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Parametrization of algebraic points on the hyperelliptic curve of affine equation $y^2 = x(x-2)(x-3)(x-6)(x-9)$

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Abstract. We describe the families of algebraic points of degree at most $\ell \geq 5$ on the hyperelliptic curve $\mathcal{C}$ with equation:

$$y^2 = x(x-2)(x-3)(x-6)(x-9).$$

First, we give a $\mathbb{Q}$-base arising from linear systems and an explicit expression for the Mordell-Weil group of rational points of the Jacobian. Then, we use one of the fundamental Abel-Jacobi theorems to describe a principal divisor of the rational and central function of this work. Finally, following the different cases associated to the integer α_k, we can exhibit families of points.

1. Introduction

Let $\mathcal{C}$ be a smooth projective plane curve defined over $\mathbb{Q}$. For all algebraic extension field $\mathbb{K}$ of $\mathbb{Q}$, we denote by $\mathcal{C}(\mathbb{K})$ the set of $\mathbb{K}$-rational points of $\mathcal{C}$ on $\mathbb{K}$ and by $\mathcal{C}^{(\ell)}(\mathbb{Q})$ the set of algebraic points of degree d over $\mathbb{Q}$ $i.e$ $\mathcal{C}^{(\ell)}(\mathbb{Q}) = \bigcup_{[\mathbb{Q}(R):\mathbb{Q}]\leq \ell} \mathcal{C}(\mathbb{K})$. The degree of an algebraic point R is the degree of its field of definition on $\mathbb{Q}$ $i.e$ $\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$. It's well known that the determintion of $\mathcal{C}(\mathbb{K})$ is a difficult problem in number theory beacause there is still no general algorithm to compute $\mathcal{C}(\mathbb{K})$. In the case $g \geq 2$, the theorem of Fatlings proves that $\mathcal{C}(\mathbb{K})$ is finite but this proof is not effective [7]. If the genus $g \geq 2$ and by the well known theorem of Mordell-Weil [2, 10, 13] for any number field $\mathbb{K}$ and any curve $\mathcal{C}$ the groupe of its $\mathbb{K}$-rational points $\mathcal{C}(\mathbb{K})$ is finitely generated. In other $\mathcal{J}(\mathbb{K}) \cong \mathcal{J}(\mathbb{K})_{tor} \times \mathbb{Z}^r$ where $\mathcal{J}(\mathbb{K})_{tor}$ is a finite torsion subgroups and r is a positive integer called the rank of $\mathcal{J}(\mathbb{K})$. If the rank is null, then we have equality $\mathcal{J}(\mathbb{K}) = \mathcal{J}(\mathbb{K})_{tor}$; in this case we can use the theorem of Riemann-Roch to determine the basis of the associate linear systems to the curve. Every linear system is a vectoriel on $\mathbb{K}$ of finite dimension. By using the Abel-Jacobi theorem [1] we can give a parametrisation of algebric points of given degree of $\mathcal{C}$ over $\mathbb{Q}$.

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In this paper, we consider the curve $\mathcal{C}$ of affine equation

$$y^2 = x(x-2)(x-3)(x-6)(x-9),$$

this curve $\mathcal{C}$ is a hyperelliptic curve of genus $g = 2$ (see [3, 9]) and the Mordell-Weil groupe $\mathcal{J}(\mathbb{Q})$ of $\mathcal{C}$ is finite, so we can give a parametrisation of $\mathcal{C}^{(\ell)}(\mathbb{Q})$. We begin by presenting the essential results, then state the main theorem and finally demonstrate it.

2. Auxiliary results

Definition 2.1. *For a divisor $D \in Div(\mathcal{C})$, we define the $\mathbb{Q}$-vector space denoted $\mathcal{L}(D)$ by:*

$$\mathcal{L}(D) \ := \ \{f \in \mathbb{K}(\mathcal{C})\setminus \{0\} \ \mid div(f) \geq -D\} \cup \{0\}.$$

Corollary 2.1. *According to* [11]*, for two divisors D and D' of $div(\mathcal{C})$, we have the following implications:*

$$D \equiv D' \Longrightarrow \mathcal{L}(D) \simeq \mathcal{L}(D') \Longrightarrow \dim \mathcal{L}(D) = \dim \mathcal{L}(D').$$

Lemma 2.1. *According to* [12]*, we have: $\mathcal{J}(\mathbb{Q}) \ \cong \ (\mathbb{Z}/2\mathbb{Z})^4$.*

Let $x = \dfrac{X}{Z}$ and $y = \dfrac{Y}{Z}$ be rational functions defined on $\mathbb{Q}$.

$$y^2 = \prod_{k=0}^{4}(x-\gamma_k), \tag{1}$$

with $\gamma_k \in \{0, 2, 3, 6, 9\}$ recpectively for $k \in \{0, \dots, 4\}$. Let x, y be the affine coordinates and X, Y and Z the projective coordinates. Let's say: $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. The projective equation of the curve is given by:

$$Z^3Y^2 = \prod_{k=0}^{4}(X-\gamma_k Z).$$

We note P_k and ∞ the points of $\mathcal{C}$ defined by: $P_k = [\gamma_k : 0 : 1]$ and $\infty = [1 : 0 : 0]$.

Lemma 2.2. *For curve $\mathcal{C}$, we have the following rationnal divisors:*

- ***i*):** *$div(x-\gamma_k) = 2P_k - 2\infty$ with $P_k = [\gamma_k : 0 : 1]$, $k \in \{0, \dots, 4\}$ and $\gamma_k \in \{0, \ 2, \ 3, \ 6, \ 9\}$.*
- ***ii*):** *$div(y) = \sum_{k=0}^{4} P_k - 5\infty$.*

Proof. We will carry out a calculation of this type:

$$div(t-\omega) = (t = \omega Z)\cdot \mathcal{C} - (Z=0)\cdot \mathcal{C},$$

where $\omega = \begin{cases} x \\ y \end{cases}$ and $\Omega = \begin{cases} X \\ Y \end{cases}$ see [6]

Corollary 2.2. *The following results are the consequences of* Lemma 2.2*:*

- **⊛:** $\sum_{k=0}^{4} j(P_k) = 0,$

⊛: $2j(P_k) = 0$ *where* $k \in \{0, \ldots, 4\}$.

Thus, The $j(P_{k_{k\in\{0,\ldots,4\}}})$ generate the same subgroup $\mathcal{J}(\mathbb{Q})$.

Remark 2.1. *The generator of the torsion group of rational points of the Jacobian* $\mathcal{J}(\mathbb{Q})_{tor}$ *described in* [8] *is given by:*

$$\mathcal{J}(\mathbb{Q})_{tor} \simeq \langle [P_0 - \infty],\ [P_1 - \infty],\ [P_2 - \infty],\ [P_3 - \infty] \rangle.$$

From Lemma 2.1 and Remark 2.1, we derive the following Lemma:

Lemma 2.3. *The mordell-weil group* $\mathcal{J}(\mathbb{Q})$ *of the curve* $\mathcal{C}$ *of affine equation* $y^2 = \prod_{k=0}^{4}(x - \gamma_k)$ *is given by:*

$$\mathcal{J}(\mathbb{Q}) = \left\{ \sum_{k=0}^{3} \alpha_k j(P_k) \;\middle|\; \alpha_k \in \{0,\ 1\} \right\}.$$

Lemma 2.4.

1: *We have the following linear systems:*
- $\mathcal{L}(\infty) = \langle 1 \rangle$,
- $\mathcal{L}(2\infty) = \mathcal{L}(3\infty) = \mathcal{L}(\infty) \oplus \langle x \rangle$,
- $\mathcal{L}(4\infty) = \mathcal{L}(3\infty) \oplus \langle x^2 \rangle$,
- $\mathcal{L}(5\infty) = \mathcal{L}(4\infty) \oplus \langle y \rangle$,
- $\mathcal{L}(6\infty) = \mathcal{L}(5\infty) \oplus \langle x^3 \rangle$,
- $\mathcal{L}(7\infty) = \mathcal{L}(6\infty) \oplus \langle yx \rangle$,
- $\mathcal{L}(8\infty) = \mathcal{L}(7\infty) \oplus \langle x^4 \rangle$,
- $\mathcal{L}(9\infty) = \mathcal{L}(8\infty) \oplus \langle yx^2 \rangle$,
- $\mathcal{L}(10\infty) = \mathcal{L}(9\infty) \oplus \langle x^5 \rangle$,
- $\mathcal{L}(11\infty) = \mathcal{L}(10\infty) \oplus \langle yx^3 \rangle$,

2: *Generaly for* $m \in \mathbb{N}$, *a* $\mathbb{Q}$*-basis of the space* $\mathcal{L}(m\infty)$ *is given by:*

$$\mathcal{B}_m = \left\{ x^i \;\middle|\; i \in \mathbb{N} \text{ and } i \leq \frac{m}{2} \right\} \bigcup \left\{ yx^j \;\middle|\; j \in \mathbb{N} \text{ and } j \leq \frac{m-5}{2} \right\}$$

Proof. see [4, 5, 6].

3. Main result

The main result of this paper is the following theorem:

Theorem 3.1. *The set of algebraic points of degree at most* $\ell \geq 5$ *over* $\mathbb{Q}$ *on the curve* $\mathcal{C}$ *of affine equation* $y^2 = \prod_{k=0}^{4}(x - \gamma_k)$ *is given by*

$$\mathcal{C}^{(\ell)}(\mathbb{Q}) = \left(\bigcup_{\substack{k=0 \\ s \in \{0,1\}}}^{4} \mathcal{K}_{s,k} \right) \bigcup \left(\bigcup_{\substack{\kappa, \vartheta = 0 \\ \kappa \neq \vartheta}}^{3} \mathcal{P}_{\kappa,\vartheta} \right) \bigcup \left(\bigcup_{\sigma=0}^{3} \mathcal{M}_\sigma \right),\ \textit{with:}$$

$$\mathcal{K}_{s,k}=\left\{\left(x,-\frac{\sum_{i=s}^{\frac{\ell+s}{2}} a_i\left(x^i+s\gamma_k^i\right)}{\sum_{j=0}^{\frac{\ell+s-5}{2}} b_jx^j}\right)\;\middle|\;\begin{array}{c}a_0 \textit{ and } b_0 \textit{ non-zero},\\ a_{\frac{\ell+s}{2}}\neq 0 \textit{ if } \ell \textit{ is even},\\ b_{\frac{\ell+s-5}{2}}\neq 0 \textit{ if } \ell \textit{ is odd}\\ \textit{and } x \textit{ is a solution}\\ \textit{of the equation:}\end{array}\right.$$
$$\left.\left(\sum_{i=s}^{\frac{\ell+s}{2}} a_i\left(x^i+s\gamma_k^i\right)\right)^2 x^{-s}=x^{1-s}\left(\sum_{j=0}^{\frac{\ell+s-5}{2}} b_jx^j\right)^2\prod_{k=1}^{4}(x-\gamma_k)\right\}$$

$$\mathcal{P}_{\kappa,\vartheta}=\left\{\left(x,-\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_jx^j}\right)\;\middle|\;\begin{array}{c}a_{\frac{\ell+2}{2}}\neq 0 \textit{ if } \ell \textit{ is even},\\ b_{\frac{\ell-3}{2}}\neq 0 \textit{ if } \ell \textit{ is odd}\\ \textit{and } x \textit{ is a solution}\\ \textit{of the equation:}\end{array}\right.$$
$$\left.\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)\right)^2 x^{-1}=\left(\sum_{j=0}^{\frac{\ell-3}{2}} b_jx^{j-\frac{1}{2}}\right)^2\prod_{k=1}^{4}(x-\gamma_k)\right\}$$

$$\mathcal{M}_{\sigma}=\left\{\left(x,-\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_jx^j}\right)\;\middle|\;\begin{array}{c}a_{\frac{\ell+2}{2}}\neq 0 \textit{ if } \ell \textit{ is even},\\ b_{\frac{\ell-3}{2}}\neq 0 \textit{ if } \ell \textit{ is odd}\\ \textit{and } x \textit{ is a root of}\\ \textit{the equation:}\end{array}\right.$$
$$\left.\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)\right)^2 x^{-1}=\left(\sum_{j=0}^{\frac{\ell-3}{2}} b_jx^{j-\frac{1}{2}}\right)^2\prod_{k=1}^{4}(x-\gamma_k)\right\}$$

Proof. Let $R\in\mathcal{C}(\bar{\mathbb{Q}})$ such that $[\mathbb{Q}(R):\mathbb{Q}]=\ell$ with $\ell\geq 5$ and $R\notin\{P_k,\ \infty\}$ for $k\in\{0,\dots,4\}$. Let's consider R_n with $n\in\{1,\dots,\ell\}$ the Galois conjugates of R and let $\lambda=\left[\sum_{n=1}^{\ell}R_n-\ell\infty\right]\in\mathcal{J}(\mathbb{Q})$. From Lemma 2.3, we have $\lambda=-\sum_{k=0}^{3}\alpha_k j(P_k)$ and hence:

$$\left[\sum_{n=1}^{\ell}R_n-\ell\infty\right]=\left[\sum_{k=0}^{3}\alpha_k\infty-\sum_{k=0}^{3}\alpha_kP_k\right]. \tag{2}$$

From the expression (2), this gives the following expression:

$$\left[\sum_{n=1}^{\ell}R_n+\sum_{k=0}^{3}\alpha_kP_k-\left(\ell+\sum_{k=0}^{3}\alpha_k\right)\infty\right]=0. \tag{3}$$

Equation (3) leads to the existence of a rational function $f(x,y)$ defined on $\mathbb{Q}$, according to the Abel-Jacobi theorem (cf. [1, 14]), such that:

$$div(f) = \sum_{n=1}^{\ell} R_n + \sum_{k=0}^{3} \alpha_k P_k - \left(\ell + \sum_{k=0}^{3} \alpha_k\right)\infty. \tag{4}$$

From expression (4) we deduct that $f \in \mathcal{L}\left(\left(\ell + \sum_{k=0}^{3} \alpha_k\right)\infty\right)$, according to the Lemma 2.4 , $f(x,y)$ can be expressed as follows:

$$f(x,y) = \sum_{i=0}^{\frac{\ell+\sum_{k=0}^{3}\alpha_k}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell+\sum_{k=0}^{3}\alpha_k - 5}{2}} b_j y x^j, \tag{5}$$

with $ord_{P_k} f = \alpha_k$ and $k \in \{0, \dots, 3\}$. Depending on the values taken by the α_k values, we draw up a table of the different cases:

α_0	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
α_1	0	0	1	0	0	1	0	0	1	0	1	1	1	0	1	1
α_2	0	0	0	1	0	0	1	0	0	1	1	1	0	1	1	1
α_3	0	0	0	0	1	0	0	1	1	1	0	0	1	1	1	1

$1^{st}-$ **:** Let's consider the cases where the α_k's are either all zero, only one is non-zero or none is zero.

- If all α_k's are zero, then the function $f(x,y)$ of the expression (5) is written:

$$f(x,y) = \sum_{i=0}^{\frac{\ell}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-5}{2}} b_j y x^j, \tag{6}$$

with $a_i, b_j \in \mathbb{Q}$, a_0 and b_0 not simultaneously null (otherwise of the R_n's should be egal to P_0, which would be absurd), $a_{\frac{\ell}{2}} \neq 0$ and $b_{\frac{\ell-5}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be egal to ∞, which would be absurd).

- If only one of the α_k's is zero, then the function $f(x,y)$ of the expression (5) is written:

$$f(x,y) = \sum_{i=0}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_i y x^j, \tag{7}$$

and since $ord_{P_k} f = 1$, which implies that $a_0 = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \gamma_k^i$, so equation (7) becomes:

$$f(x,y) = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \left(x^i + \gamma_k^i\right) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{8}$$

with $a_i, b_j \in \mathbb{Q}$, $b_0 \neq 0$ (otherwise of the R_n's should be at P_k, which would be absurd), $a_{\frac{\ell+1}{2}} \neq 0$ and $b_{\frac{\ell-4}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd).

- If none of the α_k's is zero, from Corollary 2.2, the rational divisor of $f(x,y)$ is expressed as follows:

$$div(f) = \sum_{n=1}^{\ell} R_n + P_4 - (\ell+1)\infty. \tag{9}$$

From expression (9), we deduct that $f \in \mathcal{L}\left((\ell+1)\infty\right)$, according to the Lemma 2.4 , $f(x,y)$ can be expressed as follows:

$$f(x,y) = \sum_{i=0}^{\frac{\ell+1}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-4}{2}} b_j y x^j, \tag{10}$$

and since $ord_{P_4} f = 1$, which implies that $a_0 = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \gamma_4^i$, so the expreesion (10) becomes:

$$f(x,y) = \sum_{i=1}^{\frac{\ell+1}{2}} a_i \left(x^i + \gamma_4^i\right) + \sum_{j=0}^{\frac{\ell-4}{2}} b_j y x^j, \tag{11}$$

with $a_i, b_j \in \mathbb{Q}$, $b_0 \neq 0$ (otherwise of the R_n's should be at P_4, which would be absurd), $a_{\frac{\ell+1}{2}} \neq 0$ and $b_{\frac{\ell-4}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd).

Thus, from equations (6), (8) and (11), for α_k's are either all zero, only one is non-zero or none is zero, there is exit $s \in \{0,1\}$ and $k \in \{0,\ldots,4\}$ such that:

$$f(x,y) = \sum_{i=s}^{\frac{\ell+s}{2}} a_i \left(x^i + s\gamma_k^i\right) + \sum_{j=0}^{\frac{\ell+s-5}{2}} b_j y x^j. \tag{12}$$

At points R_n, the function $f(x,y)$ of (12) gives $f(x,y) = 0$, resulting in an expression for y as a function of x of the form $y = -\frac{\sum_{i=s}^{\frac{\ell+s}{2}} a_i \left(x^i + s\gamma_k^i\right)}{\sum_{j=0}^{\frac{\ell+s-5}{2}} b_j x^j}$. By replacing the y in the expression of the equation (1), we obtain:

$$\left(\sum_{i=s}^{\frac{\ell+s}{2}} a_i \left(x^i + s\gamma_k^i\right)\right)^2 = x \left(\sum_{j=0}^{\frac{\ell+s-5}{2}} b_j x^j\right)^2 \prod_{k=1}^{4} (x - \gamma_k). \tag{13}$$

From the equation (13), becomes the following equation:

$$\left(\sum_{i=s}^{\frac{\ell+s}{2}} a_i\left(x^i+s\gamma_k^i\right)\right)^2 x^{-s}=x^{1-s}\left(\sum_{j=0}^{\frac{\ell+s-5}{2}} b_j x^j\right)^2 \prod_{k=1}^{4}(x-\gamma_k). \tag{14}$$

The equation (14) is degree ℓ. In fact, the first member is degree $2\left(\frac{\ell+s}{2}\right)-s=\ell$ and the second one is degree $2\left(\frac{\ell+s-5}{2}\right)-s+5=\ell$. This gives a first family points of degree ℓ:

$$\mathcal{K}_{s,k}=\left\{\left(x,-\frac{\sum_{i=s}^{\frac{\ell+s}{2}} a_i\left(x^i+s\gamma_k^i\right)}{\sum_{j=0}^{\frac{\ell+s-5}{2}} b_j x^j}\right)\ \middle|\ \begin{array}{c} a_0 \text{ and } b_0 \text{ non-zero,}\\ a_{\frac{\ell+s}{2}}\neq 0 \text{ if } \ell \text{ is even,}\\ b_{\frac{\ell+s-5}{2}}\neq 0 \text{ if } \ell \text{ is odd}\\ \text{and } x \text{ is a solution}\\ \text{of the equation:}\\ \left(\sum_{i=s}^{\frac{\ell+s}{2}} a_i\left(x^i+s\gamma_k^i\right)\right)^2 x^{-s}=x^{1-s}\left(\sum_{j=0}^{\frac{\ell+s-5}{2}} b_j x^j\right)^2 \prod_{k=1}^{4}(x-\gamma_k)\end{array}\right\}$$

2^{nd}– : Let's consider only the cases where two of the α_k's are zero, then the function $f(x,y)$ of the expression (5) is written:

$$f(x,y)=\sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i+\sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{15}$$

and since $ord_{P_\kappa} f = ord_{P_\vartheta} f = 1$ with $\kappa,\ \vartheta \in \{0,\ldots,3\}$ and $\kappa\neq\vartheta$, which implies that $a_0=\sum_{i=1}^{\frac{\ell+2}{2}} a_i\psi_{\kappa,\vartheta}^i$ with $\psi_{\kappa,\vartheta}^i=-\frac{1}{2}\left(\gamma_\kappa^i+\gamma_\vartheta^i\right)$, hence the equation (15) becomes:

$$f(x,y)=\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)+\sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{16}$$

with $a_i, b_j\in\mathbb{Q}$, $a_{\frac{\ell+2}{2}}\neq 0$ and $b_{\frac{\ell-3}{2}}\neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd). At points R_n, the function $f(x,y)$ of (16) gives $f(x,y)=0$, resulting in an expression for y as a function of x of the form $y=-\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}$. By replacing the expression for y in the expression of the equation (1), we obtain:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_k^i\right)\right)^2 = x\left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j\right)^2 \prod_{k=1}^{4}(x-\gamma_k). \tag{17}$$

From the equation (17), becomes the following equation:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)\right)^2 x^{-1} = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 \prod_{k=1}^{4}(x-\gamma_k). \tag{18}$$

The equation (18) is of degree ℓ. Indeed, the first member is degree $2\left(\frac{\ell+2}{2}\right)-1=\ell$ and the second one is degree $2\left(\frac{\ell-3}{2}-\frac{1}{2}\right)+4=\ell$
This gives a second family points of degree ℓ:

$$\mathcal{P}_{\kappa,\vartheta} = \left\{ \left(x, -\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j} \right) \middle| \begin{array}{c} a_{\frac{\ell+2}{2}} \neq 0 \text{ if } \ell \text{ is even,} \\ b_{\frac{\ell-3}{2}} \neq 0 \text{ if } \ell \text{ is odd} \\ \text{and } x \text{ is a solution} \\ \text{of the equation:} \end{array} \right. \\ \left. \left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\psi_{\kappa,\vartheta}^i\right)\right)^2 x^{-1} = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 \prod_{k=1}^{4}(x-\gamma_k) \right\}$$

3rd– : If only one of the α_k's is zero, from Corollary 2.2, the rational divisor of $f(x,y)$ is expressed as follows:

$$div(f) = \sum_{n=1}^{\ell} R_n + P_\sigma + P_4 - (\ell+2)\infty, \tag{19}$$

where $\sigma \in \{0,\ldots,3\}$, from expression (19), we deduct that $f \in \mathcal{L}\left((\ell+1)\infty\right)$, according to the Lemma 2.4 , $f(x,y)$ can be expressed as follows:

$$f(x,y) = \sum_{i=0}^{\frac{\ell+2}{2}} a_i x^i + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{20}$$

and since $ord_{P_\sigma} f = ord_{P_4} f = 1$, which implies that $a_0 = \sum_{i=1}^{\frac{\ell+2}{2}} a_i \zeta_\sigma^i$ with $\zeta_\sigma^i = -\frac{1}{2}\sum_{\mu=3}^{4}\left(\gamma_\sigma^i + 9^i\right)$, hence the equation (20) becomes

$$f(x,y) = \sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right) + \sum_{j=0}^{\frac{\ell-3}{2}} b_j y x^j, \tag{21}$$

with $a_i, b_j \in \mathbb{Q}$, $a_{\frac{\ell+2}{2}} \neq 0$ and $b_{\frac{\ell-3}{2}} \neq 0$ depending on whether ℓ is even or odd (otherwise one of the R_n's should be at ∞, which would be absurd). At points R_n, the function $f(x,y)$ of (21) gives $f(x,y)=0$, resulting in an expression for y as a function of x of the form $y=-\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}$. By replacing the y in the expression of the equation (1), we obtain:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)\right)^2 = x\left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j\right)^2 \prod_{k=1}^{4}(x-\gamma_k). \tag{22}$$

From the equation (22), becomes the following equation:

$$\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)\right)^2 x^{-1} = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 \prod_{k=1}^{4}(x-\gamma_k). \tag{23}$$

The equation (23) is degree ℓ. Indeed, the first member is degree $2\left(\frac{\ell+2}{2}\right)-1=\ell$ and the second one is degree $2\left(\frac{\ell-3}{2}-\frac{1}{2}\right)+4=\ell$. This gives a third family of points of degree ℓ:

$$\mathcal{M}_\sigma=\left\{\left(x,-\frac{\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)}{\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^j}\right) \;\middle|\; \begin{array}{c} a_{\frac{\ell+2}{2}}\neq 0 \text{ if } \ell \text{ is even,}\\ b_{\frac{\ell-3}{2}}\neq 0 \text{ if } \ell \text{ is odd}\\ \text{and } x \text{ is a root of}\\ \text{the equation:}\end{array} \right.$$
$$\left.\left(\sum_{i=1}^{\frac{\ell+2}{2}} a_i\left(x^i+\zeta_\sigma^i\right)\right)^2 x^{-1} = \left(\sum_{j=0}^{\frac{\ell-3}{2}} b_j x^{j-\frac{1}{2}}\right)^2 \prod_{k=1}^{4}(x-\gamma_k)\right\}$$

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