_2} = \frac{\max\left\{|b_{\Delta_2}m+c_{\Delta_2}|, |c_{\Delta_2}m-b_{\Delta_2}|, \frac{2}{3}\left(|b_{\Delta_2}m+c_{\Delta_2}|+|c_{\Delta_2}m-b_{\Delta_2}|\right)\right\}}{\left[\max\left\{1, |m|, \frac{2}{3}(1+|m|)\right\}\right]} \tag{9}$$

Proof. i) The equations (6) and (7) are is an immediate consequences of the definitions of d_{TH} and d_{TO}.

ii) Let θ is the angle CBA of the triangle ABC whose vertices labeled counterclockwise order, thus clearly θ is acute and positive. Let the slopes of sides AB and BC be m and m_1, respectively. Therefore $\tan\theta = \frac{m-m_1}{1+mm_1}$ and the slope of AC is $m' = -\frac{1}{m}$. Also according to the equation (5) $\tan\theta = \frac{b}{c} = \frac{b_{\Delta_i}}{c_{\Delta_i}}$. In that case

$$\frac{b_{\Delta_i}}{c_{\Delta_i}} = \frac{m-m_1}{1+mm_1} \tag{10}$$

By solving equation (10) for m_1 we get

$$m_1 = \frac{c_{\Delta_i}m - b_{\Delta_i}}{b_{\Delta_i}m + c_{\Delta_i}} \tag{11}$$

where $m \neq \frac{-c_{\Delta_i}}{b_{\Delta_i}}$ since $m \neq \frac{1}{m_1}$. By applying equation (3) for Δ_1 to the equation $a^2 = b^2 + c^2$ and by using the equation (4)

(12)
$$\left[\frac{\sqrt{1+m_1^2}}{(\sqrt{3}-1)(1+|m_1|)+(2-\sqrt{3})\max\{1,|m_1|\}}\right]^2 a_{\Delta_1}^2 = \left[\frac{\sqrt{1+m^2}}{(\sqrt{3}-1)(1+|m|)+(2-\sqrt{3})\max\{1,|m|\}}\right]^2 (b_{\Delta_1}^2 + c_{\Delta_1}^2)$$

is obtained and by substituting equivalent value of m_1 as given in the equation (11) and taking the square root of both sides of equation (12) it would be simplified as

$$a_{\Delta_1} = \frac{(\sqrt{3}-1)\left(|b_{\Delta_1}m+c_{\Delta_1}|+|c_{\Delta_1}m-b_{\Delta_1}|\right)+(2-\sqrt{3})\max\left\{|b_{\Delta_1}m+c_{\Delta_1}|,|c_{\Delta_1}m-b_{\Delta_1}|\right\}}{\left[(\sqrt{3}-1)(1+|m|)+(2-\sqrt{3})\max\{1,|m|\}\right]}$$

Similarly by applying equation (3) for Δ_2 to the equation $a^2 = b^2 + c^2$ and by using the equation (4)

(13)
$$\left[\frac{\sqrt{1+m_1^2}}{\max\left\{1,|m_1|,\frac{2}{3}(1+|m_1|)\right\}}\right]^2 a_{\Delta_2}^2 = \left[\frac{\sqrt{1+m^2}}{\max\left\{1,|m|,\frac{2}{3}(1+|m|)\right\}}\right]^2 (b_{\Delta_2}^2 + c_{\Delta_2}^2)$$

is obtained and by substituting equivalent value of m_1 as given in the equation (11) and taking the square root of both sides of equation (13) it would be simplified as

$$a_{\Delta_2} = \frac{\max\left\{|b_{\Delta_2}m + c_{\Delta_2}|, |c_{\Delta_2}m - b_{\Delta_2}|, \frac{2}{3}\left(|b_{\Delta_2}m + c_{\Delta_2}| + |c_{\Delta_2}m - b_{\Delta_2}|\right)\right\}}{\left[\max\left\{1,|m|,\frac{2}{3}(1+|m|)\right\}\right]}$$

Clearly the obtained results are valid when m is the slope of either AB or AC.

Corollary 3.1. *If the hypotenuse (BC side) of the triangle ABC is parallel to one of the coordinate axes, then*

$$\left[(\sqrt{3}-1)(b_{\Delta_1}+c_{\Delta_1}) + (2-\sqrt{3})\max\{b_{\Delta_1}, c_{\Delta_1}\}\right] a_{\Delta_1} = b_{\Delta_1}^2 + c_{\Delta_1}^2 \tag{14}$$

and

$$\max\left\{b_{\Delta_2}, c_{\Delta_2}, \frac{2}{3}(b_{\Delta_2}+c_{\Delta_2})\right\} a_{\Delta_2} = b_{\Delta_2}^2 + c_{\Delta_2}^2 \tag{15}$$

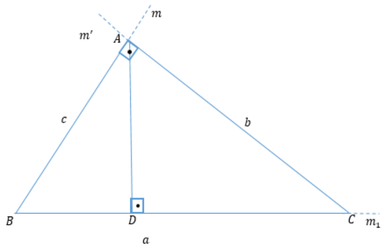
Proof. This is a direct concequence of the equations of (12) and (13). If BC side of the triangle ABC is parallel to $x-$axis, then $m_1 = 0$ and if BC is parallel to $y-$axis, then.$m_1 \longrightarrow \infty$ and for both cases $\rho(m_1) = 1$ and eqution (12) becomes

$$\left[(\sqrt{3}-1)(1+|m|) + (2-\sqrt{3})\max\{1,|m|\}\right]^2 a_{\Delta_1}^2 = \left(1+m^2\right)\left(b_{\Delta_1}^2 + c_{\Delta_1}^2\right)$$

and equation (13) becomes

$$\left[\max\left\{1,|m|,\frac{2}{3}(1+|m|)\right\}\right]^2 a_{\Delta_2}^2 = \left(1+m^2\right)\left(b_{\Delta_2}^2 + c_{\Delta_2}^2\right)$$

, where m is the slope of AB. Let AD be the altitude from A as in the Figure 4.



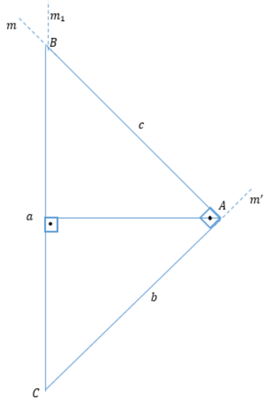


Figure 4: Altitude from A

Using similar triangles and equation (5), if BC is horizontal, then $|m| = \left|\frac{AD}{BD}\right| = \left|\frac{AC}{AB}\right| = \frac{b}{c} = \frac{b_{\Delta_i}}{c_{\Delta_i}}$ and if BC is vertical, then $|m| = \frac{c}{b} = \frac{c_{\Delta_i}}{b_{\Delta_i}}$. Thus

$$\left[(\sqrt{3}-1)\left(1+\left|\frac{b_{\Delta_1}}{c_{\Delta_1}}\right|\right)+(2-\sqrt{3})\max\left\{1,\left|\frac{b_{\Delta_1}}{c_{\Delta_1}}\right|\right\}\right]^2 a_{\Delta_1}^2 = \left[1+\left(\frac{b_{\Delta_1}}{c_{\Delta_1}}\right)^2\right]^2 (b_{\Delta_1}^2 + c_{\Delta_1}^2)$$

and

$$\left[\max\left\{1,\left|\frac{b_{\Delta_2}}{c_{\Delta_2}}\right|,\frac{2}{3}\left(1+\left|\frac{b_{\Delta_2}}{c_{\Delta_2}}\right|\right)\right\}\right]^2 a_{\Delta_2}^2 = \left(1+\left(\frac{b_{\Delta_2}}{c_{\Delta_2}}\right)^2\right)(b_{\Delta_2}^2 + c_{\Delta_2}^2)$$

when BC is horizontal and

$$\left[(\sqrt{3}-1)\left(1+\left|\frac{c_{\Delta_1}}{b_{\Delta_1}}\right|\right)+(2-\sqrt{3})\max\left\{1,\left|\frac{c_{\Delta_1}}{b_{\Delta_1}}\right|\right\}\right]^2 a_{\Delta_1}^2 = \left[1+\left(\frac{c_{\Delta_1}}{b_{\Delta_1}}\right)^2\right]^2 (b_{\Delta_1}^2 + c_{\Delta_1}^2)$$

and

$$\left[\max\left\{1,\left|\frac{c_{\Delta_2}}{b_{\Delta_2}}\right|,\frac{2}{3}\left(1+\left|\frac{c_{\Delta_2}}{b_{\Delta_2}}\right|\right)\right\}\right]^2 a_{\Delta_2}^2 = \left(1+\left(\frac{c_{\Delta_2}}{b_{\Delta_2}}\right)^2\right)(b_{\Delta_2}^2 + c_{\Delta_2}^2)$$

when BC is vertical. Obviously by simplifying each of these equations the equations (14) and (15) are obtained.

Now we give the next corollary which indicates a version of Phythagorean theorem in $\mathbb{R}^2_{TH}$ and $\mathbb{R}^2_{TO}$ that uses the slope of the hypotenuse as a parameter.

Corollary 3.2. *If none of the sides of ABC is parallel to a coordinate axis, and m_1 is the slope of the BC side (hypotenuse), then*

(16)
$$a_{\Delta_1} = \frac{\left[(\sqrt{3}-1)(1+|m_1|)+(2-\sqrt{3})\max\{1,|m_1|\}\right]}{\left[(\sqrt{3}-1)\left(|b_{\Delta_1}+c_{\Delta_1}m_1|+|c_{\Delta_1}-b_{\Delta_1}m_1|\right)+(2-\sqrt{3})\max\left\{|b_{\Delta_1}+c_{\Delta_1}m_1|,|c_{\Delta_1}-b_{\Delta_1}m_1|\right\}\right]}(b_{\Delta_1}^2+c_{\Delta_1}^2)$$

and

(17)
$$a_{\Delta_2} = \frac{\max\left\{1,|m_1|,\frac{2}{3}(1+|m_1|)\right\}}{\max\left\{|b_{\Delta_2}+c_{\Delta_2}m_1|,|c_{\Delta_2}-b_{\Delta_2}m_1|,\frac{2}{3}\left(|b_{\Delta_2}+c_{\Delta_2}m_1|+|c_{\Delta_2}-b_{\Delta_2}m_1|\right)\right\}}(b_{\Delta_2}^2+c_{\Delta_2}^2)$$

Proof. Let m is the slope of AB. If the equation (6) is solved for m, then

$$m = \frac{b+cm_1}{c-bm_1} \quad (18)$$

is obtained, where $m_1 \neq \frac{c}{b}$. Substituting this value of m in the equations (8) and (9) and doing the required calculations yields to the equations (16) and (17).

Remark 3.1. *Note that if the leg AB is parallel to the $x-$axis, its slope is $m = 0$, then the equations (8) and (9) reduce to the equations (6) and (7). Moreover, when BC is parallel to the $x-$axis, meaning $m_1 = 0$, then the equations (16) and (17) simplify to equations (14) and (15). Furthermore, equations (6) and (7) are special cases with limiting $m \longrightarrow \infty$ in equations (8) and (9), respectively and equations (14) and (15) are special cases with limiting $m_1 \longrightarrow \infty$ in equations (16) and (17), respectively. To verify these for Δ_1, let us first observe that equations (8) and (14) are derived from the equation (12). Clearly, if $m \longrightarrow \infty$, then $\rho^2_{\Delta_1}(m) = \left[\frac{\sqrt{1+m^2}}{(\sqrt{3}-1)(1+|m|)+(2-\sqrt{3})\max\{1,|m|\}}\right]^2 \longrightarrow \infty$ and $m_1 \longrightarrow \frac{c}{b}$, by equation (11). Thus as $m \longrightarrow \infty$, equation (12) reduces to the equation (8). Similarly, as $m_1 \longrightarrow \infty$, $\rho^2_{\Delta_1}(m) = \left[\frac{\sqrt{1+m_1^2}}{(\sqrt{3}-1)(1+|m_1|)+(2-\sqrt{3})\max\{1,|m_1|\}}\right]^2 \longrightarrow 1$ and $m \longrightarrow -\frac{c}{b}$ by equation (11). Thus in this case as $m_1 \longrightarrow \infty$ equation (12) reduces to the equation (14) again.*

If similar calculations are applied for Δ_2, then as $m \longrightarrow \infty$, the equation (9) reduces to the equation (7), and as $m_1 \longrightarrow \infty$, the equation (17) reduces to the equation (15).

Remark 3.2. *If ABC with a right angle A is labelled in a clockwise order, then the roles of b and c are interchanged and so the equations (8) and (9) become*

$$a_{\Delta_1} = \frac{(\sqrt{3}-1)\left(|c_{\Delta_1}m+b_{\Delta_1}|+|b_{\Delta_1}m-c_{\Delta_1}|\right)+(2-\sqrt{3})\max\left\{|c_{\Delta_1}m+b_{\Delta_1}|,|b_{\Delta_1}m-c_{\Delta_1}|\right\}}{\left[(\sqrt{3}-1)(1+|m|)+(2-\sqrt{3})\max\{1,|m|\}\right]}$$

and

$$a_{\Delta_2} = \frac{\max\left\{|c_{\Delta_2}m+b_{\Delta_2}|, |b_{\Delta_2}m-c_{\Delta_2}|, \frac{2}{3}\left(|c_{\Delta_2}m+b_{\Delta_2}|+|b_{\Delta_2}m-c_{\Delta_2}|\right)\right\}}{\left[\max\left\{1,|m|,\frac{2}{3}(1+|m|)\right\}\right]}$$

and equations (16) and (17) become

$$a_{\Delta_1} = \frac{\left[(\sqrt{3}-1)(1+|m_1|)+(2-\sqrt{3})\max\{1,|m_1|\}\right]}{\left[(\sqrt{3}-1)\left(|c_{\Delta_1}+b_{\Delta_1}m_1|+|b_{\Delta_1}-c_{\Delta_1}m_1|\right)+(2-\sqrt{3})\max\left\{|c_{\Delta_1}+b_{\Delta_1}m_1|,|b_{\Delta_1}-c_{\Delta_1}m_1|\right\}\right]}\left(b^2_{\Delta_1}+c^2_{\Delta_1}\right)$$

and

$$a_{\Delta_2} = \frac{\max\left\{1,|m_1|,\frac{2}{3}(1+|m_1|)\right\}}{\max\left\{|c_{\Delta_2}+b_{\Delta_2}m_1|,|b_{\Delta_2}-c_{\Delta_2}m_1|,,\frac{2}{3}\left(|c_{\Delta_2}+b_{\Delta_2}m_1|+|b_{\Delta_2}-c_{\Delta_2}m_1|\right)\right\}}\left(b^2_{\Delta_2}+c^2_{\Delta_2}\right)$$

, respectively.

The next example we give, shows that the converse of Theorem 3.1 and hence the converse of the Corollary 3.2 is not valid in $\mathbb{R}^2_{TH}$, that is if a triangle ABC in $\mathbb{R}^2_{TH}$ has a right angle at A, then it satisfies equation (8) or (16), which is analogue of $a^2 = b^2 + c^2$ in Euclidean geometry, but if ABC satisfies equation (8) or (16), then it doesn't have to has a right angle. That means there are triangles with no right angle in $\mathbb{R}^2_{TH}$ that satisfies equation (8) or (16).

Example 3.1. *Let* $\mathcal{C}_1$ *be the* $TH-$*circle with center* $O = (0,0)$ *and radius* r*, with vertices labeled counterclockwise as* V_1*,* V_2*,...,* V_8*, thus coordinates of these vertices are* $V_1 : (0,r)$*,* $V_2 : \left(\frac{\sqrt{3}}{3}r, \frac{\sqrt{3}}{3}r\right)$*,* $V_3 : (r,0)$*,* $V_4 : \left(\frac{\sqrt{3}}{3}r, \frac{\sqrt{3}}{3}r\right)$*,* $V_5 : (0,-r)$*,* $V_6 : \left(-\frac{\sqrt{3}}{3}r, -\frac{\sqrt{3}}{3}r\right)$*,* $V_7 : (-r,0)$ *and* $V_8 : \left(-\frac{\sqrt{3}}{3}r, \frac{\sqrt{3}}{3}r\right)$*. Let* $A = \left(-\frac{1+3\sqrt{3}}{13}r, \frac{5+2\sqrt{3}}{13}r\right)$ *be the foot of the perpendicular from* O *the line segment* $V_1 V_8$*. Consider the circle* $\mathcal{C}_2$ *with center* $O' = \left(\frac{-3+4\sqrt{3}}{39}r, \frac{15-7\sqrt{3}}{39}r\right)$ *and radius* r*, where* A *is to be the vertex* V_8' *on the line with the slope* -1 *passing through* O'*. Thus, we obtain the triangle* AOO' *with a right angle at* O *with the vertices labeled counterclockwise. Let* $d_{TH}(O,O') = c$*,* $d_{TH}(O,A) = b$ *and* $d_{TH}(O',A) = a$*, see Figure 4. Let* A' *be a point on the intersection of* $TH-$*circle* $\mathcal{C}_1$ *and the* $TH-$*circle* $\mathcal{C}_2$*, implying* $d_{TH}(O,A) = b$ *and* $d_{TH}(O',A') = a$*, which yields* $d_{TH}(O,A') = b$ *and* $d_{TH}(O',A) = a$*. Applying equation (8) in the Theorem 3.1 to the triangle* AOO'*, the equation* $\left[\left(\sqrt{3}-1\right)(1+|m|) + \left(2-\sqrt{3}\right)\max\{1,|m|\}\right] a = \left(\sqrt{3}-1\right)(|bm+c| + |cm-b|) + (2-\sqrt{3})\max\{|bm+c|, |cm-b|\}$ *is obtained. By a little observation the equation is also valid for the triangle* $A'OO'$ *which has no right angle. Thus, it is shown that the converse of the Theorem 3.1 , and hence the converse of the Corollary 3.2, is not true in* $\mathbb{R}^2_{TH}$*.*

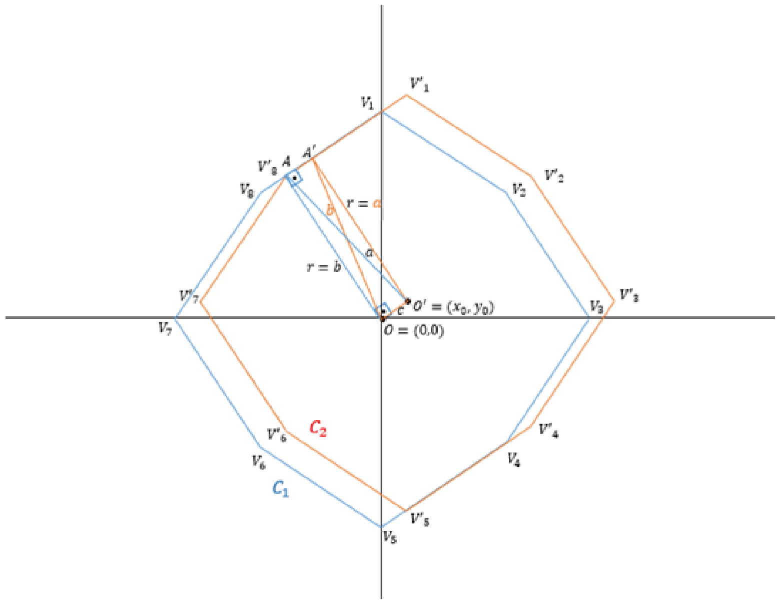


Figure 5:Obtaining the triangle $A'OO'$ *with no right angle which satisfies equation (5) in* $\mathbb{R}^2_{TH}$*.*

Next we provide an example to demonstrate that the converse of Theorem 3.1 and hence the converse of the Corollary 3.2 is not valid in $\mathbb{R}^2_{TO}$, following similar reasoning to the one given above.

Example 3.2. *Let* $\mathcal{C}_1$ *be the* $TO-$*circle with center* $A = (7,5)$ *and radius* 4 *and* $\mathcal{C}_2$ *be a* $TO-$*circle with center* $B = (9,5)$ *and radius* 4*. As in the Figure 5 these two circles intersect through two line segments. Consider the points* $C = (7,9)$*,* $C' = (8,9)$ *and the triangles* CAB *and* $C'AB$*. Since* CAB *is a right triangle with right angle at* A *equation (7) in the Theorem 3.1 holds.*

But clearly $d_{TO}(A,C) = d_{TO}(A,C') = 4$, $d_{TO}(B,C) = d_{TO}(B,C') = 4$, $d_{TO}(A,B) = 2$ *thus equation (7) is still valid for the triiangle* $C'AB$ *which is not a right triangle. Thus the converse of the Theorem 3.1 , and hence the converse of the Corollary 3.2, is not true in* $\mathbb{R}^2_{TO}$.

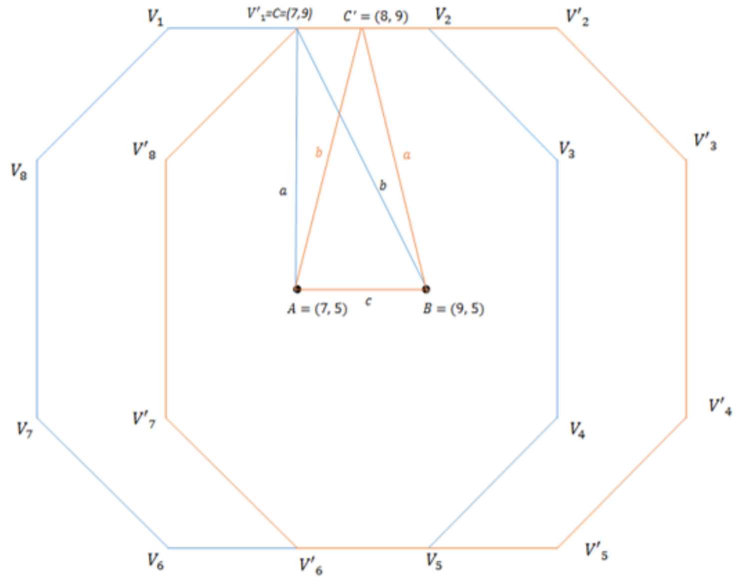


Figure 6:Obtaining the triangle ABC' *with no right angle which satisfies equation (7) in* $\mathbb{R}^2_{TO}$.

The following theorem provides the necessary and sufficient conditions for a triangle in the $\mathbb{R}^2_{\Delta_i}$ where $\Delta_1 = TH$ and $\Delta_2 = TO$ to form a right angle. The sufficient condition is essentially represents an alternative form of the converse of the Pythagorean theorem.

Theorem 3.2. *Let* ABC *be a triangle in* $\mathbb{R}^2_{\Delta_i}$, *with* $a_{\Delta_i}, b_{\Delta_i}$ *and* c_{Δ_i} *representing the tetrakis hexahedron and truncated octahedron distances of sides* BC, AC, AB *for* $\Delta_1 = TH$ *and* $\Delta_2 = TO$ *respectively. Assume that none of the sides of triangle* ABC *is parallel to the* $y-$*axis, meaning that* m_1, m *and* m' *are finite. Therefore,* CAB *is a right angle if and only if*

$$\rho_{\Delta_i}(m_1)a^2_{\Delta_i} = \rho_{\Delta_i}(m)(b^2_{\Delta_i} + c^2_{\Delta_i}) = \rho_{\Delta_i}(m')(b^2_{\Delta_i} + c^2_{\Delta_i}) \tag{19}$$

where

$$\rho_{\Delta_1}(x) = \frac{1+x^2}{\left[\left(\sqrt{3}-1\right)\{1+|x|\} + \left(2-\sqrt{3}\right)\max\{1,|x|\}\right]^2}$$

and

$$\rho_{\Delta_2}(x) = \rho_{\Delta_2}(m) = \frac{1+x^2}{\left[\max\left\{1,|x|,\frac{2}{3}(1+|x|)\right\}\right]^2}$$

Proof. If equation (19) is valid for Δ_1, then $\rho_{\Delta_1}(m) = \rho_{\Delta_1}(m')$ and

$$\rho_{\Delta_1}(m_1)a^2_{\Delta_1} = \rho_{\Delta_1}(m)b^2_{\Delta_1} + \rho_{\Delta_1}(m)c^2 = \rho_{\Delta_1}(m')b^2_{\Delta_1} + \rho_{\Delta_1}(m')c^2.$$

Therefore it is easily obtained that

$$\rho(m_1)a^2 = \rho(m)b^2 + \rho(m')c^2_{\Delta_1}. \tag{20}$$

Using the equation (3) for equation (20) it is found that $a^2 = b^2 + c^2$ where a, b, c are the Euclidean lenghts of sides BC, AC, AB . In the Euclidean case the converse of Pythagorean theorem is valid. So the angle CAB is right angle.

Now suppose that the angle CAB is right. Then clearly $m' = -\frac{1}{m}$ and thus by equation (4) $\rho_{\Delta_1}(m) = \rho_{\Delta_1}(m')$. Since $a^2 = b^2 + c^2$ in the Euclidean case, and using the equation (3) to Euclidean Pythegorean theorem one can obtaine the equation (19). By analogous calculations for Δ_2 it can be easily seen that CAB is a right angle if and only if $\rho_{\Delta_2}(m_1)a^2_{\Delta_2} = \rho(m)(b^2_{\Delta_2} + c^2_{\Delta_2}) + \rho(m')(b^2_{\Delta_2} + c^2_{\Delta_2})$.

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