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ON RIEMANNIAN CONNECTIONS AND SEMI-SIMPLICITY OF A LIE ALGEBRA

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Abstract. Using a almost product structure defined by a spray, we give a necessary and sufficient condition, for a linear connection with vanishing torsion to be Riemannian and, for the semi-simplicity of Lie algebra of projectable vector fields which commute with a spray. In the general case, we propose some properties that allow recognizing a semi-simple Lie algebra

1. Introduction

The object of this paper is a review and a complement of our results in [1], [2], [3] and [4]. All considered objects are smooth. Let M be a connected paracompact differentiable manifold of dimension $n \geq 2$, J the vector $1-$form defining the tangent structure, C the Liouville field on the tangent space TM, S a spray. We denote $\Gamma = [J, S]$, Γ is an almost product structure: $\Gamma^2 = I$, I being the identity vector 1-form. We can consider Γ [9] as a linear connection with vanishing torsion. The curvature of Γ is then the Nijenhuis tensor of h, $R = \frac{1}{2}[h, h]$, with $h = \frac{I+\Gamma}{2}$. We will give some properties of R. We then study a linear connection coming from a metric. At the end, we are interested in the Lie algebra $A_S = \{X \in \chi(TM) \mid [X, S] = 0\}$, where $\chi(TM)$ denotes the set of all vector fields on TM.

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2. Preliminaries

We recall the bracket of two vectors 1−form K and L on a manifold M [8],

$$\begin{aligned}[K,L](X,Y) &= [KX,LY]+[LX,KY]+KL[X,Y]+LK[X,Y]-K[LX,Y]\\ &\quad -L[KX,Y]-K[X,LY]-L[X,KY]\end{aligned}$$

for all $X,\ Y\in\chi(M)$.
The bracket $N_L=\frac{1}{2}[L,L]$ is called the Nijenhuis tensor of L. The Lie derivative L_X with respect to X applied to L can be written

$$[X,L]Y=[X,LY]-L[X,Y].$$

The exterior derivation d_L is defined in [5]: $d_L=[i_L,d]$.
Let Γ be an almost product structure. We denote

$$h=\frac{1}{2}(I+\Gamma) \text{ and } v=\frac{1}{2}(I-\Gamma),$$

The vector 1−form h is the horizontal projector, projector of the subspace corresponding to the eigenvalue +1, and v the vertical projector corresponding to the eigenvalue −1. The curvature of Γ is defined by $R=\frac{1}{2}[h,h]$, which is also equal to $\frac{1}{8}[\Gamma,\Gamma]$.
The Lie algebra A_Γ is defined by

$$A_\Gamma=\{X\in\chi(TM) \text{ such that } [X,\Gamma]=0\}.$$

The nullity space of the curvature R is:

$$N_R=\{X\in\chi(TM)\ \text{ such that } R(X,Y)=0,\ \forall\ Y\in\chi(TM)\}.$$

Definition 2.1. *A second order differential equation on a manifold M is a vector field S on the tangent space TM such that $JS=C$.
Such a vector field on TM is also called a semi-spray on M, S is a spray on M if S is homogeneous of degree 1: $[C,S]=S$.
In what follows, we use the notation in [9] and [15] to express a geodesic spray of a linear connection. In local natural coordinates on an open set U of M, (x^i,y^j) are the coordinates in TU, a spray S is written*

$$S=y^i\frac{\partial}{\partial x^i}-2G^i(x^1,\ldots,x^n,y^1,\ldots,y^n)\frac{\partial}{\partial y^i}.$$

For a connection $\Gamma=[J,S]$, the coefficients of Γ become $\Gamma_i^j=\frac{\partial G^j}{\partial y^i}$ and the projector horizontal is

$$h(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial x^i}-\Gamma_i^j\frac{\partial}{\partial y^j},\ h(\frac{\partial}{\partial y^j})=0$$

the projector vertical

$$v(\frac{\partial}{\partial x^i})=\Gamma_i^j\frac{\partial}{\partial y^j},\ v(\frac{\partial}{\partial y^j})=\frac{\partial}{\partial y^j}$$

The curvature $R=\frac{1}{2}[h,h]$ become

$$R=\frac{1}{2}R_{ij}^k dx^i\wedge dx^j\otimes\frac{\partial}{\partial y^k} \text{ with } R_{ij}^k=\frac{\partial\Gamma_i^k}{\partial x^j}-\frac{\partial\Gamma_j^k}{\partial x^i}+\Gamma_i^l\frac{\partial\Gamma_j^k}{\partial y^l}-\Gamma_j^l\frac{\partial\Gamma_i^k}{\partial y^l},$$

$i,j,k,l\in\{1,\ldots,n\}$.

As the functions G^k are homogeneous of degree 2, the coefficients $\Gamma^k_{ij} = \frac{\partial^2 G^k}{\partial y^i \partial y^j}$ do not depend on y^i, $i \in \{1, \ldots, n\}$. We then have $R^k_{ij} = y^l R^k_{l,ij}(x)$, the $R^k_{l,ij}(x)$ depend only on the coordinates of the manifold M.

3. Properties of curvature R

Proposition 3.1 ([14]). *The horizontal nullity space of the curvature R is involutive. The elements of A_Γ are projectable vector fields.*

Proof. From the expression of the curvature R and taking into account $h^2 = h$, we have

$$R(hX, Y) = v[hX, hY],$$

If $hX \in N_R$, we obtain $v[hX, hY] = 0\ \forall Y \in \chi(TM)$.
Using the Jacobi Identity, for all hX and $hY \in N_R$, we find $v[[hX, hY], hZ] = 0\ \forall Z \in \chi(TM)$. As we have $h[hX, hY] = [hX, hY]$, the horizontal nullity space of the curvature R is involutive.
We notice that $A_\Gamma = A_h = A_v$.
For $X \in A_h$, we obtain

$$[X, hY] = h[X, Y]\ \forall Y \in \chi(TM).$$

If Y is a vertical vector field, we have $h[X, Y] = 0$. This means that X is a projectable vector field.

Proposition 3.2 ([1]). *Let X be a projectable vector field. The two following relations are equivalent*

i) $[hX, J] = 0$
ii) $[JX, h] = 0$

Proof. See proposition 3 of [1].

Proposition 3.3 ([3]). *We assume that hN_R is generated as a module by projectable vector fields. If the rank of the nullity space hN_R of the curvature R is constant, there exists a local basis of hN_R satisfying Proposition 3.2.*

Proof. See proposition 4 of [3].

4. Riemannian manifolds

Given a function E from $\mathcal{T}M = TM - \{0\}$ in $\mathbb{R}^+$, with $E(0) = 0$, $\mathcal{C}^\infty$ on $\mathcal{T}M$, $\mathcal{C}^2$ on the null section, homogeneous of degree two, such that $dd_J E$ has a maximal rank. The function E defines a Riemannian manifold on M. The map E is called an energy function, its fundamental form $\Omega = dd_J E$ defines a spray S by $i_S dd_J E = -dE$ [10], the derivation i_S being the inner product with respect to S. The vector 1−form $\Gamma = [J, S]$ is called the canonical connection [9]. The fundamental form Ω defines a metric g on the vertical bundle by $g(JX, JY) = \Omega(JX, Y)$, for all X, $Y \in \chi(TM)$. There is [9], one and only one metric lift D of the canonical connection such that:

$$J\mathbb{T}(hX, hY) = 0,\ \mathbb{T}(JX, JY) = 0\ (\mathbb{T}(X, Y) = D_X Y - D_Y X - [X, Y]);$$
$$DJ = 0;\ DC = v; D\Gamma = 0;\ Dg = 0.$$

The linear connection D is called Cartan connection. We have

$$D_{JX}JY = [J, JY]X,\ \ D_{hX}JY = [h, JY]X.$$

From the linear connection D, we associate a curvature

$$\mathcal{R}(X,Y)Z = D_{hX}D_{hY}JZ - D_{hY}D_{hX}JZ - D_{[hX,hY]}JZ \tag{1}$$

for all X, Y, $Z \in \chi(TM)$. The relationship between the curvature $\mathcal{R}$ and R is

$$\mathcal{R}(X,Y)Z = J[Z, R(X,Y)] - [JZ, R(X,Y)] + R([JZ,X],Y) + R(X,[JZ,Y]).$$

for all X, Y, $Z \in \chi(TM)$. In particular,

$$\mathcal{R}(X,Y)S = -R(X,Y).$$

In natural local coordinates on an open set U of M, $(x^i, y^j) \in TU$, the energy function is written

$$E = \frac{1}{2}g_{ij}(x^1,\dots,x^n)y^iy^j,$$

where $g_{ij}(x^1,\dots,x^n)$ are symmetric positive functions such that the matrix $(g_{ij}(x^1,\dots,x^n))$ is invertible. And the relation $i_S dd_J E = -dE$ gives the spray S

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i(x^1,\dots,x^n,y^1,\dots,y^n)\frac{\partial}{\partial y^i},$$

with $G_k = \frac{1}{2}y^iy^j\gamma_{ikj}$,
where $\gamma_{ikj} = \frac{1}{2}(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})$ and $\gamma^k_{ij} = g^{kl}\gamma_{ilj}$.
We have $G^k = \frac{1}{2}y^iy^j\gamma^k_{ij}$.

Proposition 4.1. *Let E be an energy function, Γ a connection such that $\Gamma = [J,S]$. The following two relationship are equivalent:*

i) $i_S dd_J E = -dE$*;*
ii) $d_h E = 0$.

Proof. See proposition 1 [3].

Proposition 4.2. *For a connection satisfying the Proposition 4.1, the scalar $1-$form d_vE is completely integrable.*

Proof. The Kernel of d_vE is formed by vector fields belonging to the horizontal space Imh ($v \circ h = 0$) and vertical vector fields JY such that $L_{JY}E = 0$, $Y \in Imh$, taking into account $vJ = J$.
As we have

$$[hX, hY] = h[hX, hY] + v[hX, hY] = h[hX, hY] + R(X,Y),$$

for all X, $Y \in \chi(TM)$, and that $d_hE = 0$ implies $d_RE = 0$. We obtain

$$[hX, hY] \in Kerd_vE.$$

Its remains to show that $L_{v[hX,JY]}E = 0\ \forall X \in Imh$ and, $Y \in Imh$ satisfying $L_{JY}E = 0$. This is immediate since we have $v = I - h$.

Proposition 4.3. *On a Riemannian manifold (M,E), the horizontal nullity space hN_R of the curvature R is generated as a module by projectable vector fields belonging to hN_R and, orthogonal to the image space ImR of the curvature R and $hN_R = hN_{\mathcal{R}}$.*

Proof. If $R^\circ = i_S R$ is zero, then the curvature R is zero; in this case, the horizontal space Imh is the horizontal nullity space of R, isomorphic to $\chi(U)$, U being a open set of M [5].
In what follows, we assume that $R^\circ \neq 0$. According to relation (4.2) of [3], $JX \perp ImR \iff \mathcal{R}(S,X)Y = 0\ \forall Y \in \chi(TM)$. We obtain $R(X,Y) = R^\circ([JY,X])\ \forall Y \in \chi(TM)$. As R is a semi-basic vector $2-$form, the above relation is only possible if $X = S$ or if $X \in hN_R$, then X is generated as a module by projectable vector fields belonging to hN_R. We get $hN_R = hN_{\mathcal{R}}$.

Theorem 4.1. *Let $\Gamma = [J,S]$ be a linear connection. The connection Γ comes from a energy function if and only if*

(1) *there is an energy function E_0 such that $d_R E_0 = 0$;*
(2) *the scalar $1-$form $d_v E_0$ is completely integrable.*

Then, there exist a constant $\varphi(x)$ on the bundle such that $e^{\varphi(x)}E_0$ is the energy function of Γ.

Proof. Both conditions are necessary according to the Proposition 4.1 and 4.2.
Conversely, let E_0 be an energy function such that $d_R E_0$. We will show that, there exist a constant φ function on the bundle such that $d_h(e^\varphi E_0) = 0$.
The equation is equivalent to

$$d\varphi = -\frac{1}{E_0} d_h E_0.$$

The condition of integrability of such an equation is

$$d(\frac{1}{E_0}) \wedge d_h E_0 + \frac{1}{E_0} dd_h E_0 = 0,$$

namely

$$dd_h E_0 = \frac{dE_0}{E_0} \wedge d_h E_0.$$

As $d_v E_0$ is completely integrable, we have, according to Frobenius theorem,

$$dd_v E_0 \wedge d_v E_0 = 0$$

Applying the inner product i_C to the above equality, we get

$$dd_v E_0 = \frac{dE_0}{E_0} \wedge d_v E_0,$$

that is to say

$$dd_h E_0 = \frac{dE_0}{E_0} \wedge d_h E_0.$$

This is the condition of integrability sought.
For more information see [3].

5. Lie algebra defined by spray

Let $A_S = \{X \in \chi(TM) \text{ such that } [X,S] = 0\}$. By developing the calculation $[X,S] = 0$, we note that the projectable elements of A_S are, on an open set U of M, of the form:

$$X = X^i(x)\frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i}.$$

Denoting $\overline{\chi(M)}$ the complete lift of the vector fields $\chi(M)$ on TM, the projectable elements of A_S are in $A_S \cap \overline{\chi(M)}$. The geodesic spray of a linear connection is defined locally by

$$\ddot{x}^i = -\Gamma^i_{jk}\dot{x}^j\dot{x}^k.$$

A result from [12] shows that the dimension of the Lie algebra $\overline{A_S}$ is at most equal to $n^2 + n$. If the dimension $\overline{A_S}$ equal to $n^2 + n$, then (M, S) is isomorphic to $(\mathbb{R}^n, Z_\lambda)$ for a unique $\lambda \in \mathbb{R}$, Z_λ is given by the equations $\ddot{x}^i = \lambda\dot{x}^i$, $i = 1, \ldots, n$. This condition is equivalent to the nullity of the curvature R of Γ cf.[5]. We can see this property on example 5 of [4]. In the following, we are interested in the nature of the algebra $\overline{A_S}$. By associating the equality $[\overline{X}, S] = 0$ with the tangent structure J using the Jacobi identity [8], we can write

$$[[\overline{X}, S], J] + [[S, J], \overline{X}] + [[J, \overline{X}], S] = 0.$$

Taking into account the hypothesis $[\overline{X}, S] = 0$ and a result of [11]: $[J, \overline{X}] = 0$, we find

$$[\overline{X}, \Gamma] = 0 \text{ with } \Gamma = [J, S].$$

We notice that $[C, J] = -J$ and $[C, \overline{X}] = 0$, we then take $\Gamma = [J, S]$ with $[C, S] = S$.
The $1-$vector form Γ is a linear connection without torsion in the sense of [9].

Proposition 5.1 ([1])**.** *The Lie algebra $\overline{A_S}$ coincides with $\overline{A_\Gamma} = A_\Gamma \cap \overline{\chi(M)}$.*

Proof. See proposition 9 of [1].

Proposition 5.2. [14] *Let H° denote the set of projectable horizontal vector fields and $A_\Gamma \cap H^\circ = A^h_\Gamma$, then we have $A^h_\Gamma = N_R \cap H^\circ$ and A^h_Γ is an ideal of A_Γ.*

Proof. The curvature R is written, for all $X, Y \in \chi(TM)$

$$R(X, Y) = v[hX, hY].$$

If $hX \in A^h_\Gamma$, we have $R(X, Y) = v \circ h[hX, Y] = 0, \forall Y \in \chi(TM)$. That means $X \in N_R$.
The curvature R is written, for all $X, Y \in \chi(TM)$

$$R(X, Y) = [hX, hY] + h^2[X, Y] - h[hX, Y] - h[X, hY].$$

If $X \in N_R \cap H^\circ$, given $hX = X$ and $R(X, Y) = 0$ for all $Y \in \chi(TM)$, we find $[X, hY] = h[X, hY]$.
If Y is a vertical vector field, the above equality still holds, because it is zero.
For the ideal A^h_Γ, it is immediate from the expression of A_Γ.

Proposition 5.3. *Let $\overline{A_\Gamma}^h = A^h_\Gamma \cap \overline{A_\Gamma}$, the set of the horizontal vector fields $\overline{A_\Gamma}^h$ form a commutative ideal of $\overline{A_\Gamma}$. The dimension of $\overline{A_\Gamma}^h$ corresponds to the dimension of A^h_Γ if the rank of A^h_Γ is constant.*

Proof. By Proposition5.2, A_Γ^h is an ideal of A_Γ, so $\overline{A_\Gamma}^h = A_\Gamma^h \cap \overline{\chi(M)}$ is an ideal of $\overline{A_\Gamma} = A_\Gamma \cap \overline{\chi(M)}$, moreover $v[\overline{X},\overline{Y}] = 0$, for all $\overline{X},\overline{Y} \in \overline{A_\Gamma}^h$. Propositions 3.2 and 2 of [1] give $J[\overline{X},\overline{Y}] = 0$, for all $\overline{X},\overline{Y} \in \overline{A_\Gamma}^h$, noting that $[J,\Gamma] = 0$. The horizontal and vertical parts of $[\overline{X},\overline{Y}]$ are therefore zero, that is, $[\overline{X},\overline{Y}] = 0$.
The existence of such an element of $\overline{A_\Gamma}^h$ is given by the proposition3.3.

6. Case of constant values of $\overline{A_\Gamma}$

If we expand the equation $[\overline{X}, S] = 0$ with $S = [C, S]$, we get

$$X^l \frac{\partial \Gamma_{ij}^k}{\partial x^l} + \frac{\partial X^l}{\partial x^j}\Gamma_{il}^k + \frac{\partial X^l}{\partial x^i}\Gamma_{lj}^k + \frac{\partial^2 X^k}{\partial x^i \partial x^j} - \frac{\partial X^k}{\partial x^l}\Gamma_{ij}^l = 0.$$

we note that the constant values of $\overline{A_\Gamma}$ verify

$$X^l \frac{\partial \Gamma_{ij}^k}{\partial x^l} = 0 \tag{2}$$

Proposition 6.1. *Le Γ be a linear connection without torsion. If the constant vector fields of $\overline{A_\Gamma}$ form a commutative ideal of $\overline{A_\Gamma}$, they are at most the constant elements of an ideal I of affine vector fields containing these constants such that for all $\overline{X} \in \overline{A_\Gamma}$, $\overline{X}$ is written $\overline{X} = \overline{X_1} + \overline{X_2}$ with $\overline{X_2} \in I$ and that $[\overline{X_1}, \overline{X_2}] = 0$, the derived ideal of $\overline{A_\Gamma}$ never coincides with $\overline{A_\Gamma}$.*

Proof.

1st case:: The functions G^k do not depend on some coordinates in an open set U of M. To simplify, quite to change the numbering order of the coordinates, the spray S is such that $\frac{\partial G^k}{\partial x^{p+1}} = 0, \ldots, \frac{\partial G^k}{\partial x^n} = 0$, $k \in \{1, \ldots, n\}$ and $1 \le p \le n-1$. Then, we have $\frac{\partial}{\partial x^{p+1}}, \ldots, \frac{\partial}{\partial x^n} \in \overline{A_\Gamma}(U)$.
For any $\overline{X} \in \overline{A_\Gamma}(U)$, we can write

$$\begin{aligned} \overline{X} &= X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i} \\ &= \sum_{l=1}^{p}(X^l \frac{\partial}{\partial x^l} + y^j \frac{\partial X^l}{\partial x^j}\frac{\partial}{\partial y^l}) + \sum_{r=p+1}^{n}(X^r \frac{\partial}{\partial x^r} + y^j \frac{\partial X^r}{\partial x^j}\frac{\partial}{\partial y^r}),\ 1 \le j \le n. \end{aligned}$$

For the Lie sub-algebra generated by $\{\frac{\partial}{\partial x^{p+1}}, \ldots, \frac{\partial}{\partial x^n}\}$ form an ideal of $\overline{A_\Gamma}(U)$, we must have $[\frac{\partial}{\partial x^h}, \overline{X}]$ belong to this ideal for all h, $p+1 \le h \le n$.
That implies $\frac{\partial X^l}{\partial x^h} = 0$, for all l such that $1 \le l \le p$ and for all h such that $p+1 \le h \le n$.
We have $X^r = a_s^r x^s + b^r$, $p+1 \le r, s \le n$; $a_s^r, b^r \in \mathbb{R}$.
Denoting

$$\begin{aligned} \overline{X_1} &= \sum_{l=1}^{p}(X^l \frac{\partial}{\partial x^l} + y^j \frac{\partial X^l}{\partial x^j}\frac{\partial}{\partial y^l}),\ 1 \le j \le n \\ \overline{X_2} &= \sum_{r=p+1}^{n}(a_s^r x^s + b^r)\frac{\partial}{\partial x^r} + a_s^r y^s \frac{\partial}{\partial y^r},\ p+1 \le s \le n. \end{aligned}$$

An element $\overline{X} \in \overline{A_\Gamma}$, $\overline{X}$ is written $\overline{X} = \overline{X_1} + \overline{X_2}$ with $[\overline{X_1}, \overline{X_2}] = 0$.

2nd case:: The elements of $\overline{A_\Gamma}$ are of the form $a^l \frac{\partial}{\partial x^l}$, $l \in \{1, \ldots, p\}$. The decomposition of the elements of $\overline{A_\Gamma}$ amounts to the same way. In any case, the derived ideal of $\overline{A_\Gamma}$ never coincides with $\overline{A_\Gamma}$.

Theorem 6.1. *The Lie algebra $\overline{A_\Gamma}$ is semi-simple if and only if the horizontal and projectable vector fields of the nullity space of the curvature R is zero and the derived ideal of $\overline{A_\Gamma}$ coincides with $\overline{A_\Gamma}$.*

Proof. If the Lie algebra $\overline{A_\Gamma}$ is semi-simple, any commutative ideal of $\overline{A_\Gamma}$ reduces to zero by definition. According to the proposition 5.3, the horizontal and projectable vector fields of the nullity space of the curvature R of Γ is zero. The derived ideal of $\overline{A_\Gamma}$ coincides with $\overline{A_\Gamma}$ by a classical result.
Conversely, if $\overline{X} \in \overline{A_\Gamma}$, we have $[\overline{X}, h] = 0$. According to the Jacobi Identity cf.[8] $[\overline{X}, [h, h]] = 0$, ie. $[\overline{X}, R] = 0$. We can write $[\overline{X}, R(Y, Z)] = R([\overline{X}, Y], Z) + R(Y, [\overline{X}, Z])$, for all $Y, Z \in \chi(TM)$. If $\overline{X}$ and $\overline{Y}$ are elements of a commutative ideal of $\overline{A_\Gamma}$, we find

$$[\overline{X}, R(\overline{Y}, Z)] = R(\overline{Y}, [\overline{X}, Z]), \ \forall Z \in \chi(TM). \tag{3}$$

If the horizontal and projectable vector fields of the nullity space of the curvature R is zero, the semi-basic vector 2−form R is non-degenerate on $\overline{\chi(M)} \times \chi(TM)$. The only possible case for the equations (3) is that the commutative ideal of $\overline{A_\Gamma}$ is at most formed by constant vector fields of $\overline{A_\Gamma}$, according to the proposition 6.1, the derived ideal of $\overline{A_\Gamma}$ never coincides with $\overline{A_\Gamma}$ if this ideal formed by constant vector fields is not zero.

Example 6.1. *We take $M = \mathbb{R}^3$, a spray S:*

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - 2(e^{x^3}(y^1)^2 + y^2 y^3) \frac{\partial}{\partial y^1}.$$

and the linear connection $\Gamma = [J, S]$. The non-zero coefficients of Γ are

$$\Gamma^1_1 = 2e^{x^3} y^1, \ \Gamma^1_2 = y^3, \ \Gamma^1_3 = y^2.$$

A base of the horizontal space of Γ is written

$$\frac{\partial}{\partial x^1} - 2e^{x^3} y^1 \frac{\partial}{\partial y^1},$$
$$\frac{\partial}{\partial x^2} - y^3 \frac{\partial}{\partial y^1},$$
$$\frac{\partial}{\partial x^3} - y^2 \frac{\partial}{\partial y^1}.$$

The horizontal nullity space of the curvature is generated as a module by

$$(y^1 - y^2) \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - y^1 y^3 \frac{\partial}{\partial y^1}.$$

The horizontal nullity space is not generated as a module by projectable vector fields in hN_R. This linear connection according to the proposition 4.3 cannot come from an energy function.

The Lie algebra $\overline{A_\Gamma}$ is generated as Lie algebra by:

$$g_1 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} + y^1\frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^2},\ g_2 = \frac{\partial}{\partial x^1},\ g_3 = \frac{\partial}{\partial x^2}.$$

The Lie algebra $\overline{A_\Gamma}$ is that of affine vector fieds containing the commutative ideal $\{g_2, g_3\}$.

7. Lie algebras of infinitesimal isometries

Definition 7.1. *A vector field X on a Riemannian manifold (M, E) is called infinitesimal automorphism of the symplectic form Ω if $L_X\Omega = 0$.*
The set of infinitesimal automorphisms of Ω forms a Lie algebra. We denote this Lie algebra by A_g, in general of infinite dimension.

Theorem 7.1. *We denote $\overline{A_g} = A_g \cap \overline{\chi(M)}$. The Lie algebra $\overline{A_g}$ is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of Γ is zero and, the derived ideal of $\overline{A_g}$ coincides with $\overline{A_g}$.*

Proof. This is the application of proposition 4.3 and theorem 6.1.
For more information, see [2] and [4].

Example 7.1. *We take $M = \mathbb{R}^4$ and the energy function is written:*

$$E = \frac{1}{2}(e^{x^3}(y^1)^2 + (y^2)^2 + e^{x^1}(y^3)^2 + e^{x^2}(y^4)^2).$$

The non-zero linear connection coefficients are

$$\Gamma^1_1 = \frac{y^3}{2},\ \Gamma^1_3 = -\frac{y^3e^{x^1-x^3} - y^1}{2},\ \Gamma^2_4 = -\frac{y^4e^{x^2}}{2},$$
$$\Gamma^3_1 = -\frac{y^1e^{x^3-x^1} - y^3}{2},\ \Gamma^3_3 = \frac{y^1}{2},\ \Gamma^4_2 = \frac{y^4}{2},\ \Gamma^4_4 = \frac{y^2}{2}.$$

The horizontal nullity space of the curvature is zero.
The Lie algebra $\overline{A_\Gamma}$ is generated as Lie algebra by:

$$\begin{aligned} g_1 &= x^4\frac{\partial}{\partial x^2} - (-e^{-x^2} + \frac{(x^4)^2}{4})\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial y^2} - (\frac{x^4y^4}{2} + y^2e^{-x^2})\frac{\partial}{\partial y^4}, \\ g_2 &= -2\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial y^4},\ g_3 = \frac{\partial}{\partial x^4},\ g_4 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}. \end{aligned}$$

We see that g_4 is the center of $\overline{A_\Gamma}$ corresponding to the second case of the proposition 6.1, while the Lie algebra $\overline{A_g}$ is generated as a Lie algebra by g_1, g_2, g_3. The Lie algebra $\overline{A_g}$ is simple and isomorphic to $sl(2)$.

8. Finite dimensional Lie algebra

In this section, we consider only a finite-dimensional Lie algebra over the field $\mathbb{K}$ of zero characteristic class. The notions and notations are those of [7].

Theorem 8.1. *The Lie algebra $\mathfrak{g}$ is semi-simple if and only if the adjoint representation of $\mathfrak{g}$ is semi-simple and the derived ideal of $\mathfrak{g}$ coincides with $\mathfrak{g}$.*

Proof. The condition is necessary, see Lemma 1 p.72 [7].
Conversely, if the adjoint representation of $\mathfrak{g}$ is semi-simple, by definition, the Lie algebra is reductive. By Proposition 5 b) p.78 [7], the derived ideal of $\mathfrak{g}$ is semi-simple. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ is semi-simple.

Theorem 8.2. *The Lie algebra $\mathfrak{g}$ is semi-simple if and only if the adjoint representation of $\mathfrak{g}$ is semi-simple and the center of $\mathfrak{g}$ is reduced to $\{0\}$.*

Proof. This is a consequence of proposition 5 g) p.78 [7].

Theorem 8.3. *The Lie algebra $\mathfrak{g}$ is semi-simple if and only if the derived ideal coincides with $\mathfrak{g}$, any derivation is inner and the radical of $\mathfrak{g}$ is a commutative ideal.*

Proof. The necessary conditions are well known.
Conversely, let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ which is a commutative ideal by hypothesis. Let $e_1, \ldots, e_p, e_{p+1}, \ldots, e_n$ be a base of $\mathfrak{g}$ such that $e_1, \ldots, e_p$ $(p < n)$ belong to $\mathfrak{r}$ and $e_{p+1}, \ldots, e_n$ a basis of a Levi subalgebra of $\mathfrak{g}$. By defining a linear map D such that $D(e_i) = e_i$, $1 \leq i \leq p$, and $D(e_{p+1}) = 0, \ldots, D(e_n) = 0$, the map D is a derivation of $\mathfrak{g}$, its trace function is equal to p. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, the adjoint representation of $\mathfrak{g}$ belongs to $sl(\mathfrak{g})$ which is semi-simple [7] p.71. Its trace function is zero. We end up with a contradiction if $\mathfrak{r} \neq 0$ and if the derivation D is inner.

Remark 8.1. *We can see such reasoning in an example* [4] *§5.*

Remark 8.2. *For Lie algebras of countable dimension, see our results in* [13].

Remark 8.3. *The radical of the Lie algebra of the example* [6] *is not commutative.*

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