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INEQUALITIES FOR THE PRODUCT OF MEDIANS IN A TRIANGLE

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Abstract. In this article we study inequalities involving the product of three medians in a triangle. For an arbitrary triangle, we prove two inequality chains related to the fundamental triangle inequality. For a non-obtuse triangle, we establish several inequalities with one parameter. We also propose two interesting conjectures as open problems.

1. Introduction

Let m_a, m_b, m_c be the medians of a triangle ABC, and let h_a, h_b, h_c be its altitudes. From the monograph [13, pp.215-216], we learn that M.S.Klamin and A.Meir in 1981 proved the following beautiful inequality:

$$(h_a^2 + h_b^2 + h_c^2)\left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2}\right) \leq 9. \tag{1.1}$$

It is worth noting that the equality in (1.1) holds if and only if the triangle ABC is isosceles. This fact shows that inequality (1.1) is very strong.

By the well-known median formula:

$$4m_a^2 = 2(b^2 + c^2) - a^2, \tag{1.2}$$

where a, b, c are the sides of ABC. It is easy to prove the following identity:

$$(m_b m_c)^2 + (m_c m_a)^2 + (m_a m_b)^2 = \frac{9}{16}(b^2c^2 + c^2a^2 + a^2b^2). \tag{1.3}$$

Therefore, by the known identity

$$bc = 2Rh_a \tag{1.4}$$

(where R is the circumradius of $\triangle ABC$), we easily know that inequality (1.1) is equivalent to the following three inequalities:

$$m_a m_b m_c \geq \frac{1}{2}R(h_a^2 + h_b^2 + h_c^2), \tag{1.5}$$

$$m_a m_b m_c \geq \frac{1}{8R}(b^2c^2 + c^2a^2 + a^2b^2). \tag{1.6}$$

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$$\frac{m_bm_c}{m_a}+\frac{m_cm_a}{m_b}+\frac{m_am_b}{m_c}\le\frac{9}{2}R. \tag{1.7}$$

A direct proof of inequality (1.6) was early given by X. Z. Yang in a Chinese paper [17]. Inequality (1.5) and its equivalent form (1.6) can be used to prove some inequalities involving the medians of a triangle. The author has already applied them in the recent papers [8]-[10].

By using formula (1.2) and some known identities in triangle ABC, we can easily prove the following identity:

$$(m_am_bm_c)^2-\frac{1}{4}R^2\left(h_a^2+h_b^2+h_c^2\right)^2=\frac{t_0(s^2+2Rr+r^2)^2}{16R^2}, \tag{1.8}$$

where s and r are the semiperimeter and inradius of $\triangle ABC$ respectively, and

$$t_0=-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3.$$

Therefore, we see that inequalities (1.5)-(1.7) essentially are equivalent to the fundamental triangle inequality (cf. [3, inequality 13.8], [4], [7] and [13]):

$$s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3\le 0, \tag{1.9}$$

in which the equality holds if and only if the triangle is isosceles.

The purpose of this paper is to establish some inequalities involving the product $m_am_bm_c$ in a triangle. We also propose two interest conjectures as open problems.

2. Two inequality chains in an arbitrary triangle

For any triangle ABC, the author considered the lower bounds of $m_am_bm_c$ in terms of the elements R, r and s, and first found that

$$m_am_bm_c\ge\frac{s^2(s^2-7Rr+5r^2)}{2(4R+r)}, \tag{2.1}$$

which is stronger than inequalities (1.5) and (1.6). Essentially, this inequality also is equivalent to the fundamental triangle inequality.

Next, we shall prove that inequality (2.1) can be extended to the following chain:

Theorem 2.1. *In a triangle ABC the following inequality chain holds:*

$$\begin{aligned}&\frac{1}{2}Rs^2-2Rr(R-2r)\ge\frac{s^2(s^2+4R^2-10Rr+13r^2)}{4(4R+r)}\\&\ge m_am_bm_c\ge\frac{s^2(s^2-7Rr+5r^2)}{2(4R+r)}\ge\frac{1}{2}R(h_a^2+h_b^2+h_c^2)\\&\ge\frac{1}{2}Rs^2-\frac{1}{2}(R+r)(3R+r)(R-2r).\end{aligned} \tag{2.2}$$

The first and the second equality of (2.2) hold if and only $\triangle ABC$ is equilateral. The third and fourth equality of (2.2) hold if and only if $\triangle ABC$ is isosceles. The fifth equality of (2.2) holds if and only if $\triangle ABC$ is equilateral or right isosceles.

Next, we prove Theorem 2.1.

Proof. We have the following identity:

$$\frac{1}{2}Rs^2 - 2Rr(R-2r) - \frac{s^2(s^2+4R^2-10Rr+13r^2)}{4(4R+r)} = \frac{A_1}{4(4R+r)}, \tag{2.3}$$

where

$$A_1 = -s^4 + (4R^2+12Rr-13r^2)s^2 - 8(R-2r)(4R+r)Rr.$$

Thus, to prove the first inequality of (2.2), we need show $A_1 \geq 0$. Since A_1 can be rewritten as

$$\begin{aligned} A_1 = & - s^4 + (4R^2+20Rr-2r^2)s^2 - r(4R+r)^3 \\ & + (8R+11r)r(4R^2+4Rr+3r^2-s^2) \\ & + 4(7R+4r)(R-2r)r^2. \end{aligned} \tag{2.4}$$

Consequently, by Gerretsen's inequality (cf. [6], [7], [13] and [3, inequality 5.8]):

$$y_0 \equiv 4R^2+4Rr+3r^2-s^2 \geq 0, \tag{2.5}$$

Euler's inequality [3, inequality 5.1]):

$$e \equiv R-2r \geq 0 \tag{2.6}$$

and the fundamental triangle inequality (1.9), we deduce $A_1 \geq 0$ and the first inequality of (2.2) is proved.

We now prove the second inequality of (2.2). Firstly, by Euler's inequality and another Gerretsen's inequality (cf. [3, inequality 5.8], [6], [7] and [13]):

$$x_0 \equiv s^2 - 16Rr + 5r^2 \geq 0, \tag{2.7}$$

we have $s^2+4R^2-10Rr+13r^2 > 0$. Thus, we only need to prove that

$$A_2 \equiv \frac{s^4(s^2+4R^2-10Rr+13r^2)^2}{16(4R+r)^2} - (m_a m_b m_c)^2 \geq 0.$$

Using the following known identity:

$$\begin{aligned} & 16(m_a m_b m_c)^2 \\ & = s^6 - 3(4R-11r)rs^4 - 3(20R^2+40Rr+11r^2)r^2s^2 - (4R+r)^3r^3, \end{aligned} \tag{2.8}$$

we easily obtain

$$A_2 = \frac{A_3}{16(4R+r)^2}, \tag{2.9}$$

where

$$\begin{aligned} A_3 = & s^8 + (-8R^2-28Rr+25r^2)s^6 + 4(R^2+8Rr \\ & - 2r^2)(4R^2-4Rr-17r^2)s^4 + 3(20R^2+40Rr \\ & + 11r^2)(4R+r)^2r^2s^2 + (4R+r)^5r^3. \end{aligned}$$

So, we have to prove $A_3 \geq 0$. Note that the following inequality (cf. [3, inequality 5.5]):

$$(4R+r)^2 \geq 3s^2. \tag{2.10}$$

To prove $A_3 \geq 0$ we only need to show

$$\begin{aligned}
&s^6 + (-8R^2 - 28Rr + 25r^2)s^4 + 4(R^2 + 8Rr \\
&- 2r^2)(4R^2 - 4Rr - 17r^2)s^2 + 3(20R^2 + 40Rr \\
&+ 11r^2)(4R + r)^2 r^2 + 3(4R + r)^3 r^3 \geq 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
A_4 \equiv &s^6 + (-8R^2 - 28Rr + 25r^2)s^4 + 4(R^2 + 8Rr \\
&- 2r^2)(4R^2 - 4Rr - 17r^2)s^2 + 12(5R^2 + 11Rr \\
&+ 3r^2)(4R + r)^2 r^2 \geq 0. \qquad (2.11)
\end{aligned}$$

For proving $A_4 \geq 0$, we shall use the following Kooi's inequality (cf. [3, inequality 5.7])):

$$s^2 \leq \frac{R(4R + r)^2}{4R - 2r}, \qquad (2.12)$$

that is

$$K_0 \equiv R(4R + r)^2 - (4R - 2r)s^2 \geq 0. \qquad (2.13)$$

After analysis, we obtain the following identity (which can be checked by expanding):

$$8(2R - r)^3 A_4 = p_1 + q_1 K_0, \qquad (2.14)$$

where

$$\begin{aligned}
p_1 = &(R - 2r)(256R^4 - 528R^3 r + 669R^2 r^2 \\
&- 536Rr^3 + 144r^4)(4R + r)^2 r^2, \\
q_1 = &- 4(2R - r)^2 s^4 + 2(16R^3 + 88R^2 r - 157Rr^2 \\
&+ 50r^3)(2R - r)s^2 + (32R^2 + 61Rr \\
&- 34r^2)(112R^2 - 97Rr + 16r^2)r^2.
\end{aligned}$$

Since $R \geq 2r$, we have $p_1 \geq 0$. Thus, by (2.14) and $K_0 \geq 0$, it remains to prove the strict inequality $q_1 > 0$, which can be proved by using the fundamental triangle inequality (1.9), Gerretsen's inequalities (2.5), (2.7) and Euler's inequality (2.6). In fact, we can rewrite q_1 as follows:

$$q_1 = 4t_0(2R - r)^2 + r(m_1 x_0 + m_2 y_0 + m_3), \qquad (2.15)$$

where t_0 is the same as in (1.8), and

$$\begin{aligned}
m_1 = &2(2R - r)(16R^2 + 46r^2), \\
m_2 = &218Rr(2R - r), \\
m_3 = &1024R^5 + 2608R^4 r + 1704R^3 r^2 \\
&- 6529R^2 r^3 + 2568Rr^4 - 80r^5.
\end{aligned}$$

By Euler's inequality one sees that $m_1 > 0, m_2 > 0$ and $m_3 > 0$. And, then by inequalities (1.9), (2.5) and (2.7) we deduce $q_1 > 0$. So, inequality (2.11) and the second inequality of (2.2) are proved.

We now prove the third inequality of (2.2), i.e., inequality (2.1). Using the above identity (2.8), we easily obtain

$$(m_a m_b m_c)^2 - \frac{s^4(s^2-7Rr+5r^2)^2}{4(4R+r)^2} = \frac{t_0[4s^4+(16R+31r)rs^2+(4R+r)^2r^2]}{16(4R+r)^2}, \tag{2.16}$$

where t_0 is the same as in the previous identity (1.8). Note that $s^2-7Rr+5r^2>0$ and $t_0 \ge 0$, we immediately deduce from (2.16) that inequality (2.1) holds.

The forth inequality of (2.2) also is easily proved. Using the relation $2Rh_a = bc$ and the following known identity:

$$b^2c^2+c^2a^2+a^2b^2 = s^4-2(4R-r)rs^2+(4R+r)^2r^2, \tag{2.17}$$

we easily obtain

$$\frac{s^2(s^2-7Rr+5r^2)}{8R+2r} - \frac{1}{2}R(h_a^2+h_b^2+h_c^2) = \frac{rt_0}{8R(4R+r)}. \tag{2.18}$$

Since $t_0 \ge 0$ and $s^2-7Rr+5r^2>0$, we deduce that the forth inequality of (2.2) holds.

Finally, we prove the fifth inequality of (2.2). It is easy to obtain the following identity:

$$\frac{1}{2}R(h_a^2+h_b^2+h_c)^2 - \frac{1}{2}Rs^2 - \frac{1}{2}(R+r)(3R+r)(R-2r) = \frac{A_5}{8R}, \tag{2.19}$$

where

$$A_5 = s^4-2(2R^2+4Rr-r^2)s^2+(2R^2-2Rr-r^2)(6R^2+2Rr-r^2).$$

Thus we have to prove that

$$A_5 \ge 0. \tag{2.20}$$

We shall consider the following two cases to finish the proof of this inequality.

Case 1. $R^2-4Rr+r^2>0$.

Note that A_5 is a quadratic function in s^2 and it is easy to obtain its discriminant as follows:

$$F_0 = -32Rr(R^2-4Rr+r^2).$$

Thus, we have $F_0<0$ and then inequality $A_5>0$ follows immediately.

Case 2. $R^2-4Rr+r^2 \le 0$.

Let $A_5 = f(s^2)$, then

$$f'(s^2) = 2s^2-2(2R^2+4Rr-r^2).$$

We first show that $f'(s^2)>0$, i.e.,

$$s^2-2R^2-4Rr+r^2>0.$$

By Gerretsen's inequality (2.7), Euler's inequality and the hypothesis we have

$$\begin{aligned}&s^2-2R^2-4Rr+r^2\\&\geq 16Rr-5r^2-2R^2-4Rr+r^2\\&=-2(R^2-4Rr+r^2)+2r(2R-r)>0.\end{aligned}$$

Hence, $f'(s^2)>0$ holds under Case 2.

We now recall that from the fundamental triangle inequality (1.9) one can easily obtain the following inequality (cf. [1], [2] and [15]):

$$s^2\geq 2R^2+10Rr-r^2-2(R-2r)d_0, \tag{2.21}$$

where $d_0=\sqrt{R^2-2Rr}$. Thus, by $f'(s^2)>0$, to prove $A_5\geq 0$ we only need to prove

$$s_0^4+(-4R^2-8Rr+2r^2)s_0^2+(2R^2-2Rr-r^2)(6R^2+2Rr-r^2)\geq 0, \tag{2.22}$$

where $s_0^2=2R^2+10Rr-r^2-2(R-2r)d_0$. After the calculation, the above inequality becomes

$$4(R-2r)[(R-2r)d_0^2-6Rd_0r+R(2R^2-2Rr-r^2)]\geq 0.$$

Since $R\geq 2r$, it remains to show that

$$(R-2r)d_0^2+R(2R^2-2Rr-r^2)-6Rrd_0\geq 0.$$

Note that

$$(R-2r)d_0^2+R(2R^2-2Rr-r^2)=3R(R-r)^2>0.$$

So, we only need to prove

$$9R^2(R-r)^4-36R^2r^2d_0^2\geq 0,$$

which is equivalent to

$$9R^2(R^2-2Rr-r^2)^2\geq 0.$$

Therefore, we finish the proof of $A_5\geq 0$ under Case 2.

Combining the arguments of the above two cases, we conclude that $A_5\geq 0$ holds for all triangles. This completes the proof of the last inequality of (2.2).

We have known that all the equalities of (2.5)-(2.7), (2.12) hold if and only if the triangle ABC is equilateral. Therefore, by the equality condition of the fundamental inequality (1.9), we easily determine the equality condition for each inequality in (2.2). This completes the proof of Theorem 2.1.

Remark 2.1. *The second term of (2.2) is greater or equal to the fourth term. From this fact, one can easily obtain Gerretsen's inequality (2.5).*

Remark 2.2. *From inequality chain (2.2) we see that*

$$\begin{aligned}&\frac{1}{2}Rs^2-2Rr(R-2r)\geq m_am_bm_c\\&\geq\frac{1}{2}Rs^2-\frac{1}{2}(R+r)(3R+r)(R-2r),\end{aligned}\tag{2.23}$$

which together with Gerretsen's inequalities (2.5) *and* (2.7) *gives us*

$$\frac{1}{2}R(4R^2+11r^2) \geq m_am_bm_c \geq -\frac{3}{2}R^3+9R^2r+Rr^2+r^3. \tag{2.24}$$

In fact, one can prove the following stronger double inequality (we omit the details here):

$$(R+r)(2R^2-2Rr+5r^2) \geq m_am_bm_c \geq 9(2R-r)r^2. \tag{2.25}$$

Next, we shall give a double inequality whose low bound improves the previous inequality (2.1). Also, the equalities of this double inequality both hold if and only if the triangle is isosceles.

Theorem 2.2. *In a triangle ABC, let*

$$M_1 = \frac{-5s^4+(128R^2+424Rr-10r^2)s^2-(128R+5r)(4R+r)^2r}{216R},$$

$$M_2 = \frac{(9R-23r)s^4+2(28R^2+293Rr-23r^2)rs^2-(119R+23r)(4R+r)^2r^2}{72R(R+r)}.$$

Then

$$M_1 \geq m_am_bm_c \geq M_2. \tag{2.26}$$

Both equalities of (2.26) *hold if and only if $\triangle ABC$ is isosceles.*

Proof. To prove $M_1 \geq m_am_bm_c$, we first show $M_1 > 0$. We set

$$x_0 = s^2-16Rr+5r^2,$$
$$t_0 = -s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3.$$

It is easy to check that

$$\begin{aligned} &-5s^4+(128R^2+424Rr-10r^2)s^2-(128R+5r)(4R+r)^2r \\ &= 5t_0+108R[(R+3r)x_0+(35R-16r)r^2]. \end{aligned} \tag{2.27}$$

Thus, by the fundamental triangle inequality $t_0 \geq 0$, Euler's inequality (2.6) and Gerretsen's inequality (2.7), one sees that the right hand side of (2.27) is positive. So we have $M_1 > 0$. To prove $M_1 \geq m_am_bm_c$ we need to show that $M_1^2-(m_am_bm_c)^2 \geq 0$.

Using the previous identity (2.8) we obtain

$$M_1^2-(m_am_bm_c)^2 = \frac{t_0B_1}{46656R^2}, \tag{2.28}$$

where

$$\begin{aligned} B_1 = &-25s^4+(4096R^2+3740Rr-50r^2)s^2 \\ &-(65536R^3+24420R^2r+1380Rr^2+25r^3)r. \end{aligned}$$

Since $t_0 \geq 0$, we have to show $B_1 \geq 0$, which can be rewritten as follows:

$$B_1 = 25t_0+108R(37R+30r)x_0+8640Rr^2(R-2r). \tag{2.29}$$

Thus by inequalities (1.9), (2.6) and (2.7) we deduce that $B_1 \geq 0$ and inequality $M_1 \geq m_a m_b m_c$ is proved.

We next prove $m_a m_b m_c \geq M_2$. For this we first show the strict inequality $M_2 > 0$. It is easy to obtain

$$\begin{aligned} &72R(R+r)M_2 \\ &= 23rt_0 + 9R[x_0^2 + 4(7R+r)rx_0 + 12(12R^2+5Rr-4r^2)r^2]. \end{aligned} \tag{2.30}$$

By $t_0 \geq 0$, Gerretsen's inequality $x_0 \geq 0$ and Euler's inequality (2.6) we deduce that $M_2 > 0$. Therefore, to prove $m_a m_b m_c \geq M_2$ we only need to show that $(m_a m_b m_c)^2 \geq M_2^2$. Note that

$$(m_a m_b m_c)^2 - M_2^2 = \frac{t_0 B_2}{5184R^2(R+r)^2}, \tag{2.31}$$

where

$$\begin{aligned} B_2 =&(9R-23r)^2 s^4 + 2(162R^3 + 661R^2 r - 8188Rr^2 + 529r^3)rs^2 \\ &+ (324R^4 + 57292R^3 r + 36381R^2 r^2 + 7590Rr^3 + 529r^4)r^2. \end{aligned}$$

Since $t_0 \geq 0$, it remains to prove the strict inequality $B_2 > 0$. We set $e = R - 2r$, then it is easy to check the following identity:

$$B_2 = (9R-23r)^2 x_0^2 + 4r(n_1 x_0 + n_2 y_0 + n_3), \tag{2.32}$$

where

$$\begin{aligned} n_1 &= (729e + 1190r)e^2, \\ n_2 &= (2815e + 5616r)r^2, \\ n_3 &= er(6561e^3 + 30698e^2 r + 53421er^2 + 35424r^3). \end{aligned}$$

In view of Gerretsen's inequalities $x_0 \geq, y_0 \geq 0$ and Euler's inequality $e \geq 0$, we see that $B_2 \geq 0$ holds and inequality $m_a m_b m_c \geq M_2$ is proved. From the above proof, we easily deduce that both equality conditions of inequality chain (2.26) are the same as that of inequality (1.8). This completes the proof of Theorem 2.2.

Remark 2.3. *Note that*

$$M_2 - \frac{s^4(s^2 - 7Rr + 5r^2)^2}{(8R+2r)^2} = \frac{(119R+23r)rt_0}{R(R+r)(4R+r)}, \tag{2.33}$$

where $t_0 \geq 0$ is the same as in (1.8). We see that inequality $m_a m_b m_c \geq M_2$ is stronger than the previous inequality (2.1).

Remark 2.4. *It is easy to obtain Note that*

$$M_1 - M_2 = \frac{4(R-2r)t_0}{27R(R+r)}, \tag{2.34}$$

where t_0 is the same as in (1.8). Therefore, by $M_1 \geq M_2$ and $R \geq 2r$ we can deduce $t_0 \geq 0$, which is equivalent to the fundamental triangle inequality (1.9).

3. Inequalities with one parameter in non-obtuse triangles

From Theorem 2.1 we see that

$$m_a m_b m_c \geq \frac{1}{2}Rs^2 - \frac{1}{2}(R+r)(3R+r)(R-2r). \tag{3.1}$$

From this inequality, using Walker's non-obtuse triangle inequality (see [14] and [13]):

$$s^2 \geq 2R^2 + 8Rr + 3r^2 \tag{3.2}$$

(with equality if and only if $\triangle ABC$ is equilateral or right isosceles), we obtain a lower bound of $m_a m_b m_c$ in terms of R and r, i.e.

$$m_a m_b m_c \geq -\frac{1}{2}R^3 + 5R^2 r + 5Rr^2 + r^3, \tag{3.3}$$

which is valid for non-obtuse triangle ABC.

In this section, we shall give three generalizations (with one parameter) of inequality (3.3).

We first give the following generalization of Walker's inequality (3.1).

Lemma 3.1. *Let $k \geq -3 + 2\sqrt{2}$ be a real number, then for a non-obtuse triangle ABC the following inequality holds:*

$$s^2 \geq \frac{4kR^3 - 4(k-4)R^2 r - (7k+3)Rr^2 - 2(k+2)r^3}{(k+1)R - 2kr}, \tag{3.4}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Proof. In [11], the author has proved that inequality (3.4) holds for $k \geq 0$. It remains to prove the case when $0 > k \geq -3 + \sqrt{2}$. Putting $k = -3 + \sqrt{2} + t (t \geq 0)$, then by Euler's inequality one has

$$(k+1)R - 2kr = (R-2r)t + 2(-1+\sqrt{2})(R - r + r\sqrt{2}) > 0.$$

Hence inequality (3.4) is equivalent to

$$\begin{aligned}&[(k+1)R - 2kr]s^2\\ &\geq 4kR^3 - 4(k-4)R^2 r - (7k+3)Rr^2 - 2(k+2)r^3,\end{aligned}$$

i.e.,

$$k(R-2r)[s^2 - (2R+r)^2] + Rs^2 R + 3Rr^2 - 16R^2 r + 4r^3 \geq 0. \tag{3.5}$$

We recall that for non-obtuse triangles we have the following fundamental Ciamberlini inequality (cf. [5], [13] and [16]):

$$s \geq 2R + r, \tag{3.6}$$

with equality if and only if $\triangle ABC$ is a right triangle. Also, we have $R \geq 2r$, thus when $k \geq -3 + 2\sqrt{2}$ we only need to show that

$$\begin{aligned}&(-3+2\sqrt{2})(R-2r)[s^2 - (2R+r)^2]\\ &+ Rs^2 R + 3Rr^2 - 16R^2 r + 4r^3 \geq 0,\end{aligned}$$

which is equivalent to

$$2(3 - 2\sqrt{2})(R - r + \sqrt{2}r)T_0 \geq 0,$$

where

$$T_0 = (1+\sqrt{2})s^2 + 2R^2 - 6(4+3\sqrt{2})rR + (13+9\sqrt{2})r^2.$$

It remains to prove that

$$T_0 \geq 0, \tag{3.7}$$

which actually holds for all triangles. In deed, by the previous inequality (2.21) we have

$$\begin{aligned}T_0 \geq &(1+\sqrt{2})[2R^2 + 10Rr - r^2 - 2(R-2r)d_0]\\ &+ 2R^2 - 6(4+3\sqrt{2})rR + (13+9\sqrt{2})r^2\\ &= (2+\sqrt{2})(2R - 2r - \sqrt{2}r - \sqrt{2}d_0)(R-2r).\end{aligned}$$

where $d_0 = \sqrt{R^2 - 2Rr}$. Since $R \geq 2r$, to prove $T_0 \geq$ we need to show that

$$(2R - 2r - r\sqrt{2})^2 - (\sqrt{2}d_0)^2 \geq 0.$$

But we have

$$(2R - 2r - r\sqrt{2})^2 - 2d_0^2 = 2(R - r - \sqrt{2}r)^2 \geq 0,$$

So, inequality (3.7) is proved and we proved that when $0 > k \geq -3 + \sqrt{2}$ inequality (3.4) holds for non-obtuse triangle ABC.

It is easy to determine that equality in (3.4) holds if and only if $\triangle ABC$ is equilateral or right isosceles. Lemma 3.1 is proved.

Now, we apply Lemma 3.1 to inequality (3.1), the following inequality is obtained immediately.

Theorem 3.1. *Let $k \geq -3 + 2\sqrt{2}$ be a real number, then for a non-obtuse triangle ABC the following inequality holds:*

$$\begin{aligned}&m_a m_b m_c\\ &\geq \frac{(k-3)R^4 + 2r(2k+9)R^3 - 4r^2(k-1)R^2 - 2r^3(7k+1)R - 4kr^4}{2(k+1)R - 4kr},\end{aligned} \tag{3.8}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Putting $k = 1$ in (3.8), simplifying gives us inequality (3.3). So, Theorem 3.1 is a generalization of inequality (3.3). In (3.8), taking $k = 0$ we easily get

$$m_a m_b m_c \geq -\frac{3}{2}R^3 + 9R^2 r + 2Rr^2 - r^3. \tag{3.9}$$

In (3.8), putting $k = R/r$ and then simplifying gives us

$$m_a m_b m_c \geq \frac{1}{2}R^3 + R^2 r + 8Rr^2 + 3r^3. \tag{3.10}$$

Next, we give another generalization of inequality (3.3):

Theorem 3.2. *Let k be a real number such that $k \geq (5 - 4\sqrt{2})/7$ or $k \leq -7/3$, then for a non-obtuse triangle ABC the following inequality holds:*

$$\begin{aligned}&m_a m_b m_c\\ &\geq \frac{(k-3)R^3 + 2(k+9)R^2 r + 4(4k+1)Rr^2 + 2(3k-1)r^3}{2(k+1)},\end{aligned} \tag{3.11}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

For $k = 0$ in (3.11), inequality (3.3) follows immediately. Hence Theorem 3.2 is a generalization of inequality (3.3).

Next, we shall use the previous inequality (2.1) to prove Theorem 3.2.

Proof. We first consider the following case:

Case 1. $k \geq \dfrac{5-4\sqrt{2}}{7}$.

Denote the right hand side of (3.11) by C_0, then it is easy to obtain

$$\frac{s^2(s^2-7Rr+5r^2)}{2(4R+r)} - C_0 = \frac{kC_1+C_2}{2(k+1)(4R+r)}, \tag{3.12}$$

where

$$C_1 = s^4 - (7R-5r)rs^2 - (4R+r)(R^3+2R^2r+16Rr^2+6r^3),$$
$$C_2 = s^4 - (7R-5r)rs^2 + (4R+r)(3R^3-18R^2r-4Rr^2+2r^3).$$

As $k \geq (5-4\sqrt{2})/7$, so that $k+1>0$. According to identity (3.12) and inequality (2.1), to prove $m_am_bm_c \geq C_0$ we need to prove that

$$kC_1 + C_2 \geq 0. \tag{3.13}$$

We first show $C_1 \geq 0$. We set

$$Q_0 = (s^2-2R^2-8Rr-3r^2)[s^2-(2R+r)^2].$$

Walker's inequality (3.2) and Ciamberlini's inequality (3.6) show that $Q_0 \geq 0$. One can rewrite C_1 as follows:

$$\begin{aligned} C_1 =& Q_0 + (6R^2+5Rr+9r^2)s^2 \\ & - (12R^4+49R^3r+112R^2r^2+60Rr^3+9r^4). \end{aligned} \tag{3.14}$$

Thus, it remains to prove

$$\begin{aligned} &(6R^2+5Rr+9r^2)s^2 \\ &- (12R^4+49R^3r+112R^2r^2+60Rr^3+9r^4) \geq 0, \end{aligned} \tag{3.15}$$

which is easily obtained by using the inequality of Lemma 3.1. In fact, if we let

$$\begin{aligned} &\frac{12R^4+49R^3r+112R^2r^2+60Rr^3+9r^4}{6R^2+5Rr+9r^2} \\ &= \frac{4kR^3-4(k-4)R^2r-(7k+3)Rr^2-2(k+2)r^3}{(k+1)R-2kr}, \end{aligned}$$

then solve k to get

$$k = \frac{12R^2+Rr+18r^2}{R(12R+19r)}.$$

Therefore, if we take k as above in inequality (3.4), then inequality (3.15) can be obtained after simplifying. We thus deduce that $C_1 \geq 0$ holds.

Since $C_1 \geq 0$ and $7k \geq 5-4\sqrt{2}$, to prove inequality (3.13) we only need to prove that

$$(5-4\sqrt{2})C_1 + 7C_2 \geq 0,$$

which is equivalent to

$$C_3 \equiv (12-4\sqrt{2})s^4 + 4(-3+\sqrt{2})(7R-5r)rs^2 + 2(4+\sqrt{2})(4R+r)[2R^3 + 2(-10+3\sqrt{2})R^2r + (13\sqrt{2}-20)Rr^2 + 4(\sqrt{2}-1)r^3] \geq 0. \tag{3.16}$$

We can rewrite C_3 as follows:

$$C_3 = (12-4\sqrt{2})Q_0 + 2(3\sqrt{2}-2)(p_1s^2+q_1), \tag{3.17}$$

where

$$p_1 = \sqrt{2}(6R^2+5Rr+9r^2),$$
$$q_1 = 8R^4 - 2(47\sqrt{2}+15)rR^3 - 8(11\sqrt{2}-2)r^2R^2 - (27\sqrt{2}-22)r^3R - (3\sqrt{2}-4)r^4.$$

Hence, to prove inequality (3.16) it remains to prove that

$$p_1s^2 + q_1 \geq 0. \tag{3.18}$$

We shall use the previous inequality (2.21) to prove this inequality holds for all triangles. By (2.21), we have

$$\begin{aligned} & p_1s^2+q_1 \\ & \geq p_1[2R^2+10Rr-r^2-2(R-2r)d_0]+q_1 \\ & = (R-2r)(p_2-q_2d_0), \end{aligned}$$

where

$$p_2 = (8+12\sqrt{2})R^3 - 14R^2r - 2(6+13\sqrt{2})Rr^2 + 2(3\sqrt{2}-1)r^3,$$
$$q_2 = 2\sqrt{2}(6R^2+5Rr+9r^2).$$

Thus, to prove $p_1s^2+q_1 \geq 0$ we only need to prove that

$$p_2s^2 - q_2 \geq 0. \tag{3.19}$$

Let $e = R-2r$, then it is easy to get

$$p_2 = (8+12\sqrt{2})e^3 + 2(17+36\sqrt{2})e^2r + 2(14+59\sqrt{2})er^2 + 2(25\sqrt{2}-9)r^3 > 0.$$

So, we only need to prove $p_2^2 - q_2^2d_0^2 \geq 0$. Using $d_0 = \sqrt{R^2-2Rr}$, we get

$$p_2^2 - q_2^2d_0^2 = (R-r+\sqrt{2}r)(R-r-\sqrt{2}r)^2C_4, \tag{3.20}$$

where

$$C_4 = (192\sqrt{2}+64)R^3 + 16(19\sqrt{2}+28)R^2r + 12(28\sqrt{2}-19)Rr^2 + 4(25\sqrt{2}-31)r^3 > 0.$$

Since $R \geq 2r$, we have $p_2^2 - q_2^2d_0^2 \geq 0$ and proved that inequality (3.18) holds for all triangles. This completes the proof of inequality (3.11) under the case $k \geq (5-4\sqrt{2})/7$.

Next, we consider the following case:

Case 2. $k \leq -\dfrac{7}{3}$.

In this case we have $k+1<0$. Consequently, by identity (3.12) we have to prove that $kC_1+C_2\le 0$. Putting $k=-\frac{7}{3}-p(p\ge 0)$, then we need to prove

$$-(\frac{7}{3}+p)C_1+C_2\le 0,$$

i.e.,

$$(7+3p)C_1-3C_2\ge 0,$$

which is equivalent to

$$pD_1+D_2\ge 0, \tag{3.21}$$

where

$$\begin{aligned}D_1 =&3s^4-3(7R-5r)rs^2-3(4R+r)(R^3+2R^2r+16Rr^2+6r^3),\\ D_2 =&4s^4-4(7R-5r)rs^2-4(4R+r)(4R^3-10R^2r+25Rr^2+12r^3).\end{aligned}$$

As $p\ge 0$, we need to prove $D_1\ge 0$ and $D_2\ge 0$. Note that

$$D_1=3Q_0+D_3, \tag{3.22}$$

where Q_0 is the same as in (3.14) and

$$\begin{aligned}D_3 =&(18R^2+15Rr+27r^2)s^2\\ &-(36R^4+147R^3r+336R^2r^2+180Rr^3+27r^4).\end{aligned}$$

Since $Q_0\ge 0$, to prove $D_1\ge 0$ we need to show $D_3\ge 0$, which can be obtained by using the inequality of Lemma 3.1. In deed, if we let

$$\begin{aligned}&\frac{36R^4+147R^3r+336R^2r^2+180Rr^3+27r^4}{18R^2+15Rr+27r^2}\\ =&\frac{4kR^3-4(k-4)R^2r-(7k+3)Rr^2-2(k+2)r^3}{(k+1)R-2kr},\end{aligned}$$

then solve k to get

$$k=\frac{12R^2+Rr+8r^2}{R(12R+19r)}.$$

Therefore, we deduce that $D_3\ge 0$ can be obtained from inequality (3.4). Inequality $D_1\ge 0$ is proved.

In the same way, one can also prove inequality $D_2\ge 0$. Firstly, note that

$$D_2=4Q_0+D_4, \tag{3.23}$$

where

$$\begin{aligned}D_4 =&(24R^2+20Rr+36r^2)s^2\\ &-(96R^4+16R^3r+544R^2r^2+372Rr^3+60r^4).\end{aligned}$$

Consequently, we only need to prove $D_4\ge 0$. In inequality (3.4), we take

$$k=\frac{12R^2+2Rr+9r^2}{(20R+3r)r},$$

then it is easy to get inequality $D_4\ge 0$. This completes the proof of inequality (3.11) when $k\le -\frac{7}{3}$.

According to the equality conditions of (1.9), (3.4) and (3.6), we easily determine the equality condition of (3.11). Theorem 3.2 is proved.

In Theorem 3.2, for $k = 3, -9, -\frac{1}{4}, \frac{1}{3}$, we respectively obtain the following inequalities:

$$m_a m_b m_c \geq \frac{1}{2} r(6R^2 + 13Rr + 4r^2), \tag{3.24}$$

$$m_a m_b m_c \geq \frac{3}{4} R^3 + \frac{35}{4} Rr^2 + \frac{7}{2} r^3, \tag{3.25}$$

$$m_a m_b m_c \geq -\frac{13}{6} R^3 + \frac{35}{3} R^2 r - \frac{7}{3} r^3, \tag{3.26}$$

$$m_a m_b m_c \geq \frac{1}{2} R(-2R^2 + 14Rr + 7r^2). \tag{3.27}$$

Next, we shall give the third generalization of inequality (3.3). We first give another generalization of Walker's inequality. That was given by the author in the recent paper [12]:

Lemma 3.2. *Let k be a real number such that $0 \leq k \leq 3 + \sqrt{2}$, then for a non-obtuse triangle ABC the following inequality holds:*

$$s^2 \geq (4-k)R^2 + 4kRr + 3(3-k)r^2 + \frac{2(2-k)r^3}{R}, \tag{3.28}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Inequality (3.28) prompt us to obtain the following conclusion:

Theorem 3.3. *Let k be a real number such that $-3 \leq k \leq 3 + \sqrt{2}$, then for a non-obtuse triangle ABC the following inequality holds:*

$$\begin{aligned} & m_a m_b m_c \\ & \geq \frac{1}{2}(1-k)R^3 + (2k+1)R^2 r + \frac{1}{2}(16-3k)Rr^2 + (3-k)r^3, \end{aligned} \tag{3.29}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

In Theorem 3.3, for $k = 2$, we get inequality (3.3). Consequently, Theorem 3.3 is a generalization of (3.3).

We now prove Theorem 3.3.

Proof. Applying inequality (3.28) to the previous inequality (3.1), we immediately know that inequality (3.29) holds for $0 \leq k \leq 3 + \sqrt{2}$.

We next prove that inequality (3.29) holds for $-3 \leq k < 0$. Denoting the right hand side of (3.29) by E_0, then we have

$$\frac{s^2(s^2 - 7Rr + 5r^2)}{8R + 2r} - E_0 = \frac{E_1}{8R + 2r}, \tag{3.30}$$

where

$$\begin{aligned} E_1 = & (4R + r)(R - 2r)(R^2 - 2Rr - r^2)k \\ & + s^4 - (7R - 5r)rs^2 - (4R + r)(R^3 + 2R^2 r + 16Rr^2 + 6r^3). \end{aligned}$$

Thus, by the previous inequality (2.1), to prove inequality (3.29) we need to show $E_1 \geq 0$. When $k = 0$, the inequality becomes

$$E_2 \equiv s^4 - (7R - 5r)rs^2 - (4R + r)(R^3 + 2R^2r + 16Rr^2 + 6r^3) \geq 0. \tag{3.31}$$

When $k = -3$, inequality $E_1 \geq 0$ becomes

$$E_3 \equiv s^4 - (7R - 5r)rs^2 - (4R + r)(4R^3 - 10R^2r + 25Rr^2 + 12r^3) \geq 0. \tag{3.32}$$

According to the property of linear functions, to prove inequality $E_1 \geq 0$ we only need to prove (3.31) and (3.32). Putting

$$Q_0 = (s^2 - 2R^2 - 8Rr - 3r^2)[s^2 - (2R + r)^2],$$

Walker's inequality (3.2) and Ciamberlini's inequality (3.6) show that $Q_0 \geq 0$. Note that

$$E_2 = Q_0 + (6R^2 + 5Rr + 9r^2)s^2 - (12R^4 + 49R^3r + 112R^2r^2 + 60Rr^3 + 9r^4). \tag{3.33}$$

To prove $E_2 \geq 0$, we need to show that

$$(6R^2 + 5Rr + 9r^2)s^2 - (12R^4 + 49R^3r + 112R^2r^2 + 60Rr^3 + 9r^4) \geq 0. \tag{3.34}$$

In Lemma 3.1, we take

$$k = \frac{12R^2 + Rr + 18r^2}{R(12R + 19r)}.$$

Simplifying gives us inequality (3.34). Thus inequality (3.31) is proved.

In the same way, we can prove inequality (3.32). Note that

$$E_3 = Q_0 + (6R^2 + 5Rr + 9r^2)s^2 - (24R^4 + 4R^3r + 136R^2r^2 + 93Rr^3 + 15r^4). \tag{3.35}$$

As $Q_0 \geq 0$, to prove $E_3 \geq 0$ we need to prove that

$$(6R^2 + 5Rr + 9r^2)s^2 - (24R^4 + 4R^3r + 136R^2r^2 + 93Rr^3 + 15r^4) \geq 0. \tag{3.36}$$

In Lemma 3.1, if we take

$$k = \frac{12R^2 + 2Rr + 9r^2}{(20R + 3r)r},$$

then it is easy to obtain inequality (3.36). So, inequality $E_3 \geq 0$ is proved and we proved that inequality (3.29) holds for $-3 \leq k < 0$.

Combining the above arguments, we proved that for non-obtuse triangle ABC inequality (3.29) holds for $-3 \leq k \leq 3 + \sqrt{2}$. It is easy to determine the equality condition of (3.29). This completes the proof of Theorem 3.3.

In Theorem 3.3, if we take $k = -3/2$ and $k = 3/2$ respectively, then the following two inequalities are obtained:

$$m_a m_b m_c \geq \frac{1}{4}(5R + 2r)(R^2 - 2Rr + 9r^2), \tag{3.37}$$

$$m_a m_b m_c \geq \frac{1}{4}(R + r)(-R^2 + 17Rr + 6r^2). \tag{3.38}$$

In [11], the author obtained the following non-obtuse triangle inequality from the previous inequality (3.4):

Lemma 3.3. *For any positive number k and a non-obtuse triangle ABC the following inequality holds:*

$$s^2 \geq \frac{2(2k+1)R^2 + 4(k+2)Rr + (k+3)r^2}{k+1}, \tag{3.39}$$

with equality if and only if $\triangle ABC$ is right isosceles.

Applying inequality (3.39) to inequality (3.1), we immediately obtain the following conclusion:

Theorem 3.4. *For a non-obtuse triangle ABC and positive real number k the following inequality holds:*

$$\begin{aligned} & m_a m_b m_c \\ & \geq \frac{(k-1)R^3 + 2(3k+5)R^2 r + 2(4k+5)Rr^2 + 2(k+1)r^3}{2(k+1)}, \end{aligned} \tag{3.40}$$

with equality if and only if $\triangle ABC$ is right isosceles.

In (3.40), taking $k = 1$ and $k = 3$ we get the following two inequalities respectively:

$$m_a m_b m_c \geq \frac{1}{2}r(8R^2 + 9Rr + 2r^2), \tag{3.41}$$

$$m_a m_b m_c \geq \frac{1}{4}(R + r)(R^2 + 13Rr + 4r^2). \tag{3.42}$$

Adding inequality (3.41) and the previous inequality (3.24), we can obtain

$$m_a m_b m_c \geq \frac{1}{2}r(7R^2 + 11Rr + 3r^2). \tag{3.43}$$

In addition, adding (3.38) and (3.42) gives us

$$m_a m_b m_c \geq \frac{5}{4}(R + r)(3R + r)r. \tag{3.44}$$

Both equalities in (3.43) and (3.44) hold if and only if $\triangle ABC$ is right isosceles.

4. Two conjectures

Inequality (1.1) is a beautiful geometry inequality involving the altitudes and medians of a triangle. Recently, the author found a new inequality which also involves the altitudes and medians of a triangle, i.e.

$$1 + 2\frac{h_a h_b h_c}{m_a m_b m_c} \geq \frac{h_a^2}{m_a^2} + \frac{h_b^2}{m_b^2} + \frac{h_c^2}{m_c^2}. \tag{4.1}$$

Also, the equality condition of this inequality is the same as that of (1.1).

By using inequality (1.5), it is easy to prove inequality (4.1). In fact, we easily prove the following identity:

$$1+2\frac{h_a h_b h_c(h_a^2+h_b^2+h_c^2)R}{(m_a m_b m_c)^2}-\frac{h_a^2}{m_a^2}-\frac{h_b^2}{m_b^2}-\frac{h_c^2}{m_c^2}$$
$$=\frac{-t_0(s^2+2Rr+r^2)^2}{64(m_a m_b m_c)^2R^2}, \tag{4.2}$$

where t_0 is the same as in (1.8). Consequently, by inequality (1.5) and the fundamental inequality (1.9) we see that inequality (4.1) holds for all triangles.

Considering exponential generalizations of (4.1), we propose the following general conjecture:

Conjecture 1. *Let $k \geq -3$ be a real number, and let*

$$x_1=\left(\frac{h_a}{m_a}\right)^k,\ x_2=\left(\frac{h_b}{m_c}\right)^k,\ x_3=\left(\frac{h_c}{m_c}\right)^k.$$

Then for any triangle ABC the following inequality holds:

$$1+2x_1x_2x_3 \geq x_1^2+x_2^2+x_3^2. \tag{4.3}$$

When $k \neq 0$, the equality in (4.3) holds if and only if $\triangle ABC$ is isosceles.

We also propose the following similar conjectures:

Conjecture 2. *Let $k \geq -12$ be a real number and let*

$$y_1=\left(\frac{r_b+r_c}{2m_a}\right)^k,\ y_2=\left(\frac{r_c+r_a}{2m_b}\right)^k,\ y_3=\left(\frac{r_a+r_b}{2m_c}\right)^k,$$

where r_a, r_b, r_c are the raddi of excircles of triangle ABC. Then

$$1+2y_1y_2y_3 \leq y_1^2+y_2^2+y_3^2. \tag{4.4}$$

When $k \neq 0$, the equality in (4.4) holds if and only if $\triangle ABC$ is isosceles.

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