INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 14 (2025), No. 3, 5 - 19

THE DUAL FUHRMANN POINT LOCUS

YU CHEN AND R.J. FISHER

Abstract. In this paper, the dual Fuhrmann triangle $\triangle F_a^* F_b^* F_c^*$ and the dual Fuhrman point F^* of a triangle $\triangle ABC$ are introduced and the barycentric coordinates of F_a^*, F_b^*, F_c^*, and F^* with respect to $\triangle ABC$ are calculated. Fix a circle $\mathcal{C}$ with center O and radius r and fix a point I lying inside $\mathcal{C}$ with $d = OI > 0$. The set of all triangles inscribed in $\mathcal{C}$ and having the incenter I is called a $(\mathcal{C}, I)$-locus. We prove that the locus $\mathcal{C}_{f^*}$ of dual Fuhrmann points, the locus $\mathcal{C}_{g^*}$ of centroids of the dual Fuhrmann triangles, and the locus $\mathcal{C}_{h^*}$ of orthocenters of the dual Fuhrmann triangles associated to the $(\mathcal{C}, I)$-locus are circles with centers O_{f^*}, O_{g^*}, and O_{h^*} and radii $\frac{2d^2r^2-d^4}{4r^3-d^2r}$, $\frac{2d^2}{3r}$, and $\frac{4rd^2}{4r^2-d^2}$, respectively, where $\overrightarrow{OO_{f^*}} = \frac{8r^2-3d^2}{4r^2-d^2}\overrightarrow{OI}$, $\overrightarrow{OO_{g^*}} = \frac{5}{3}\overrightarrow{OI}$, and $\overrightarrow{OO_{h^*}} = \frac{4r^2+d^2}{4r^2-d^2}\overrightarrow{OI}$.

1. Introduction

Given a triangle $\triangle ABC$, let I be its incenter and I_a, I_b, and I_c be its excenters opposite to A, B, and C, respectively. Let $\mathcal{C}$ be the circumcircle of $\triangle ABC$ and A', B', and C' be the second points where the lines ℓ_{AI}, ℓ_{BI}, and ℓ_{CI} intersect $\mathcal{C}$, respectively. The reflections F_a, F_b, and F_c of A' across $\overline{BC}$, B' across $\overline{AC}$, and C' across $\overline{AB}$ are by definition the vertices of the **Fuhrmann triangle** of $\triangle ABC$ and the circumcenter F of $\triangle F_a F_b F_c$ is called the **Fuhrmann point** of $\triangle ABC$. On the other hand, let $\overline{A'A''}$, $\overline{B'B''}$, and $\overline{C'C''}$ be diameters of the circle $\mathcal{C}$. The reflections F_a^*, F_b^*, and F_c^* of A'' across $\overline{BC}$, B'' across $\overline{AC}$, and C'' across $\overline{AB}$ are by definition the vertices of the **dual Fuhrmann triangle** of $\triangle ABC$ and the circumcenter F^* of $\triangle F_a^* F_b^* F_c^*$ is called the **dual Fuhrmann point** of $\triangle ABC$.

Let $\mathcal{C}$ be a circle with center O and radius r. Let I be a point lying inside $\mathcal{C}$ but not O. The converse to the Euler's Incenter Theorem states that for each $A \in \mathcal{C}$,

---

Keywords and phrases: Affine Combination, Barycentric Coordinates, Circumcircle, Circumcenter, Centroid, Excenters, Dual Fuhrmann Point, Fuhrmann Point, Incenter, Orthocenter

(2020)Mathematics Subject Classification: 51M04

Received: 16.09.2024. In revised form: 21.02.2025. Accepted: 13.02.2025.

there is a unique triangle inscribed in $\mathcal{C}$ having A as a vertex and having I as its incenter. See [2, Theorem 155] or [1, §2]. The locus of these triangles is referred to as the $(\mathcal{C}, I)$**-locus**.

This paper establishes two geometric properties of the locus $\mathcal{C}_{f^*}$ of dual Fuhrmann points associated to the $(\mathcal{C}, I)$-locus. In §5, we prove that $\mathcal{C}_{f^*}$ is a circle whose center lies on the ray $\overrightarrow{OI}$ and whose radius is $\frac{d^2(2r^2-d^2)}{r(4r^2-d^2)}$, where $d = OI$. Specifically, $\mathcal{C}_{f^*}$ is the image of the locus $\mathcal{C}_{h'}$ of Negal points associated to the $(\mathcal{C}, I)$-locus under a dilation.

In [3], Dung proves that the incenter of a triangle is the orthocenter of the Fuhrmann triangle of the given triangle. Dually, Theorem 6.1 establishes that the locus of orthocenters of the dual Fuhrmann triangles associated to the $(\mathcal{C}, I)$-locus is a circle with radius $\frac{4rd^2}{4r^2-d^2}$ and center O_{h^*} defined by the vector equation $\overrightarrow{OO_{h^*}} = \frac{4r^2+d^2}{4r^2-d^2}\overrightarrow{OI}$.

The paper uses the concept of an affine combination of geometric vectors to prove the two main results, Theorems 5.1 and 6.1. Specifically, given $\triangle ABC$ with circumcenter O and a point P in the plane, the vector $\overrightarrow{OP}$ can be written uniquely in the form

$$\overrightarrow{OP} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC} \tag{1.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $\alpha + \beta + \gamma = 1$. Commonly, (1.1) is called an **affine combination** with respect to $\triangle ABC$ and the coefficients α, β, and γ are called the (normalized) **barycentric coordinates** of P with respect to $\triangle ABC$. The notation $[\alpha : \beta : \gamma]$ is used as a shorthand for the affine combination (1.1).

Let $\triangle F_a^* F_b^* F_c^*$ be the dual Fuhrmann triangle and F^* be the dual Fuhrmann point of $\triangle ABC$ in the $(\mathcal{C}, I)$-locus. A critical part of proving the two main theorems involves determining the barycentric coordinates of the vertices of $\triangle F_a^* F_b^* F_c^*$, its circumcenter F^*, and its centroid G^* with respect to $\triangle ABC$. To this end, §3 proves some general facts about calculating the barycentric coordinates of a point subjected to both orthogonal projection and reflection across a line.

Note that the paper freely uses the canonical vector space of geometric vectors associated to the plane. In particular, the geometric form of the dot product enters. Finally, Heron's formula for the area of a triangle appears frequently. Let $\triangle ABC$ be a triangle with $BC = a$, $AC = b$, and $AB = c$. Let $S = \text{Area}(\triangle ABC)$. Heron's formula can be expressed as

$$\begin{aligned} 16S^2 &= (a+b+c)(a+b-c)(a+c-b)(b+c-a) \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4. \end{aligned} \tag{1.2}$$

We also need to use the barycentric coordinates of the incenter I, the three excenters I_a, I_b, and I_c, and the Nagel point H' of $\triangle ABC$. All are well known, e.g., in [7], and are conveniently expressed by the vector equations

$$\begin{aligned}\overrightarrow{OI} &= \tfrac{a}{a+b+c}\overrightarrow{OA} + \tfrac{b}{a+b+c}\overrightarrow{OB} + \tfrac{c}{a+b+c}\overrightarrow{OC},\\ \overrightarrow{OI_a} &= \tfrac{-a}{b+c-a}\overrightarrow{OA} + \tfrac{b}{b+c-a}\overrightarrow{OB} + \tfrac{c}{b+c-a}\overrightarrow{OC},\\ \overrightarrow{OI_b} &= \tfrac{a}{a+c-b}\overrightarrow{OA} + \tfrac{-b}{a+c-b}\overrightarrow{OB} + \tfrac{c}{a+c-b}\overrightarrow{OC},\\ \overrightarrow{OI_c} &= \tfrac{a}{a+b-c}\overrightarrow{OA} + \tfrac{b}{a+b-c}\overrightarrow{OB} + \tfrac{-c}{a+b-c}\overrightarrow{OC},\\ \overrightarrow{OH'} &= \tfrac{b+c-a}{a+b+c}\overrightarrow{OA} + \tfrac{a+c-b}{a+b+c}\overrightarrow{OB} + \tfrac{a+b-c}{a+b+c}\overrightarrow{OC}.\end{aligned} \tag{1.3}$$

2. From a Linear Combination to an Affine Combination

The following two results though basic are needed in the later parts of the paper.

Theorem 2.1. *Let O be the circumcenter of $\triangle ABC$. Let $a = BC$, $b = AC$, $c = AB$, and $S = \text{Area}(\triangle ABC)$. The barycentric coordinates of O with respect to $\triangle ABC$ are*

$$\left[\tfrac{a^2(b^2+c^2-a^2)}{16S^2} : \tfrac{b^2(a^2+c^2-b^2)}{16S^2} : \tfrac{c^2(a^2+b^2-c^2)}{16S^2}\right].$$

In particular, the zero vector can be expressible as

$$\overrightarrow{OO} = \tfrac{a^2(b^2+c^2-a^2)}{16S^2}\overrightarrow{OA} + \tfrac{b^2(a^2+c^2-b^2)}{16S^2}\overrightarrow{OB} + \tfrac{c^2(a^2+b^2-c62)}{16S^2}\overrightarrow{OC}.$$

Proof. Denote the radius of the circumcircle of $\triangle ABC$ by r. Write

$$\overrightarrow{OO} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC}, \tag{2.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ satisfies $\alpha + \beta + \gamma = 1$.

By applying the Law of Cosines,

$$\overrightarrow{OB}\cdot\overrightarrow{OC} = r^2 - \tfrac{a^2}{2}, \quad \overrightarrow{OA}\cdot\overrightarrow{OC} = r^2 - \tfrac{b^2}{2}, \quad \text{and} \quad \overrightarrow{OA}\cdot\overrightarrow{OB} = r^2 - \tfrac{c^2}{2}. \tag{2.2}$$

The vector $\overrightarrow{OO}$ is determined by the equations $\overrightarrow{OO}\cdot\overrightarrow{OC} = 0$, $\overrightarrow{OO}\cdot\overrightarrow{OB} = 0$, and $\overrightarrow{OO}\cdot\overrightarrow{OA} = 0$. The condition $\alpha + \beta + \gamma = 1$ leads to the following linear system.

$$\begin{pmatrix} b^2 & a^2 & 0 \\ 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \begin{pmatrix}2r^2 \\ 2r^2 \\ 2r^2\end{pmatrix}$$

The coefficient matrix of the linear system has determinant $2a^2b^2c^2$ and the inverse of the coefficient matrix is

$$\begin{pmatrix} \frac{1}{2b^2} & -\frac{a^2}{2b^2c^2} & \frac{1}{2c^2} \\ \frac{1}{2a^2} & \frac{1}{2c^2} & -\frac{b^2}{2a^2c^2} \\ -\frac{c^2}{2a^2b^2} & \frac{1}{2b^2} & \frac{1}{2a^2} \end{pmatrix}.$$

Using the equation $4rS = abc$, we can write the solution of the linear system as

$$\begin{aligned}\alpha &= \tfrac{r^2(b^2+c^2-a^2)}{b^2c^2} = \tfrac{a^2(b^2+c^2-a^2)}{16S^2},\\ \beta &= \tfrac{r^2(a^2+c^2-b^2)}{a^2c^2} = \tfrac{b^2(a^2+c^2-b^2)}{16S^2},\\ \gamma &= \tfrac{r^2(a^2+b^2-c^2)}{a^2b^2} = \tfrac{c^2(a^2+b^2-c^2)}{16S^2}.\end{aligned}$$

Hence,

$$\overrightarrow{OO} = \tfrac{a^2(b^2+c^2-a^2)}{16S^2}\overrightarrow{OA} + \tfrac{b^2(a^2+c^2-b^2)}{16S^2}\overrightarrow{OB} + \tfrac{c^2(a^2+b^2-c^2)}{16S^2}\overrightarrow{OC}. \qquad \square$$

Theorem 2.2. *Let O be the circumcenter of $\triangle ABC$. Let $a = BC$, $b = AC$, $c = AB$, and $S = \text{Area}(\triangle ABC)$. Let P be a point given by a vector equation $\overrightarrow{OP} = x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}$, where $x, y, z \in \mathbb{R}$. The barycentric coordinates of P with respect to $\triangle ABC$ are $[\alpha\colon \beta\colon \gamma]$, where*

$$\begin{aligned}\alpha &= \tfrac{[a^2(b^2+c^2)-(b^2-c^2)^2]x+[a^2(b^2+c^2-a^2)](1-y-z)}{16S^2},\\ \beta &= \tfrac{[b^2(a^2+c^2)-(a^2-c^2)^2]y+[b^2(a^2+c^2-b^2)](1-x-z)}{16S^2},\\ \gamma &= \tfrac{[c^2(a^2+b^2)-(a^2-b^2)^2]z+[c^2(a^2+b^2-c^2)](1-x-y)}{16S^2}.\end{aligned}$$

Proof. Let $[\alpha_0\colon \beta_0\colon \gamma_0]$ denote the barycentric coordinates of O with respect to $\triangle ABC$ in Theorem 2.1. Note that

$$\overrightarrow{OP} = \overrightarrow{OP} + \lambda\overrightarrow{OO} = (x + \lambda\alpha_0)\overrightarrow{OA} + (y + \lambda\beta_0)\overrightarrow{OB} + (z + \lambda\gamma_0)\overrightarrow{OC}$$

for all $\lambda \in \mathbb{R}$. Choose a $\lambda_0 \in \mathbb{R}$ that satisfies the equation

$$(x + \lambda_0\alpha_0) + (y + \lambda\beta_0) + (z + \lambda_0\gamma_0) = 1.$$

The condition $\alpha_0 + \beta_0 + \gamma_0 = 1$ leads to $\lambda_0 = 1 - x - y - z$. Using (1.2), we get

$$\begin{aligned}\alpha &= x + (1 - x - y - z)\alpha_0 = \tfrac{[a^2(b^2+c^2)-(b^2-c^2)^2]x+[a^2(b^2+c^2-a^2)](1-y-z)}{16S^2}\\ \beta &= y + (1 - x - y - z)\beta_0 = \tfrac{[b^2(a^2+c^2)-(a^2-c^2)^2]y+[b^2(a^2+c^2-b^2)](1-x-z)}{16S^2},\\ \gamma &= z + (1 - x - y - z)\gamma_0 = \tfrac{[c^2(a^2+b^2)-(a^2-b^2)^2]z+[c^2(a^2+b^2-c^2)](1-x-y)}{16S^2}.\end{aligned} \qquad \square$$

Corollary 2.1. *Let H be the orthocenter of $\triangle ABC$. The barycentric coordinates of H with respect to $\triangle ABC$ are*

$$\left[\tfrac{a^4-(b^2-c^2)^2}{16S^2} : \tfrac{b^4-(a^2-c^2)^2}{16S^2} : \tfrac{c^4-(a^2-b^2)^2}{16S^2}\right]. \tag{2.3}$$

Proof. Note that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ by Sylvester's Orthocenter Law. Take $x = 1$, $y = 1$, and $z = 1$ in Theorem 2.2. The barycentric coordinates of H follow. $\square$

3. Affine Combinations Under Reflection and Orthogonal Projection

Let $\mathcal{C}$ be a circle with center O and radius r. Let $\triangle ABC$ be a triangle inscribed in $\mathcal{C}$. Set $a = BC$, $b = AC$, and $c = AB$. Given a point P in the plane, write $\overrightarrow{OP} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC}$, where $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $\alpha + \beta + \gamma = 1$.

Lemma 3.1. *Let P_a be the orthogonal projection of P onto ℓ_{BC}. The barycentric coordinates of P_a with respect to $\triangle ABC$ are*

$$\left[0 : \beta + \tfrac{\alpha}{2a^2}(a^2 + b^2 - c^2) : \gamma + \tfrac{\alpha}{2a^2}(a^2 + c^2 - b^2)\right].$$

Moreover,

$$\overrightarrow{OP_a} = \overrightarrow{OP} + \tfrac{\alpha}{2a^2}\left[-2a^2\overrightarrow{OA} + (a^2 + b^2 - c^2)\overrightarrow{OB} + (a^2 + c^2 - b^2)\overrightarrow{OC}\right].$$

Proof. Since $P_a \in \ell_{BC}$, the vector $\overrightarrow{OP_a}$ can be expressed as $\overrightarrow{OP_a} = (1-t)\overrightarrow{OB} + t\overrightarrow{OC}$ for some $t \in \mathbb{R}$. First, since $\ell_{PP_a} \perp \ell_{BC}$, we have $\overrightarrow{PP_a} \cdot \overrightarrow{BC} = 0$. Next,

$$\overrightarrow{PP_a} = \overrightarrow{OP_a} - \overrightarrow{OP} = -\alpha\overrightarrow{OA} + (1 - \beta - t)\overrightarrow{OB} + (t - \gamma)\overrightarrow{OC}.$$

Using (2.2) and $1 - \beta = \alpha + \gamma$, we get

$$\begin{aligned}
0 &= \overrightarrow{PP_a} \cdot \overrightarrow{BC} \\
&= \left[-\alpha\overrightarrow{OA} + (1 - \beta - t)\overrightarrow{OB} + (t - \gamma)\overrightarrow{OC}\right] \cdot \left(\overrightarrow{OC} - \overrightarrow{OB}\right) \\
&= -\alpha\left[\left(r^2 - \tfrac{b^2}{2}\right) - \left(r^2 - \tfrac{c^2}{2}\right)\right] + (1 - \beta - t)\left[\left(r^2 - \tfrac{a^2}{2}\right) - r^2\right] \\
&\quad + (t - \gamma)\left[r^2 - \left(r^2 - \tfrac{a^2}{2}\right)\right] \\
&= \tfrac{1}{2}\{\alpha(b^2 - c^2) + a^2[2t - (1 - \beta) - \gamma]\} \\
&= \tfrac{1}{2}\{\alpha(b^2 - c^2) + a^2[2t - (\alpha + \gamma) - \gamma]\} \\
&= \tfrac{1}{2}[\alpha(b^2 - c^2) + a^2(2t - \alpha - 2\gamma)] \\
&= \tfrac{1}{2}[\alpha(b^2 - a^2 - c^2) + 2a^2(t - \gamma)].
\end{aligned}$$

Then $t = \gamma + \frac{a^2+c^2-b^2}{2a^2}\alpha$ and

$$\begin{aligned}
\overrightarrow{OP_a} &= \left[(1 - \gamma) - \tfrac{a^2+b^2-c^2}{2a^2}\alpha\right]\overrightarrow{OB} + \left(\gamma + \tfrac{a^2+c^2-b^2}{2a^2}\alpha\right)\overrightarrow{OC} \\
&= \left[(\alpha + \beta) - \tfrac{a^2+b^2-c^2}{2a^2}\alpha\right]\overrightarrow{OB} + \left(\gamma + \tfrac{a^2+c^2-b^2}{2a^2}\alpha\right)\overrightarrow{OC} \\
&= \left(\beta + \tfrac{a^2+b^2-c^2}{2a^2}\alpha\right)\overrightarrow{OB} + \left(\gamma + \tfrac{a^2+c^2-b^2}{2a^2}\alpha\right)\overrightarrow{OC} \\
&= \overrightarrow{OP} + \tfrac{\alpha}{2a^2}\left[-2a^2\overrightarrow{OA} + (a^2 + b^2 - c^2)\overrightarrow{OB} + (a^2 + c^2 - b^2)\overrightarrow{OC}\right].
\end{aligned}$$

The barycentric coordinates of P_a are

$$\left[0 : \beta + \tfrac{\alpha}{2a^2}(a^2 + b^2 - c^2) : \gamma + \tfrac{\alpha}{2a^2}(a^2 + c^2 - b^2)\right]. \qquad \square$$

Lemma 3.2. *Let P'_a be the reflection of P across ℓ_{BC}. The barycentric coordinates of P'_a with respect to $\triangle ABC$ are*

$$\left[-\alpha : \beta + \tfrac{a^2+b^2-c^2}{a^2}\alpha : \gamma + \tfrac{a^2+c^2-b^2}{a^2}\alpha\right].$$

Moreover,

$$\overrightarrow{OP'_a} = \overrightarrow{OP} + \tfrac{\alpha}{a^2}\left[-2a^2\overrightarrow{OA} + (a^2+b^2-c^2)\overrightarrow{OB} + (a^2+c^2-b^2)\overrightarrow{OC}\right].$$

Proof. Since P'_a is the midpoint of $\overline{PP_a}$, we have $\overrightarrow{OP_a} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OP'_a}\right)$. By Lemma 3.1,

$$\begin{aligned}\overrightarrow{OP'_a} &= 2\overrightarrow{OP_a} - \overrightarrow{OP}\\ &= \overrightarrow{OP} + \tfrac{\alpha}{a^2}\left[-2a^2\overrightarrow{OA} + (a^2+b^2-c^2)\overrightarrow{OB} + (a^2+c^2-b^2)\overrightarrow{OC}\right]\\ &= -\alpha\overrightarrow{OA} + \left(\beta + \tfrac{a^2+b^2-c^2}{a^2}\alpha\right)\overrightarrow{OB} + \left(\gamma + \tfrac{a^2+c^2-b^2}{a^2}\right)\overrightarrow{OC}.\end{aligned}$$ □

Lemma 3.3. *Given a point P with barycentric coordinates $[\alpha\colon \beta\colon \gamma]$ with respect to $\triangle ABC$, define the point P' by the vector equation $\overrightarrow{OP'} = -\overrightarrow{OP}$. The barycentric coordinates of P' with respect to $\triangle ABC$ are*

$$\left[\tfrac{a^2(b^2+c^2-a^2)}{8S^2} - \alpha\colon \tfrac{b^2(a^2+c^2-b^2)}{8S^2} - \beta\colon \tfrac{c^2(a^2+b^2-c^2)}{8S^2} - \gamma\right],$$

where $S = \text{Area}(\triangle ABC)$.

Proof. Write $[\alpha'\colon \beta'\colon \gamma']$ for the barycentric coordinates of P' and $[\alpha_0\colon \beta_0\colon \gamma_0]$ for the barycentric coordinates of O with respect to $\triangle ABC$. Since O is the midpoint of $\overline{PP'}$, we have $\overrightarrow{OO} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OP'}\right)$, or equivalently, $\overrightarrow{OP'} = 2\overrightarrow{OO} - \overrightarrow{OP}$. The barycentric coordinates of P' follow from Theorem 2.1. □

4. The Barycentric Coordinates of the Dual Fuhrmann Point

In this section, Lemma 3.2 and Lemma 3.3 are used to calculate the barycentric coordinates of the vertices of the dual Fuhrmann triangle $\triangle F^*_a F^*_b F^*_c$ as well as its circumcenter F^*, aka, the dual Fuhrmann point of $\triangle ABC$.

Lemma 4.1. *Let I be the incenter and I_a, I_b, and I_c be the excenters opposite A, B, and C of $\triangle ABC$, respectively. Let A', B', and C' be the second points where ℓ_{AI}, ℓ_{BI}, and ℓ_{CI} intersect the circumcircle of $\triangle ABC$, respectively. Let O be the circumcenter of $\triangle ABC$. Then*

$$\begin{aligned}\overrightarrow{OA'} &= \tfrac{-a^2}{(b+c)^2-a^2}\overrightarrow{OA} + \tfrac{b(b+c)}{(b+c)^2-a^2}\overrightarrow{OB} + \tfrac{c(b+c)}{(b+c)^2-a^2}\overrightarrow{OC},\\ \overrightarrow{OB'} &= \tfrac{a(a+c)}{(a+c)^2-b^2}\overrightarrow{OA} + \tfrac{-b^2}{(a+c)^2-b^2}\overrightarrow{OB} + \tfrac{c(a+c)}{(a+c)^2-b^2}\overrightarrow{OC},\\ \overrightarrow{OC'} &= \tfrac{a(a+b)}{(a+b)^2-c^2}\overrightarrow{OA} + \tfrac{b(a+b)}{(a+b)^2-c^2}\overrightarrow{OB} + \tfrac{-c^2}{(a+b)^2-c^2}\overrightarrow{OC}.\end{aligned}$$

Proof. Since $\overrightarrow{OA'} = \frac{1}{2}(\overrightarrow{OI} + \overrightarrow{OI_a})$, $\overrightarrow{OB'} = \frac{1}{2}(\overrightarrow{OI} + \overrightarrow{OI_b})$, and $\overrightarrow{OC'} = \frac{1}{2}(\overrightarrow{OI} + \overrightarrow{OI_c})$, the equations follow from (1.3). ☐

Lemma 4.2. *Given A', B', and C' as above, let A'', B'', and C'' be the points of the circumeircle $\mathcal{C}$ of $\triangle ABC$ such that $\overline{A'A''}$, $\overline{B'B''}$, and $\overline{C'C''}$ are diameters of $\mathcal{C}$. Then*

$$\begin{aligned}
\overrightarrow{OA''} &= \tfrac{a^2}{a^2-(b-c)^2}\overrightarrow{OA} + \tfrac{b(c-b)}{a^2-(b-c)^2}\overrightarrow{OB} + \tfrac{c(b-c)}{a^2-(b-c)^2}\overrightarrow{OC},\\
\overrightarrow{OB''} &= \tfrac{a(c-a)}{b^2-(a-c)^2}\overrightarrow{OA} + \tfrac{b^2}{b^2-(a-c)^2}\overrightarrow{OB} + \tfrac{c(a-c)}{b^2-(a-c)^2}\overrightarrow{OC},\\
\overrightarrow{OC''} &= \tfrac{a(b-a)}{c^2-(a-b)^2}\overrightarrow{OA} + \tfrac{b(a-b)}{c^2-(a-b)^2}\overrightarrow{OB} + \tfrac{c^2}{c^2-(a-b)^2}\overrightarrow{OC}.
\end{aligned} \tag{4.1}$$

Proof. Our proof of the lemma uses Theorem 2.1, Lemma 4.1 and the proof of Lemma 3.3. First of all, Heron's formula for the area S of $\triangle ABC$ can be written in the following equivalent ways.

$$\begin{aligned}
16S^2 &= (a+b+c)(a+b-c)(a+c-b)(b+c-a)\\
&= [(a+b)^2 - c^2][c^2 - (a-b)^2]\\
&= [(a+c)^2 - b^2][b^2 - (a-c)^2]\\
&= [(b+c)^2 - a^2][a^2 - (b-c)^2]
\end{aligned}$$

Next, by Theorem 2.1 and Lemma 4.1,

$$\begin{aligned}
\overrightarrow{OA''} &= 2\overrightarrow{OO} - \overrightarrow{OA'}\\
&= \tfrac{a^2(b^2+c^2-a^2)}{8S^2}\overrightarrow{OA} + \tfrac{b^2(a^2+c^2-b^2)}{8S^2}\overrightarrow{OB} + \tfrac{c^2(a^2+b^2-c^2)}{8S^2}\overrightarrow{OC}\\
&+ \tfrac{a^2[a^2-(b-c)^2]}{16S^2}\overrightarrow{OA} - \tfrac{b(b+c)[a^2-(b-c)^2]}{16S^2}\overrightarrow{OB} - \tfrac{c(b+c)[a^2-(b-c)^2]}{16S^2}\overrightarrow{OC}\\
&= \tfrac{a^2[(b+c)^2-a^2]}{16S^2}\overrightarrow{OA} + \tfrac{b(c-b)[(b+c)^2-a^2]}{16S^2}\overrightarrow{OB} + \tfrac{c(b-c)[(b+c)^2-a^2]}{16S^2}\overrightarrow{OC}\\
&= \tfrac{a^2}{a^2-(b-c)^2}\overrightarrow{OA} + \tfrac{b(c-b)}{a^2-(b-c)^2}\overrightarrow{OB} + \tfrac{c(b-c)}{a^2-(b-c)^2}\overrightarrow{OC}.
\end{aligned}$$

In the same way, the second and third equations of (4.1) also hold. ☐

Lemma 4.3. *The vertices of the dual Fuhrmann triangle $\triangle F_a^* F_b^* F_c^*$ are given by*

$$\begin{aligned}
\overrightarrow{OF_a^*} &= \tfrac{-a^2}{a^2-(b-c)^2}\overrightarrow{OA} + \tfrac{a^2+c(b-c)}{a^2-(b-c)^2}\overrightarrow{OB} + \tfrac{a^2+b(c-b)}{a^2-(b-c)^2}\overrightarrow{OC},\\
\overrightarrow{OF_b^*} &= \tfrac{b^2+c(a-c)}{b^2-(a-c)^2}\overrightarrow{OA} + \tfrac{-b^2}{b^2-(a-c)^2}\overrightarrow{OB} + \tfrac{b^2+a(c-a)}{b^2-(a-c)^2}\overrightarrow{OC},\\
\overrightarrow{OF_c^*} &= \tfrac{c^2+b(a-b)}{c^2-(a-b)^2}\overrightarrow{OA} + \tfrac{c^2+a(b-a)}{c^2-(a-b)^2}\overrightarrow{OB} + \tfrac{-c^2}{c^2-(a-b)^2}\overrightarrow{OC}.
\end{aligned}$$

Proof. The lemma is an immediate consequence of Lemma 3.2 and Lemma 4.2. By Lemma 4.2, the barycentric coordinates of A'' with respect to $\triangle ABC$ are

$$\left[\tfrac{a^2}{a^2-(b-c)^2} : \tfrac{b(c-b)}{a^2-(b-c)^2} : \tfrac{c(b-c)}{a^2-(b-c)^2}\right].$$

Take $P = A''$ in Lemma 3.2, i.e., $\alpha = \frac{a^2}{a^2-(b-c)^2}$, $\beta = \frac{b(c-b)}{a^2-(b-c)^2}$, and $\gamma = \frac{c(b-c)}{a^2-(b-c)^2}$. Since

$$\beta + \frac{a^2+b^2-c^2}{a^2}\alpha = \frac{a^2+c(b-c)}{a^2-(b-c)^2} \quad \text{and} \quad \gamma + \frac{a^2+c^2-b^2}{a^2-(b-c)^2}\alpha = \frac{a^2+b(c-b)}{a^2-(b-c)^2},$$

the affine combination of $\overrightarrow{OF_a^*}$ is as written above by Lemma 3.2. By analogy, we get the affine combinations of $\overrightarrow{OF_b^*}$ and $\overrightarrow{OF_c^*}$. □

Next, we calculate the barycentric coordinates $[x\colon y\colon z]$ of the dual Fuhrmann point F^* with respect to $\triangle ABC$. Define

$$\begin{aligned} \alpha_a &= \frac{-a^2}{a^2-(b-c)^2}, & \beta_a &= \frac{a^2+c(b-c)}{a^2-(b-c)^2}, & \gamma_a &= \frac{a^2+b(c-b)}{a^2-(b-c)^2} \\ \alpha_b &= \frac{b^2+c(a-c)}{b^2-(a-c)^2}, & \beta_b &= \frac{-b^2}{b^2-(a-c)^2}, & \gamma_b &= \frac{b^2+a(c-a)}{b^2-(a-c)^2} \\ \alpha_c &= \frac{c^2+b(a-b)}{c^2-(a-b)^2}, & \beta_c &= \frac{c^2+a(b-a)}{c^2-(a-b)^2}, & \gamma_c &= \frac{-c^2}{c^2-(a-b)^2}. \end{aligned} \tag{4.2}$$

Since F^* is the circumcenter of $\triangle F_a^*F_b^*F_c^*$,

$$(F^*F_a^*)^2 = (F^*F_b^*)^2 = (F^*F_c^*)^2. \tag{4.3}$$

On the other hand, for each $i \in \{a,b,c\}$,

$$(F^*F_i^*)^2 = \left(\overrightarrow{OF_i^*} - \overrightarrow{OF^*}\right)\cdot\left(\overrightarrow{OF_i^*} - \overrightarrow{OF^*}\right) = (OF_i^*)^2 - 2\left(\overrightarrow{OF_i^*}\cdot\overrightarrow{OF^*}\right) + (OF^*)^2.$$

Then $(OF_i^*)^2 - 2\left(\overrightarrow{OF_i^*}\cdot\overrightarrow{OF^*}\right) = (OF_j^*)^2 - 2\left(\overrightarrow{OF_j^*}\cdot\overrightarrow{OF^*}\right)$ for all $i,j \in \{a,b,c\}$.

$$\begin{aligned} (OF_i^*)^2 &= \left(\alpha_i\overrightarrow{OA} + \beta_i\overrightarrow{OB} + \gamma_i\overrightarrow{OC}\right)\cdot\left(\alpha_i\overrightarrow{OA} + \beta_i\overrightarrow{OB} + \gamma_i\overrightarrow{OC}\right) \\ &= (\alpha_i^2+\beta_i^2+\gamma_i^2)r^2 + 2\alpha_i\beta_i\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right) + 2\alpha_i\gamma_i\left(\overrightarrow{OA}\cdot\overrightarrow{OC}\right) \\ &\qquad + 2\beta_i\gamma_i\left(\overrightarrow{OB}\cdot\overrightarrow{OC}\right) \\ &= (\alpha_i^2+\beta_i^2+\gamma_i^2)r^2 + 2\alpha_i\beta_i\left(r^2-\frac{c^2}{2}\right) + 2\alpha_i\gamma_i\left(r^2-\frac{b^2}{2}\right) + 2\beta_i\gamma_i\left(r^2-\frac{a^2}{2}\right) \\ &= r^2(\alpha_i+\beta_i+\gamma_i)^2 - (\alpha_i\beta_ic^2 + \alpha_i\gamma_ib^2 + \beta_i\gamma_ia^2) \\ &= r^2 - (\alpha_i\beta_ic^2 + \alpha_i\gamma_ib^2 + \beta_i\gamma_ia^2) \end{aligned}$$

$$\begin{aligned} 2\left(\overrightarrow{OF_i^*}\cdot\overrightarrow{OF^*}\right) &= 2\left(\alpha_i\overrightarrow{OA} + \beta_i\overrightarrow{OB} + \gamma_i\overrightarrow{OC}\right)\cdot\left(x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}\right) \\ &= 2r^2(\alpha_i+\beta_i+\gamma_i)(x+y+z) \\ &\qquad - [(\beta_ix+\alpha_iy)c^2 + (\gamma_ix+\alpha_iz)b^2 + (\gamma_iy+\beta_iz)a^2] \\ &= 2r^2 - [(\beta_ix+\alpha_iy)c^2 + (\gamma_ix+\alpha_iz)b^2 + (\gamma_iy+\beta_iz)a^2] \end{aligned}$$

Using $z = 1-x-y$,

$$\begin{aligned} -2\left(\overrightarrow{OF_i^*}\cdot\overrightarrow{OF^*}\right) &= [\beta_ic^2 + (\gamma_i-\alpha_i)b^2 + \beta_ia^2]x + [\alpha_ic^2 + \alpha_ib^2 + (\gamma_i-\beta_i)a^2]y \\ &\qquad + (\alpha_ib^2 + \beta_ia^2) - 2r^2. \end{aligned}$$

Then

$$(OF_i^*)^2 - 2\left(\overrightarrow{OF_i^*}\cdot\overrightarrow{OF^*}\right) = [\beta_i(a^2+c^2)+(\gamma_i-\alpha_i)b^2]x + [\alpha_i(b^2+c^2)+(\gamma_i-\beta_i)a^2]y \\ - (\alpha_i\beta_i c^2 + \alpha_i\gamma_i b^2 + \beta_i\gamma_i a^2) + \alpha_i b^2 + \beta_i a^2 - r^2. \tag{4.4}$$

In turn, (4.2), (4.3), and (4.4) lead to the following linear system.

$$\begin{pmatrix} 2b^2+ab+bc & a^2+b^2-c^2+ab \\ a^2+b^2-c^2+ab & 2a^2+ab+ac \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a^2+b^2-c^2+ab+\frac{abc(a-c)}{b^2-(a-c)^2} \\ a^2+b^2-c^2+ab+\frac{abc(b-c)}{a^2-(b-c)^2} \end{pmatrix}$$

Call the coefficient matrix M. Let r and r_i be the radii of the circumcircle and the incircle of $\triangle ABC$, respectively. Note that

$$abc = 4rS \quad \text{and} \quad a+b+c = \tfrac{2S}{r_i}.$$

By the Euler's Incenter Theorem, we get $d^2 = r^2 - 2r_i r$. Then Heron's formula (1.2) leads to

$$\begin{aligned} \det M &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 + 3abc(a+b+c) \\ &= 16S^2 + 3\cdot 4Sr\cdot \tfrac{2S}{r_i} \\ &= 16S^2 + \tfrac{48S^2r^2}{r^2-d^2} \\ &= 16S^2\tfrac{4r^2-d^2}{r^2-d^2} > 0. \end{aligned}$$

The unique solution of the above linear system yields the barycentric coordinates of F^*. In summary, we have the following theorem.

Theorem 4.1. *Let $\mathcal{C}$ be a circle with radius r and center O. Let I be a point lying inside $\mathcal{C}$ but not O. Let $\triangle ABC$ be a triangle in the $(\mathcal{C}, I)$-locus with $a = BC$, $b = AC$, $c = AB$, and $S = \text{Area}(\triangle ABC)$. The dual Fuhrmann point F^* of $\triangle ABC$ is given by*

$$\overrightarrow{OF^*} = x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}, \tag{4.5}$$

where

$$\begin{aligned} x &= \tfrac{r^2-d^2}{4r^2-d^2}\cdot\tfrac{a^4-(b^2-c^2)^2+ac(a^2-c^2)+ab(a^2-b^2)+2abc(a+b+c)}{16S^2}, \\ y &= \tfrac{r^2-d^2}{4r^2-d^2}\cdot\tfrac{b^4-(a^2-c^2)^2+bc(b^2-c^2)+ab(b^2-a^2)+2abc(a+b+c)}{16S^2}, \\ z &= \tfrac{r^2-d^2}{4r^2-d^2}\cdot\tfrac{c^4-(a^2-b^2)^2+ac(c^2-a^2)+bc(c^2-b^2)+2abc(a+b+c)}{16S^2}. \end{aligned}$$

5. The Locus of Dual Fuhrmann Points Associated to a $(\mathcal{C}, I)$-Locus

Recall that I_a, I_b, and I_c represent the three excenters of $\triangle ABC$. The Nagel point of $\triangle ABC$ is defined as follows. Let T_a be the orthogonal projection of I_a on ℓ_{BC}, T_b be the orthogonal projection of I_b on ℓ_{AC}, and T_c be the orthogonal

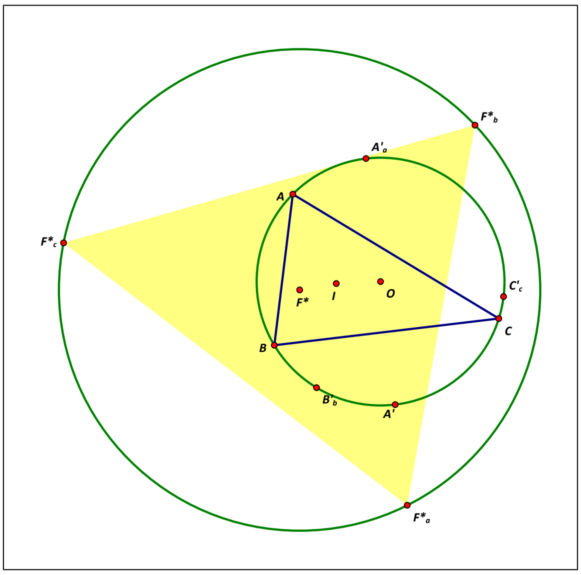

FIGURE 1. Dual Fuhrmann Triangle and Dual Fuhrmann Point

projection of I_c on ℓ_{AB}. The three Cevians ℓ_{AT_a}, ℓ_{BT_b}, and ℓ_{CT_c} are concurrent and their intersection point H' is called the **Negal point** of $\triangle ABC$. See [4, Page 5].

Let $\mathcal{C}$ be a circle with radius r and center O. Let I be a point lying inside $\mathcal{C}$ but not O. In [1, Theorem 6.2], we proved the locus of Negal points associated to the $(\mathcal{C}, I)$-locus is a circle with center at O, the center of $\mathcal{C}$, and radius $\frac{d^2}{r}$, where $d = OI$.

Theorem 5.1. *Let H' and F^* be the Nagel point and the dual Fuhrmann point of $\triangle ABC$ in the $(\mathcal{C}, I)$-locus, respectively. Then*

$$\overrightarrow{OF^*} = \frac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'} + \frac{8r^2-3d^2}{4r^2-d^2}\overrightarrow{OI}.$$

Let O_{f^*} be the point defined by the vector equation

$$\overrightarrow{OO_{f^*}} = \frac{8r^2-3d^2}{4r^2-d^2}\overrightarrow{OI}. \tag{5.1}$$

Then

$$\overrightarrow{O_{f^*}F^*} = \frac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'},$$

so the vectors $\overrightarrow{O_{f^*}F^*}$ and $\overrightarrow{OH'}$ are parallel with the same direction.

Let I^* be the point given by the vector equation

$$\overrightarrow{OI^*} = \frac{8r^2-3d^2}{2r^2}\overrightarrow{OI}$$

and

$$\delta = \frac{2r^2-d^2}{4r^2-d^2}.$$

Let $\mathcal{D}_{I^*,\delta}$ be the dilation with center I^* and dilation factor δ. Let $\mathcal{C}_{h'}$ be the locus of Negal points and $\mathcal{C}_{f^*}$ be the locus of dual Fuhrmann points associated to the $(\mathcal{C}, I)$-locus. Then $\mathcal{D}_{I^*,\delta}(\mathcal{C}_{h'}) = \mathcal{C}_{f^*}$, so $\mathcal{C}_{f^*}$ is the circle with center O_{f^*} and radius $\frac{d^2(2r^2-d^2)}{r(4r^2-d^2)}$.

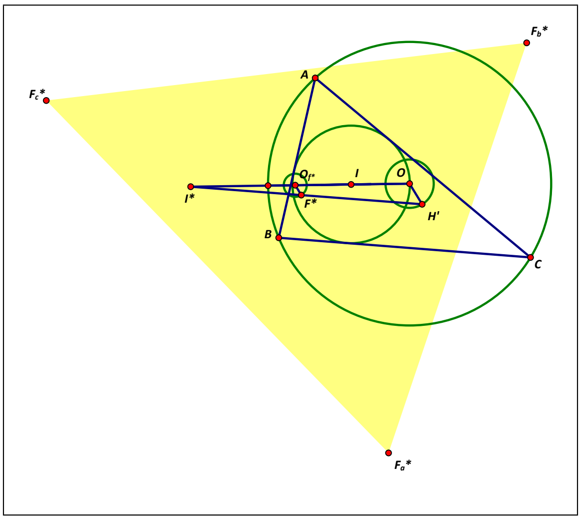

FIGURE 2. Loci of Negal Points and Dual Fuhrmann Points

Givem $\triangle ABC$ in the $(\mathcal{C}, I)$-locus, let r and r_i be the radii of its circumcircle and its incircle, respectively. Recall that

$$r = \tfrac{abc}{4S}, \quad r_i = \tfrac{2S}{a+b+c}, \quad \text{and} \quad d^2 = r^2 - 2r_i r.$$

Then

$$r^2 = \tfrac{a^2b^2c^2}{16S^2} = \tfrac{a^2b^2c^2}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}, \tag{5.2}$$

$$d^2 = \left(\tfrac{abc}{4S}\right)^2 - 2 \cdot \tfrac{abc}{4S} \cdot \tfrac{2S}{a+b+c} = \tfrac{a^2b^2c^2 - abc(a+b-c)(a+c-b)(b+c-a)}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}, \tag{5.3}$$

$$r^2 - d^2 = \tfrac{abc(a+b-c)(a+c-b)(b+c-a)}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} = \tfrac{abc}{a+b+c}, \tag{5.4}$$

$$4r^2 - d^2 = \tfrac{3a^2b^2c^2 + abc(a+b-c)(a+c-b)(b+c-a)}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}. \tag{5.5}$$

The proof of Theorem 5.1 uses the following result.

Lemma 5.1. *Let H, H', and F^* be the orthocenter, the Nagel point, and the dual Fuhrmann point of $\triangle ABC$ in the $(\mathcal{C}, I)$-locus. Define*

$$x = \tfrac{a[c(a^2-c^2)+b(a^2-b^2)]}{16S^2},$$
$$y = \tfrac{b[c(b^2-c^2)+a(b^2-a^2)]}{16S^2},$$
$$z = \tfrac{c[a(c^2-a^2)+b(c^2-b^2)]}{16S^2}.$$

Then

$$\overrightarrow{OF^*} = \tfrac{3r^2-d^2}{4r^2-d^2}\overrightarrow{OH} + \tfrac{r^2-d^2}{4r^2-d^2}\left(x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}\right), \tag{5.6}$$

where

$$\tfrac{r^2-d^2}{4r^2-d^2}\left(x\overrightarrow{OA} + y\overrightarrow{OC} + z\overrightarrow{OC}\right) = \overrightarrow{OI} - \tfrac{r^2}{4r^2-d^2}\overrightarrow{OH}. \tag{5.7}$$

Proof. By Corollary 2.1, the orthocenter H of $\triangle ABC$ is given by

$$\overrightarrow{OH} = \tfrac{a^4-(b^2-c^2)^2}{16S^2}\overrightarrow{OA} + \tfrac{b^4-(a^2-c^2)^2}{16S^2}\overrightarrow{OB} + \tfrac{c^4-(a^2-b^2)^2}{16S^2}\overrightarrow{OC}.$$

Additionally, by Theorem 2.1,

$$\overrightarrow{OO} = \tfrac{a^2(b^2+c^2-a^2)}{16S^2}\overrightarrow{OA} + \tfrac{b^2(a^2+c^2-b^2)}{16S^2}\overrightarrow{OB} + \tfrac{c^2(a^2+b^2-c^2)}{16S^2}\overrightarrow{OC}.$$

Note that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. By (5.3), (5.4), and Theorem 4.1,

$$\begin{aligned}\overrightarrow{OF^*} &= \tfrac{r^2-d^2}{4r^2-d^2}\overrightarrow{OH} + \tfrac{r^2-d^2}{4r^2-d^2}\left(x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC}\right) + \tfrac{r^2-d^2}{4r^2-d^2}\cdot\tfrac{2r^2}{r^2-d^2}\overrightarrow{OH}\\ &= \tfrac{3r^2-d^2}{4r^2-d^2}\overrightarrow{OH} + \tfrac{r^2-d^2}{4r^2-d^2}\left(x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OZ}\right).\end{aligned}$$

To prove (5.7), we use Theorem 2.2 to show that

$$\tfrac{r^2-d^2}{4r^2-d^2}\left(x\overrightarrow{OA} + y\overrightarrow{OC} + z\overrightarrow{OC}\right) \quad \text{and} \quad \overrightarrow{OI} - \tfrac{r^2}{4r^2-d^2}\overrightarrow{OH}$$

have the same affine combination.

The affine combination of the first vector is $\alpha_1\overrightarrow{OA} + \beta_1\overrightarrow{OB} + \gamma_1\overrightarrow{OC}$, where

$$\begin{aligned}\alpha_1 &= \tfrac{(r^2-d^2)[ac(a^2-c^2)+ab(a^2-b^2)]+(4r^2-d^2)a^2(b^2+c^2-a^2)}{16S^2(4r^2-d^2)},\\ \beta_1 &= \tfrac{(r^2-d^2)[bc(b^2-c^2)+ab(b^2-a^2)]+(4r^2-d^2)b^2(a^2+c^2-b^2)}{16S^2(4r^2-d^2)},\\ \gamma_1 &= \tfrac{(r^2-d^2)[ac(c^2-a^2)+bc(c^2-b^2)]+(4r^2-d^2)c^2(a^2+b^2-c^2)}{16S^2(4r^2-d^2)}.\end{aligned}$$

The affine combination of the second vector is $\alpha_2\overrightarrow{OA} + \beta_2\overrightarrow{OB} + \gamma_2\overrightarrow{OC}$, where

$$\begin{aligned}\alpha_2 &= \tfrac{(4r^2-d^2)a(a+b-c)(a+c-b)(b+c-a)+r^2[a^2(b^2+c^2-a^2)+(b^2-c^2)^2-a^4]}{16S^2(4r^2-d^2)},\\ \beta_2 &= \tfrac{(4r^2-d^2)b(a+b-c)(a+c-b)(b+c-a)+r^2[b^2(a^2+c^2-b^2)+(a^2-c^2)^2-b^4]}{16S^2(4r^2-d^2)},\\ \gamma_2 &= \tfrac{(4r^2-d^2)c(a+b-c)(a+c-b)(b+c-a)+r^2[c^2(a^2+b^2-c^2)+(a^2-b^2)^2-c^4]}{16S^2(4r^2-d^2)}.\end{aligned}$$

Using (5.2), (5.4), (5.5), and probably a compute algebra system, e.g., Mathematica, we can check that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$. □

Proof of Theorem 5.1

Proof. First of all, by (5.6) and (5.7), we get

$$\overrightarrow{OF^*} = \tfrac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH} + \overrightarrow{OI}.$$

Next, it is well known that the Negal point H' of $\triangle ABC$ is obtained by dilating the incenter I about the centroid by the factor -2. See [4, Page 7] or [1, Theorem 7.2]. We have the vector equation $\overrightarrow{H'H} = -2\overrightarrow{OI}$, or equivalently,

$$\overrightarrow{OH} = \overrightarrow{OH'} + \overrightarrow{H'H} = \overrightarrow{OH'} - 2\overrightarrow{OI}.$$

Then

$$\begin{aligned}\overrightarrow{OF^*} &= \tfrac{2r^2-d^2}{4r^2-d^2}(\overrightarrow{OH'} - 2\overrightarrow{OI}) + \overrightarrow{OI} \\ &= \tfrac{8r^2-3d^2}{4r^2-d^2}\overrightarrow{OI} + \tfrac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'} \\ &= \overrightarrow{OO_{f^*}} + \tfrac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'}.\end{aligned}$$

Hence,

$$\overrightarrow{O_{f^*}F^*} = \tfrac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'}.$$

From [1, Theorem 7.2], the radius of the circle $\mathcal{C}_{h'}$ is $\frac{d^2}{r}$. The proof of the theorem is now complete. ☐

6. The Loci of Centroids and Orthocenters of the Dual Fuhrmann Triangles

Theorem 6.1. *The locus $\mathcal{C}_{g*}$ of centroids of the dual Fuhrmann triangles associated to the $(\mathcal{C}, I)$-locus is a circle with radius $\frac{2d^2}{3r}$ and center O_{g^*} defined by the vector equation*

$$\overrightarrow{OO_{g^*}} = \tfrac{5}{3}\overrightarrow{OI}.$$

Consequently, the locus $\mathcal{C}_{h^*}$ of orthocenters of the dual Fuhrmann triangles associate to the $(\mathcal{C}, I)$-locus is the circle with radius $\frac{4rd^2}{4r^2-d^2}$ and center O_{h^*} defined by the vector equation

$$\overrightarrow{OO_{h^*}} = \tfrac{4r^2+d^2}{4r^2-d^2}\overrightarrow{OI}.$$

Proof. Using Lemma 4.3, the centroid G^* of $\triangle F_a^*F_b^*F_c^*$ is given by the vector equation

$$\overrightarrow{OG^*} = \tfrac{1}{3}\left(\overrightarrow{OF_a^*} + \overrightarrow{OF_b^*} + \overrightarrow{OF_c^*}\right) = \alpha^*\overrightarrow{OA} + \beta^*\overrightarrow{OB} + \gamma^*\overrightarrow{OC},$$

where

$$\begin{aligned}\alpha^* &= \tfrac{a[a^2+(b-c)^2]-2(b+c)(b-c)^2}{3(a+b-c)(a+c-b)(b+c-a)}, \\ \beta^* &= \tfrac{b[b^2+(a-c)^2]-2(a+c)(a-c)^2}{3(a+b-c)(a+c-b)(b+c-a)}, \\ \gamma^* &= \tfrac{c[c^2+(a-b)^2]-2(a+b)(a-b)^2}{3(a+b-c)(a+c-b)(b+c-a)}.\end{aligned}$$

Next,

$$\overrightarrow{O_{g^*}G^*} = \overrightarrow{OG^*} - \overrightarrow{OO_{g^*}} = \overrightarrow{OG^*} - \tfrac{5}{3}\overrightarrow{OI}.$$

To prove that $\mathcal{C}_{g^*}$ is a circle, it suffices to verify the vector equation

$$\overrightarrow{OG^*} - \tfrac{5}{3}\overrightarrow{OI} = \tfrac{2}{3}\overrightarrow{OH'} - \tfrac{4}{3}\overrightarrow{OO}. \tag{6.1}$$

First of all, (6.1) is equivalent to

$$16S^2\left(3\overrightarrow{OG^*}+4\overrightarrow{OO}\right)=16S^2\left(5\overrightarrow{OI}+2\overrightarrow{OH'}\right). \tag{6.2}$$

Using Heron's formula (1.2), (1.3), and Corollary 2.1, the $\overrightarrow{OA}$-coefficients of the left and the right sides of (6.2) are

$$\begin{aligned}16S^2\left[3\alpha^*+\tfrac{4a^2(b^2+c^2-a^2)}{16S^2}\right]&=(a+b+c)\{a[a^2+(b-c)^2]-2(b+c)(b-c)^2\}\\&\quad+4a^2(b^2+c^2-a^2),\\16S^2\left[\tfrac{5a}{a+b+c}+\tfrac{2(b+c-a)}{a+b+c}\right]&=(a+b-c)(a+c-b)(b+c-a)[5a+2(b+c-a)].\end{aligned}$$

By direct calculation, we can check that the two coefficients equal

$$\begin{aligned}&-3a^4-2b^4-2c^4+a^3b-ab^3+a^3c-ac^3\\&\qquad+5a^2b^2+5a^2c^2+4b^2c^2-2a^2bc+ab^2c+abc^2.\end{aligned}$$

Similarly, we can check that $\overrightarrow{OB}$-coefficents (resp., $\overrightarrow{OC}$-coefficients) of the two sides of (6.2) are equal.

Finally, let H^* denote the orthocenter of $\triangle F_a^*F_b^*F_c^*$. By Sylvester's Orthocenter Law,

$$\begin{aligned}\overrightarrow{F^*H^*}&=\overrightarrow{F^*F_a^*}+\overrightarrow{F^*F_b^*}+\overrightarrow{F^*F_c^*}\\&=\overrightarrow{OF_a^*}+\overrightarrow{OF_b^*}+\overrightarrow{OF_c^*}-3\overrightarrow{OF^*}\\&=3\overrightarrow{OG^*}-3\overrightarrow{OF^*}.\end{aligned}$$

By Theorem 5.1 and (6.1),

$$\begin{aligned}\overrightarrow{OH^*}&=3\overrightarrow{OG^*}-2\overrightarrow{OF^*}\\&=3\overrightarrow{OG^*}-2\left(\tfrac{2r^2-d^2}{4r^2-d^2}\overrightarrow{OH'}+\tfrac{8r^2-3d^2}{4r^2-d^2}\overrightarrow{OI}\right)\\&=5\overrightarrow{OI}+2\overrightarrow{OH'}-\left[\tfrac{2(2r^2-d^2)}{4r^2-d^2}\overrightarrow{OH'}+\tfrac{2(8r^2-3d^2)}{4r^2-d^2}\overrightarrow{OI}\right]\\&=\tfrac{4r^2+d^2}{4r^2-d^2}\overrightarrow{OI}+\tfrac{4r^2}{4r^2-d^2}\overrightarrow{OH'}\\&=\overrightarrow{OO_{h^*}}+\tfrac{4r^2}{4r^2-d^2}\overrightarrow{OH'}.\end{aligned}$$

Then

$$\overrightarrow{O_{h^*}H^*}=\tfrac{4r^2}{4r^2-d^2}\overrightarrow{OH'}$$

implies that

$$(O_{h^*}H^*)^2=\tfrac{16r^4}{(4r^2-d^2)^2}(OH')^2=\tfrac{16r^4}{(4r^2-d^2)^2}\left(\tfrac{d^2}{r}\right)^2=\left(\tfrac{4rd^2}{4r^2-d^2}\right)^2.$$

Hence, $\mathcal{C}_{h^*}$ is the circle with center O_{h^*} and radius $\frac{4rd^2}{4r^2-d^2}$. □

References

[1] Chen, Y. and Fisher, R.J., *A Natural Family of Parallel Circles*, International Journal of Geometry, **1**4(3) 2025, 47 - 75.

[2] Altshiller-Court, N., *College Geometry*, Dover Publications, Inc., 2007.

[3] Dung, N.T., *The Feuerbach Point and the Fuhrmann Triangle*, Forum Geometricorum, **16(2016)**, 299 - 311.

[4] Honsberger, R., *Episodes in Nineteenth and Twentieth Century Euclidean Geometry,* Washington: Mathematical Association of America, 1995.

[5] Kimberling, C. *Encyclopedia of Triangle Centers,* available at the web site http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[6] Johnson, R.A., *Advanced Euclidean Geometry: An Elementary Treatise on the Geometry of the Triangle and Circle*, Dover Publications, Inc., New York, 1960.

[7] Weisstein, E.W., *Barycentric Coordinates*, available at the web site https://mathworld.wolfram.com/BarycentricCoordinates.html.

[8] Yiu, P., *Homogeneous Barycentric Coordinates*, Int. J. Math. Educ. Sci. Technol., **31(2000)**, 569 - 578.

Department of Mathematics and Statistics

Idaho State University, Pocatello, ID 83209-8085, U.S.A.

E-mail address: `chenyu@isu.edu`

E-mail address: `robertfisher@isu.edu`