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G.O. Kropina metrics on a specific space

$$SO(n)/SO(n_1) \times \cdots \times SO(n_s)$$

Milad Zeinali Laki and Dariush Latifi

Abstract. The space (M, F) is called a Finsler $g.o.$ space if every geodesic of (M, F) is the orbit of a one-parameter subgroup of G. In this paper, we study the $g.o.$ Kropina spaces $(G/H, F)$ such that G is $SO(n)$ and H is a diagonally embedded product $H_1 \times \cdots \times H_s$, where H_k is of the same type as G. Indeed, we study the space $SO(n)/SO(n_1) \times \cdots \times SO(n_s)$ with $0 < n_1 + \cdots + n_s \leq n$. In addition, we will examine some properties of Kropina spaces.

1. Introduction

Matsumoto is the one who first introduced the concept of (α, β)-metrics in 1972 [11]. These metrics are actually an extension of a metric called the Randers metric, which is defined in the form $F = \alpha + \beta$. The (α, β)-metrics have many applications in various sciences such as Physics, Mechanics, Seismology, Biology, Control Theory and etc [1]. If we have $F = \alpha\varphi(s), s = \frac{\beta}{\alpha}$, then F is called an (α, β)-metric where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. We note that, Randers metrics ($F = \alpha + \beta$), Matsumoto metrics ($F = \frac{\alpha^2}{\alpha - \beta}$) and square metrics ($F = \frac{(\alpha+\beta)^2}{\alpha}$) are examples of (α, β)-metrics. But when we have $\varphi(s) = \frac{1}{s}$, we will arrive at a well-known (α, β)-metric called the Kropina metric and we have

$$F = \frac{\alpha^2}{\beta}. \tag{1}$$

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Now we consider the Kropina metric $F = \alpha^2/\beta$ on a n-dimensional differential manifold M. Another representation of this metric can also be written. Indeed, a Kropina metric F is characterized by a new Riemannian metric $h = \sqrt{h_{ij}(x)y^i y^j}$ on M and a vector field $W = W^i(\partial/\partial x^i)$ of constant length 1 with respect to h and we have:

$$F(x,y) = \frac{h^2(y,y)}{2h(y,W)}. \tag{2}$$

Also in a local coordinate system we can write:

$$\begin{aligned} &a_{ij} := e^{-k(x)}h_{ij}, \quad 2W_i := e^{k(x)}b_i,\\ &b^2 = 4e^{-k(x)}, \quad b_i = 2e^{-k(x)}W_i, \end{aligned} \tag{3}$$

for a function $k(x)$ of (x^i). In this equations, we used $W_i(x) := h_{ij}(x)W^j(x)$ and $b^2 := a^{ij}b_i b_j$, where (a^{ij}) is the inverse matrix of (a_{ij}). The pair (h, W) is called the navigation data of the Kropina metric $F = \alpha^2/\beta$ or the Kropina space (M, F). Consider that (M, F) be a homogeneous Kropina space. We note that this space can be written as a coset space G/H with a G-invariant Kropina metric $F = \frac{\alpha^2}{\beta}$, where both the Riemannian metric α and the form β are invariant under the action of G. In particular, the Lie algebra of G, has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, such that $Ad\,(h)\,(\mathfrak{m}) \subset \mathfrak{m}$, $h \in H$. Identifying $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the origin o, we get an H-invariant inner product on $\mathfrak{m}$ [10].

Now let $(G/H, g)$ be a homogeneous Riemannian manifold. Suppose that $X \neq 0 \in \mathfrak{g}$ be a vector. Then we say that $(M = G/H, g)$ is a geodesic orbit space (or in short $g.o.$ space) if for any geodesic $\sigma(t)$ we have $\sigma(t) = \exp aX.o$, $a \in \mathbb{R}$. Geodesic orbit spaces were first studied and presented by Kowalski and Vanhecke in 1991 [8]. Recall that, geodesic orbit spaces is a generalization of Riemannian symmetric spaces. Indeed, the class of geodesic orbit spaces is larger than the class of symmetric spaces [10].

In [10], We study results on Kropina $g.o.$ spaces and we investigate Kropina $g.o.$ metrics on compact homogeneous spaces with two isotropy summands. There was also some discussion about navigation data of non-Riemannian Kropina $g.o.$ metrics.

In this paper, we consider Kropina $g.o.$ spaces $F = G/H$. Here we have $G = SO(n)$ and $H = H_1 \times \cdots \times H_s$. Recall that H_i is of the same type as G. Indeed, we study the space

$$SO(n)/SO(n_1) \times \cdots \times SO(n_s),$$

with $0 < n_1 + \cdots + n_s \leq n$. It is worth noting that, this space include the sphere $SO(n)/SO(n-1)$, the Stiefel manifold $SO(n)/SO(n-k)$, the Grassmann manifold $SO(n)/SO(k) \times SO(n-k)$ and the flag manifold $SO(n)/SO(n_1) \times \cdots \times SO(n_s)$ with $n_1 + \cdots + n_s = n$.

2. Preliminaries

Suppose that M be a smooth n-dimensional C^∞ manifold. Also, let for every $a\in M$, T_aM be the tangent space of M at a and $TM=\cup_{a\in M}T_aM$ be the tangent bundle of M. A Finsler metric on a manifold M is a non-negative function $F:TM\to\mathbb{R}$ with the following properties [4]:

(1) F is smooth on $TM\backslash\{0\}$.
(2) $F(x,\lambda y)=\lambda F(x,y)$ for any $x\in M$, $y\in T_xM$ and $\lambda>0$.
(3) The following bilinear symmetric form $g_y : T_xM\times T_xM\to\mathbb{R}$ is positive definite

$$g_y(u,v)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2(x,y+su+tv)|_{s=t=0}.$$

The geometrical data in Finsler geometry consists of a smoothly varying family of Minkowski norms, rather than a family of inner products. (α,β)-metrics are a good example of Finsler metrics, which, as mentioned earlier, have many uses. The following definition is a comprehensive definition of these metrics.

Definition 2.1. [10] Suppose that $\alpha=\sqrt{\widetilde{a}_{ij}\,(x)\,y^iy^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\,(x,y)=b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Let

$$b:=\|\beta(x)\|_\alpha:=\sqrt{\widetilde{a}^{ij}(x)b_i(x)b_j(x)}.$$

Now, let the function F is defined as follows

$$F:=\alpha\varphi(s),\quad s=\frac{\beta}{\alpha}, \tag{4}$$

where $\varphi=\varphi(s)$ is a positive C^∞ function on $(-b_0,b_0)$ satisfying

$$\varphi\,(s)-s\varphi'\,(s)+\left(b^2-s^2\right)\varphi''\,(s)>0,\qquad |s|\leq\ b<b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha<b_0$ for any $x\in M$. A Finsler metric in the form (4) is called an (α,β)- metric.

We recall that, a Finsler space having the Finsler function

$$F\,(x,y)=\frac{\alpha^2(x,y)}{\beta(x,y)}, \tag{5}$$

is called a Kropina space. We note that, the Riemannian metric $\widetilde{a}$ induces an inner product on any cotangent space T_x^*M such that

$$\langle dx^i(x),dx^j(x)\rangle=\widetilde{a}^{ij}(x).$$

This induced inner product on T_x^*M induced a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\widetilde{X}$ on M such that $\widetilde{a}\big(y,\widetilde{X}\,(x)\,\big)=\beta\,(x,y)$ and also, we have

$$\|\beta(x)\|_\alpha=\|\widetilde{X}(x)\|_\alpha.$$

Thus for Kropina metric F we can write

$$F\,(x,y)=\frac{\alpha^2(x,y)}{\tilde{a}(\tilde{X}(x),y)}, \tag{6}$$

where for any $x \in M$, we have $\sqrt{\widetilde{a}(\widetilde{X}(x), \widetilde{X}(x))} = \|\widetilde{X}(x)\|_\alpha < b_0$.

Definition 2.2. [9] Consider a Finsler space (M, F). We say that (M, F) is a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$, acts transitively on M.

In the following we give a definition of a Finsler $g.o.$ space.

Definition 2.3. Let (M, F) be a Finsler space and $G = I(M, F)$ the full group of isometries. The space (M, F) is called a Finsler $g.o.$ space if every geodesic of (M, F) is the orbit of a one-parameter subgroup of G. That is, if σ is a geodesic, then there exist $W \in \mathfrak{g} = Lie(G)$ and $o \in M$, such that

$$\sigma(t) = \exp(tW).o.$$

Now suppose that $(G/H, F)$ be a homogeneous Finsler space, and $p = eH \in G/H$. A vector $X \in \mathfrak{g} - \{0\}$ is called a geodesic vector if the curve $\exp(tX).p$ is a geodesic. For a geodesic vector the second author give the following lemma:

Lemma 2.4. [9] *A vector $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_{X_\mathfrak{m}}\big(X_\mathfrak{m}, [X, Z]_\mathfrak{m}\big) = 0, \qquad \forall Z \in \mathfrak{m},$$

where the subscript $\mathfrak{m}$ means the corresponding projection, and g is the fundamental tensor of F on $\mathfrak{m}$.

The S-curvature of a Finsler space is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{b_i\}$ of V, let

$$\sigma_F = \frac{Vol(B^n)}{Vol\{(y^i) \in \mathbb{R}^n| \quad F(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{b_i\}$. On the other hand, for every $y \in V - \{0\}$ we have

$$\tau(y) = ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F},$$

is independent of the choice of the basis where (g_{ij}) is the fundamental tensor of F. $\tau = \tau(y)$ is called the distortion of (V, F).

Definition 2.5. Let (M, F) be a Finsler space and $\tau(x, y)$ be the distortion of the Minkowski norm F_x on T_xM. For $y \in T_xM \backslash \{0\}$, let $\sigma(t)$ be the geodesic with

$$\sigma(0) = x \quad and \quad \dot{\sigma}(0) = y.$$

Then the quantity

$$S(x, y) = \frac{d}{dt}\left[\tau\big(\sigma(t), \dot{\sigma}(t)\big)\right]\Bigg|_{t=0},$$

is called the S-curvature of the Finsler space (M, F).

Recall that, the S-curvature of Kropina metric $F=\alpha\phi(s)=\frac{\alpha^2}{\beta}$ has the form:

$$S(o,y)=-\frac{(n+1)s}{b^2\alpha(y)}\left\{-c\langle[u,y]_{\mathfrak{m}},y\rangle+\frac{\alpha(y)}{2s}\langle[u,y]_{\mathfrak{m}},u\rangle\right\}, \tag{7}$$

where u is the vector in $\mathfrak{m}$ corresponding to the 1-form β.

3. Homogeneous Kropina spaces

In this section, we study some properties of homogeneous Kropina spaces. Suppose that $(M,F=\alpha^2/\beta)$ be a homogeneous Kropina space. Then we can written M as $M=G/H$, where G is a connected transitive subgroup of the full isometry group and H is the isotropy subgroup of G. Assume that $\mathfrak{g}=Lie(G)$, $\mathfrak{h}=Lie(H)$. Therefore there is a reductive decomposition of $\mathfrak{g}$ as a direct sum $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, where $\mathfrak{m}$ is an $Ad(H)$-invariant subspace of $\mathfrak{g}$. In this case it is easily seen that both the underlying Riemannian metric α and the 1-form β are invariant under G. Also, in the navigation data (h,W), both h and W are also G-invariant. This reduces the study of homogeneous Kropina spaces to the study of invariant Kropina metrics on reductive homogeneous manifolds. Fixed a G-invariant Riemannian metric α on G/H. Then there is an one-to-one correspondence between the G-invariant 1-form on G/H and the H-invariant vector in $\mathfrak{m}$. Thus we have the following Theorem from [5]:

Theorem 3.1. *There exists a bijection between the set of invariant vector fields on G/H and the subspace*

$$V=\{u\in\mathfrak{m}|\quad Ad(h)u=u,\ \forall h\in H\}.$$

In the following, we examine the relationship between G-invariant Kropina metrics and Riemannian metrics on a homogeneous space G/H. Recall that the following Proposition from [7]:

Proposition 3.2. *Assume that G is a connected Lie group and H is a closed subgroup of G such that G/H is a reductive homogeneous space with a reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Let h be a G-invariant Riemannian metric on G/H and suppose $z\in\mathfrak{m}$ is an H-fixed vector. Then the corresponding invariant vector field z on G/H is a Killing vector field with respect to h if and only if z satisfies*

$$h([z,z_1]_{\mathfrak{m}},z_2)+h(z_1,[z,z_2]_{\mathfrak{m}})=0,\quad\forall z_1,z_2\in\mathfrak{m}. \tag{8}$$

Theorem 3.3. *Let $(G/H,F=\alpha^2/\beta)$ be a homogeneous Kropina space with navigation data (h,W). Then the following three conditions are equivalent:*

(1) *F has vanishing S-curvature.*
(2) *The linear mapping $ad(u)_{\mathfrak{m}}$, where u is the vector in $\mathfrak{m}$ corresponding to β, is skew symmetric with respect to α, i.e.,*

$$\langle[u,x]_{\mathfrak{m}},y\rangle+\langle x,[u,y]_{\mathfrak{m}}\rangle=0,\quad\forall x,y\in\mathfrak{m} \tag{9}$$

Furthermore, this is also equivalent to the condition that the invariant vector field $\tilde{u}$ generated by u is a Killing vector field with respect to α.

(3) *W is a Killing vector field with respect to h, i.e., the linear mapping $ad(w)_{\mathfrak{m}}$, where $w = W|_o \in \mathfrak{m}$, is skew symmetric with respect to h,*

$$\langle [w, x]_{\mathfrak{m}}, y\rangle_h + \langle x, [w, y]_{\mathfrak{m}}\rangle_h = 0, \quad \forall x, y \in \mathfrak{m}$$

where h, denotes the inner product on $\mathfrak{m}$ induced by h.

Proof. First assume that $ad_{\mathfrak{m}}(u)$ is skew symmetric with respect to α, i.e.,

$$\langle [u, x]_{\mathfrak{m}}, x\rangle = 0, \quad \text{and} \quad \langle [u, x]_{\mathfrak{m}}, u\rangle = -\langle x, [u, u]_{\mathfrak{m}}\rangle = 0.$$

Thus by using the formula (7), we conclude that the S-curvature is vanishing. The converse is obvious.

Now suppose that, $z_1 = z_2 = y$ in relation (8) and then we get the relation (9) or equivalently we get

$$\langle [x, y]_{\mathfrak{m}}, y\rangle = 0, \quad \forall y \in \mathfrak{m}. \tag{10}$$

Conversely, set $y = z_1 + z_2$ and by replacing it in equation (10) we get

$$\begin{aligned} 0 &= \langle [x, z_1 + z_2]_{\mathfrak{m}}, z_1 + z_2\rangle \\ &= \langle [x, z_1]_{\mathfrak{m}}, z_1\rangle + \langle [x, z_2]_{\mathfrak{m}}, z_1\rangle + \langle [x, z_1]_{\mathfrak{m}}, z_2\rangle + \langle [x, z_2]_{\mathfrak{m}}, z_2\rangle \\ &= \langle [x, z_2]_{\mathfrak{m}}, z_1\rangle + \langle [x, z_1]_{\mathfrak{m}}, z_2\rangle. \end{aligned} \tag{11}$$

The following theorem shows the relationship between Riemannian $g.o.$ spaces with Kropina $g.o.$ spaces and was proved by the authors.

Theorem 3.4. [10] *Suppose that $(M = G/H, F = \frac{\alpha^2}{\beta})$ be a homogeneous Kropina space with navigation data (h, W). If $(G/H, h)$ is a Riemannian $g.o.$ manifold and W is a G-invariant Killing vector field of $(G/H, h)$, then (M, F) is a Kropina $g.o.$ space.*

Definition 3.5. Let $(G/H, g)$ be a homogeneous Riemannian manifold. The manifold $(G/H, g)$ is called naturally reductive if there is an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that for any $X, Y, Z \in \mathfrak{m}$, we have $\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0$, or, equivalently, for any $X, Y \in \mathfrak{m}$, the below relation fulifuld

$$\langle [X, Y]_{\mathfrak{m}}, X\rangle = 0.$$

We note that naturally reductive Riemannian homogeneous spaces are geodesic orbit spaces.

In Finsler setting, there are two versions of the definition of such spaces. The first one was given by Deng and Hou in [6]. The second definition was given by Latifi in [9] as follows:

Definition 3.6. A homogeneous manifold G/H with an invariant Finsler metric F is called naturally reductive if there exists an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that

$$g_y([x, u]_{\mathfrak{m}}, v) + g_y(u, [x, v]_{\mathfrak{m}}) + 2C_y([x, y]_{\mathfrak{m}}, u, v) = 0,$$

where $y \neq 0, x, u, v \in \mathfrak{m}$ and C_y is the Cartan tensor of F at y.

Now we have the following lemma for Kropina spaces:

Lemma 3.7. [10] *Let $(M = G/H, F)$ be a homogeneous Kropina space with navigation data (h, W). Then the Kropina metric F is of Berwald type if and only if $ad(w)_{\mathfrak{m}}$, where $w = W|_o \in \mathfrak{m}$, is skew-symmetric with respect to h and $h(w, [m, m]_{\mathfrak{m}}) = 0$.*

4. Kropina geodesic orbit property on the space $SO(n)/SO(n_1) \times \cdots \times SO(n_s)$, $\Sigma n_i \leq n$

The goal of this section is to investigate the invariant Kropina geodesic orbit metrics on the following space:

$$G/H = SO(n)/SO(n_1) \times \cdots \times SO(n_s).$$

Let

$$n_0 := n - (n_1 + \cdots + n_s) \quad \text{and} \quad H = SO(n_1) \times \cdots \times SO(n_s).$$

Recall that H can be embedded diagonally in $SO(n)$, so that we can write:

$$H \cong \begin{bmatrix} Id_{n_0} & 0 \\ 0 & H \end{bmatrix}.$$

Thus if 0_{n_0} is the $n_0 \times n_0$ zero matrix, then we have:

$$\mathfrak{h} = \begin{bmatrix} 0_{n_0} & & & 0 \\ & \mathfrak{so}(n_1) & & \\ & & \ddots & \\ 0 & & & \mathfrak{so}(n_s) \end{bmatrix}. \tag{12}$$

We note that the above embedding of H is equivalent $\big($via conjugation in $SO(n)\big)$ to any block-diagonal embedding of the factors $SO(n_j)$.

In the following we give the isotropy representation of G/H. Let $\varsigma = G \to Aut(V)$, $\varsigma' = G \to Aut(W)$ are two representations of G for a subspace W of a vector space V that we can write $V = W \oplus W^{\perp}$ with respect to some invariant inner product on V. Then we remark that for the second exterior power, the following identity is valid:

$$\Lambda^2(\varsigma + \varsigma') = \Lambda^2\varsigma \oplus \Lambda^2\varsigma' \oplus (\varsigma \otimes \varsigma').$$

Let denote the standard representation of $SO(n)$ by

$$\lambda_n : SO(n) \to Aut(\mathbb{R}^n).$$

Then the adjoint representation $Ad^{SO(n)}$ of $SO(n)$ is equivalent to $\Lambda^2\lambda_n$.

Now assume that

$$\sigma_{n_i} : SO(n_1) \times \cdots \times SO(n_s) \to SO(n_i)$$

and $p_i = \lambda_{n_i} \circ \sigma_{n_i}$ be the projection onto the i-factor and the standard representation of H respectively. Indeed we have:

$$SO(n_1) \times \cdots \times SO(n_s) \xrightarrow{\sigma_{n_i}} SO(n_i) \xrightarrow{\lambda_{n_i}} Aut(\mathbb{R}^{n_i}).$$

So

$$\begin{aligned} Ad^G|_H =& \Lambda^2\lambda_n|_H = \Lambda^2(p_1 \oplus \cdots \oplus p_s \oplus \mathbb{L}_{n_0}) = \Lambda^2 p_1 \oplus \cdots \oplus \Lambda^2 p_s \oplus \Lambda^2 \mathbb{L}_{n_0} \\ & \oplus [(p_1 \otimes p_2) \oplus \cdots \oplus (p_1 \otimes p_s)] \oplus [(p_2 \otimes p_3) \oplus \cdots \oplus (p_2 \otimes p_s)] \\ & \oplus \cdots \oplus (p_{s-1} \otimes p_s) \oplus (p_1 \otimes \mathbb{L}_{n_0}) \oplus (p_2 \otimes \mathbb{L}_{n_0}) \oplus \cdots \oplus (p_s \otimes \mathbb{L}_{n_0}), \end{aligned}$$

where $\Lambda^2\mathbb{L}_{n_0}$ is the sum of $\binom{n_0}{2}$ trivial representations. We note that the dimension of the $\Lambda^2 p_1 \oplus \Lambda^2 p_2 \oplus \cdots \oplus \Lambda^2 p_s$ is $\binom{n_1}{2} + \cdots + \binom{n_s}{2}$ and is equal to the dimension of $Ad^H = \Lambda^2 p_1 \oplus \cdots \oplus \Lambda^2 p_s$. The author in [2] showed that for $h \in H, X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$, the adjoint representation of G decompose as

$$Ad^G(h)(X+Y) = Ad^G(h)X + Ad^{G/H}(h)Y$$

. Therefore, if we denote the representation $Ad^{G/H}$ by ϱ, then the isotropy representation of G/H is given by

$$\begin{aligned} \varrho =& \Lambda^2\mathbb{L}_{n_0} \oplus (p_1 \otimes p_2) \oplus \cdots \oplus (p_1 \otimes p_s) \oplus (p_2 \otimes p_3) \oplus \cdots \oplus (p_2 \otimes p_s) \\ & \oplus \cdots \oplus (p_{s-1} \otimes p_s) \oplus (p_1 \otimes \mathbb{L}_{n_0}) \oplus (p_2 \otimes \mathbb{L}_{n_0}) \oplus \cdots \oplus (p_s \otimes \mathbb{L}_{n_0}). \end{aligned} \tag{13}$$

It is worth noting that, the dimension of each of $p_i \otimes p_j$ is $n_i \times n_j$ and each of $p_i \otimes \mathbb{L}_{n_0}$, $i = 1, 2, \ldots, s$, contains n_0 equivalent representations of dimension n_i.

By the isotropy representation of G/H given in above, the decomposition of the tangent space $\mathfrak{m}$ of G/H can be written as follows:

$$\mathfrak{m} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_{\binom{n_0}{2}} \bigoplus_{0 \leqslant i < j}^{s} \mathfrak{m}_{ij}, \tag{14}$$

where we have:

$$\begin{aligned} & dim(\mathfrak{n}_i) = 1, \\ & \mathfrak{m}_{0j} = \mathfrak{m}_1^j \oplus \mathfrak{m}_2^j \oplus \mathfrak{m}_3^j \oplus \cdots \oplus \mathfrak{m}_{n_0}^j, \\ & dim(\mathfrak{m}_l^j) = n_j, \quad l = 1, 2, \ldots, n_0, \\ & \mathfrak{so}(n_0) \cong \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_{\binom{n_0}{2}}. \end{aligned}$$

Assume that $B : \mathfrak{so}(n) \times \mathfrak{so}(n) \to \mathbb{R}$ given by

$$B(X,Y) = -Trace(XY), \quad X, Y \in \mathfrak{so}(n),$$

be the $Ad(SO(n))$-invariant inner product. We obtain a B-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h} = \mathfrak{so}(n_1) \oplus \cdots \oplus \mathfrak{so}(n_s)$, $\mathfrak{m} \cong T_o(G/H)$. Then we consider the Lie brackets on $\mathfrak{g} = \mathfrak{so}(n)$ as follows. Assume that $M_n\mathbb{R}$ be the set of real $n \times n$ matrices and let $E_{ab} \in M_n\mathbb{R}$ be the matrix with 1 in the (a, b)-entry and zero elsewhere. Now, for $1 \leq a < b \leq n$ we set

$$e_{ab} := E_{ab} - E_{ba}. \tag{15}$$

Note that $e_{ab} = -e_{ba}$. Recall that the set

$$\mathbb{B} := \{e_{ab} : 1 \leq a < b \leq n\},$$

constitutes a basis of $\mathfrak{so}(n)$, which is orthogonal with respect to B. The proof of the following lemma is immediate.

Lemma 4.1. *For a, b, c distinct, the only non-zero bracket relations among the vectors (15) are $[e_{ab}, e_{bc}] = e_{ac}$.*

We note that, a choice for the modules in the decomposition (14) is:

$$\begin{aligned}
&\text{for } 1\le i<j\le s:\\
&\mathfrak{m}_{ij}=span\{e_{ab}: n_0+n_1+\cdots+n_{i-1}+1\le a\le n_0+n_1+\cdots+n_i,\\
&\qquad\qquad n_0+n_1+\cdots+n_{j-1}+1\le b\le n_0+n_1+\cdots+n_j\},\\
&\text{for } 1\le j\le s:\\
&\mathfrak{m}_{0j}=span\{e_{ab}: 1\le a\le n_0,\\
&\qquad\qquad n_0+n_1+\cdots+n_{j-1}+1\le b\le n_0+n_1+\cdots+n_j\},
\end{aligned}$$

and

$$\mathfrak{so}(n_0)=span\{e_{ab:}:\ 1\le a<b\le n_0\}.$$

In this case, the equivalent modules in the decomposition of $\mathfrak{m}_{0j}$ are given by:

$$\mathfrak{m}_l^j=span\{e_{lb}:\ n_0+n_1+\cdots+n_{j-1}+1\le b\le n_0+n_1+\cdots+n_j\},\ \ l=1,\ldots,n_0.$$

Also we have:

$$\mathfrak{so}(n_j)=span\{e_{ab}:\ n_0+n_1+\cdots+n_{j-1}+1\le a<b\le n_0+n_1+\cdots+n_j\},\ \ j=1,\ldots,s.$$

Hence, for the B-orthogonal we have the following decomposition:

$$\mathfrak{m}=\mathfrak{n}\oplus\mathfrak{p}, \tag{16}$$

where

$$\mathfrak{n}=\mathfrak{so}(n_0),\quad \mathfrak{p}=\bigoplus_{0\le i<j}^{s}\mathfrak{m}_{ij}.$$

Using the above decomposition, we can obtain the following matrix, which just shows the upper triangular part of $\mathfrak{so}(n)$:

$$\begin{bmatrix}
\mathfrak{so}(n) & \mathfrak{m}_{01} & \mathfrak{m}_{02} & \ldots & \mathfrak{m}_{0s}\\
\mathfrak{m}_{01} & 0_{n_1} & \mathfrak{m}_{12} & \ldots & \mathfrak{m}_{1s}\\
\mathfrak{m}_{02} & \mathfrak{m}_{12} & 0_{n_2} & \ldots & \mathfrak{m}_{2s}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\mathfrak{m}_{0s} & \mathfrak{m}_{1s} & \mathfrak{m}_{2s} & \ldots & 0_{n_s}
\end{bmatrix},$$

The matrices $\mathfrak{m}_{0j}$ are of size $n_0\times n_j$, the matrices $\mathfrak{m}_{ij}$ are of size $n_i\times n_j$, and the matrices $\mathfrak{so}(n_i)$ of size $n_i\times n_i$. We note that, if $n_0=0$ or $n_0=1$, then $\mathfrak{n}=\{0\}$. In the former case, the submodules $\mathfrak{m}_{0j}$ are zero while in the latter case they are non-zero and irreducible. Moreover, by using Lemma (4.1), we have:

$$[\mathfrak{so}(n_i),\mathfrak{m}_{lm}]=\begin{cases} m_{lm}, & \text{if } i=l \text{ or } i=m,\\ 0, & \text{otherwise}\end{cases},\ 0\le i\le s,\ 0\le l<m\le s, \tag{17}$$

and

$$[\mathfrak{m}_{ij},\mathfrak{m}_{jl}]=\mathfrak{m}_{il},\quad \text{for all}\ \ 0\le i<j<l\le s. \tag{18}$$

In the Riemannian spaces, we have the following Theorem [3]:

Theorem 4.2. *Assume that G/H be the space*

$$SO(n)/SO(n_1)\times\cdots\times SO(n_s)$$

where $0<n_1+\cdots+n_s\leq n$, and $n_j>1$, $j=1,\ldots,s$. A G-invariant Riemannian metric on G/H is geodesic orbit if and only if it is a normal metric, i.e. it is induced from an Ad-invariant inner product on the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$.

Now, we have the following Theorem for Kropina *g.o.* spaces:

Theorem 4.3. *Assume that $(G/H,F)$ be a Kropina space where*

$$G/H=SO(n)/SO(n_1)\times\cdots\times SO(n_s),\ \ 0<n_1+\cdots+n_s\leq n$$

and $n_j>1$, $j=1,\ldots,s$. In this case, the kropina metric $F=\frac{\alpha^2}{\beta}$ on G/H with the navigation data (h,W) is non-naturally reductive, non-Riemannian G'-geodesic orbit if and only if h is a G-invariant Riemannian normal metric and W is induced by $w\in\mathfrak{so}(n_0)$ with $\langle w,w\rangle=1$. We note that, G' is generated by G and ψ_t, the flow of W.

Proof. By Theorem 4.2, for geodesic orbit Riemannian metric g on G/H, the metric endomorphism A can be written as:

$$A=\lambda Id|_{\mathfrak{m}},\quad \lambda>0.$$

The inner product $B:\mathfrak{so}(n)\times\mathfrak{so}(n)\to\mathbb{R}$ where it is $Ad\big(SO(n)\big)$-invariant, can be written by

$$B(X,Y)=-Trace(XY),\quad X,Y\in\mathfrak{so}(n).$$

On the other hand, the geodesic orbit Riemannian metric on G/H is normal metric by Theorem 4.2. Therefore, for the geodesic orbit Riemannian metric h we have:

$$h_\lambda(X,Y)=\lambda B(X,Y),\quad \lambda>0. \tag{19}$$

We note that, the trivial $Ad(H)$-submodule is $\mathfrak{n}=\mathfrak{so}(n_0)$. Now, for every $w\in\mathfrak{so}(n_0)$, assume that W be the G-invariant vector field on

$$G/H=SO(n)/SO(n_1)\times\cdots\times SO(n_s)$$

with respect to a G-invariant Riemannian *g.o.* metric $\langle,\rangle$ on G/H of the form (19). By (17) we have the following relation:

$$h_\lambda\big([w,X_1]_{\mathfrak{m}},X_1\big)=\lambda B([w,X_1]_{\mathfrak{m}},X_1)=\lambda B([w,X_1],X_1)=0,\quad \forall X_1\in\mathfrak{m}.$$

Therefore, by Theorem 3.3 W is a G-invariant Killing vector field of $\langle,\rangle$. Therefor by Theorem 3.4 we have $(SO(n)/SO(n_1)\times\cdots\times SO(n_s),F)$ with navigation data (h_λ,W) is a geodesic orbit Kropina space. Then, a Kropina metric F on the homogeneous manifold

$$M=SO(n)/SO(n_1)\times\cdots\times SO(n_s)$$

with navigation data (h,W) is a non-Riemannian Kropina $G'-g.o.$ metric if and only if h is a Riemannian $G-g.o.$ metric on M and W is induced by every non-zero $w\in\mathfrak{so}(n_0)$ satisfying $\langle w,w\rangle=1$, where G' is generated by G and ψ_t, the flow of W.

Now, by consider the naturally reductivity and Lemma 4.1, we have $[\mathfrak{p},\mathfrak{p}]=\mathfrak{so}(n_0)\oplus\mathfrak{so}(n_1+\cdots+n_s)$. Hence, there exist non-zero elements $u,v\in\mathfrak{p}$ such that $\langle [u,v]_{\mathfrak{m}},w\rangle\neq 0$, where $w=W|_o\neq 0$. Now, by Lemma (3.7), the above G'-invariant Kropina metrics F with navigation data (h_λ,W) are non-naturally reductive.

Now we have the following results:

Corollary 4.4. *Assume that $(G/H,F)$ be a Kropina space where*

$$G/H=SO(n)/SO(n_1)\times\cdots\times SO(n_s),\ 0<n_1+\cdots+n_s\leq n$$

and $n_j>1$, $j=1,\ldots,s$. If $n_0=0$ or $n_0=1$, then any g.o. Kropina metric F on G/H with respect to G must be Riemannian.

Proof. In this case, there are no trivial submodules and then we have $\mathfrak{n}=\{0\}$ and then $W=0$. Thus F must be Riemannian.

Corollary 4.5. *Consider $(G/H,F)$ be a Kropina space with the navigation data (h,W), where*

$$G/H=SO(n)/SO(n_1)\times\cdots\times SO(n_s),\ 0<n_1+\cdots+n_s\leq n$$

and $n_j>1$, $j=1,\ldots s$. Then the non-naturally reductive, non-Riemannian G'-geodesic orbit Kropina metric F can be written as:

$$F(x,Y)=\frac{F^2}{2\lambda\sum_{1\leq i<j\leq n_0} w_{ij}a_{ij}},$$

where, G' is generated by G, ψ_t is the flow of W, and

$$F\triangleq 2\lambda\Big(\sum_{1\leq i<j\leq n_0} a_{ij}^2+\sum_{1\leq j\leq s}\sum_{\substack{1\leq k\leq n_0\\ n_0+\cdots+n_{j-1}<l\leq n_0+\cdots+n_j}} b_{kl}^2 + \sum_{1\leq i<j\leq s}\sum_{\substack{n_0+\cdots+n_{j-1}<p\leq n_0+\cdots+n_i\\ n_0+\cdots+n_{j-1}<q\leq n_0+\cdots+n_j}} c_{pq}^2\Big).$$

Proof. For a Riemanniann metric h and vector field W with $h(W,W)=1$, we Know that:

$$F(x,y)=\frac{h^2(y,y)}{2h(y,W)}.$$

From Theorem 4.3, for the G-invariant Riemannian $g.o.$ metric h_λ, the navigation data of the non-naturally reductive, non-Riemannian G'-geodesic orbit Kropina metric F is (h_λ,W). We note that here, W is induced by $w\in\mathfrak{so}(n_0)$ with $\langle w,w\rangle=1$. Now suppose that, $M_n\mathbb{R}$ be the set of $n\times n$ matrices and $E_{ab}\in M_n\mathbb{R}$ be the matrix with 1 in the (a,b)-entry and zero elsewhere. Let $e_{ij}=E_{ij}-E_{ji}$. Also, assume that

$$W = \sum_{1\le i<j\le n_0} w_{ij}e_{ij} \in \mathfrak{so}(n_0),$$

and

$$Y = \sum_{1\le i<j\le n_0} a_{ij}e_{ij} + \sum_{1\le j\le s} \sum_{\substack{1\le k\le n_0 \\ n_0+\cdots+n_{j-1}<l\le n_0+\cdots+n_j}} b_{kl}e_{kl}$$
$$+ \sum_{1\le i<j\le s} \sum_{\substack{n_0+\cdots+n_{j-1}<p\le n_0+\cdots+n_i \\ n_0+\cdots+n_{j-1}<q\le n_0+\cdots+n_j}} c_{pq}c_{pq} \in \mathfrak{m}.$$

Since

$$e_{ij}e_{kl} = \delta_{jk}E_{il} - \delta_{jl}E_{ik} - \delta_{ik}E_{jl} + \delta_{il}E_{jk},$$

and then we have:

$$h_\lambda(W,Y) = 2\lambda \sum_{1\le i<j\le n_0} w_{ij}a_{ij},$$

and hence

$$h_\lambda(X,Y) = 2\lambda\Big(\sum_{1\le i<j\le n_0} a_{ij}^2 + \sum_{1\le j\le s} \sum_{\substack{1\le k\le n_0 \\ n_0+\cdots+n_{j-1}<l\le n_0+\cdots+n_j}} b_{kl}^2$$
$$+ \sum_{1\le i<j\le s} \sum_{\substack{n_0+\cdots+n_{j-1}<p\le n_0+\cdots+n_i \\ n_0+\cdots+n_{j-1}<q\le n_0+\cdots+n_j}} c_{pq}^2\Big) \triangleq F.$$

Therefore, we have

$$F(x,Y) = \frac{F^2}{2\lambda \sum_{1\le i<j\le n_0} w_{ij}a_{ij}}.$$

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DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF MOHAGHEGH ARDABILI
P.O.Box. 5619911367, ARDABIL, IRAN
E-mail address: miladzeinali@gmail.com

DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF MOHAGHEGH ARDABILI
P.O.Box. 5619911367, ARDABIL, IRAN
E-mail address: latifi@uma.ac.ir