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ON ARBELI OVER THE SAME CHORD

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Abstract. In this article we study some structures related to a family of arbeli possessing a chord of fixed length. We discuss two main aspects. The first deals with a group permuting the arbeli over a common chord. This is connected with an aspect of the arbelos in the framework of hyperbolic geometry. The second aspect deals with certain curves described by noticeable points in the arbelos configuration. In particular we notice the intimate relation of the arbelos with the right strophoid.

1. Introduction

The *arbelos* is a sort of curvilinear triangle ABC whose *sides* $\{\alpha, \beta, \gamma\}$ are three semicircles with common endpoints $\{A, B, C\}$ called *vertices* and defining the *"chord"* AC of the arbelos, which is the diameter of the biggest involved circle (see Figure 1).

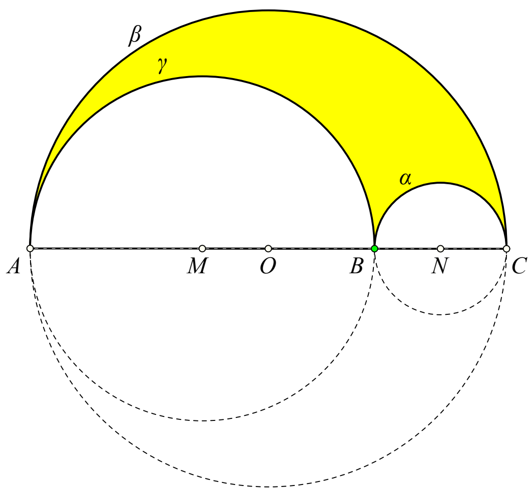

Figure 1. Arbelos

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Its shape originates from a special knife, used by ancient shoemakers for the scratching and cutting of skin. With the arbelos are connected the names of Archimedes, Pappus and Steiner ([6], [17, p.115], [19]). In the sequel we'll denote the arbelos through its vertices: ABC, or through its circles: $\alpha\beta\gamma$ if we want to stress the role of its sides. Most of the time we'll work with the entire circle carrying the particular semicircle-side of the arbelos, denoting it also with the same symbol.

In this article we study the *"family"* of arbeli possessing the same chord AC, i.e. having the same maximal circle β with diameter AC, which call the *"big"* circle or *side* of the arbelos and the two other circles or *sides* $\{\alpha, \gamma\}$, called *"small"* and are defined by a point B of the segment AC. A basic observation is that two arbeli of this family are connected by an inversion or a reflection, which fixes the circle β. The inversion can be computed easily from the data of the two arbeli and can be also represented as composition of two particular inversions, which interchange each of the two arbeli with the *"symmetric arbelos"* AOC on the segment AC.

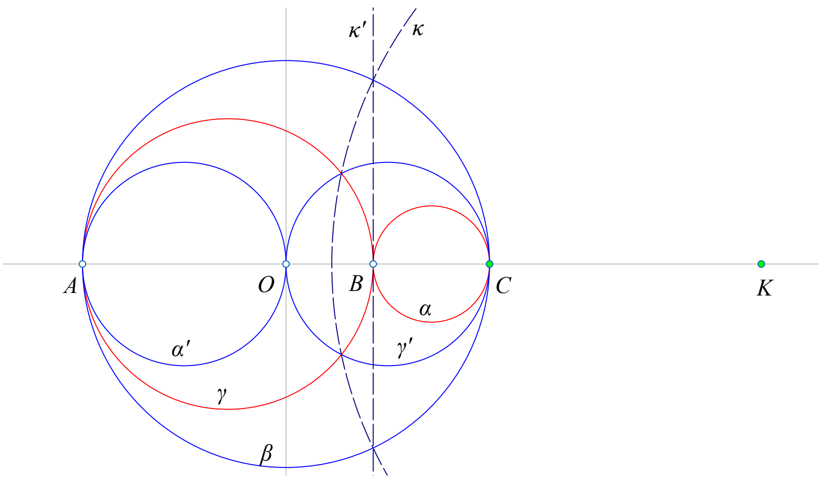

FIGURE 2. κ-inversion mapping ABC to the symmetric AOC

To fix the ideas we discuss briefly the case of such a particular inversion interchanging the symmetric arbelos AOC with an arbitrary ABC (see Figure 2). Considering a Cartesian coordinate system and identifying the common chord AC of the arbeli with the interval $[-r, r]$, the inversion f_κ w.r.t. the circle κ centered at $K(k, 0)$ with radius $R_\kappa^2 = k^2 - r^2$ maps the point $B(r^2/k, 0)$ to the origin O and the arbelos with vertices at the points $\{A(-r, 0), B(\frac{r^2}{k}, 0), C(r, 0)\}$ to the symmetric arbelos of the segment $[-r, r]$. The line κ' parallel to the y-axis through B is the β-polar of K.

2. The group Permuting the arbeli

As we noticed in the introduction, the family of arbeli over the same chord AC is connected with a group of transformations that interchanges any two members of the family. Figure 3 illustrates the case. In this the two arbeli are defined respectively through the points $\{B, B'\}$. There is then an inversion on the circle $\kappa(K)$, orthogonal to β, which interchanges the arbeli $\{\alpha\beta\gamma, \alpha'\beta\gamma'\}$. The circle κ is determined from the couple of

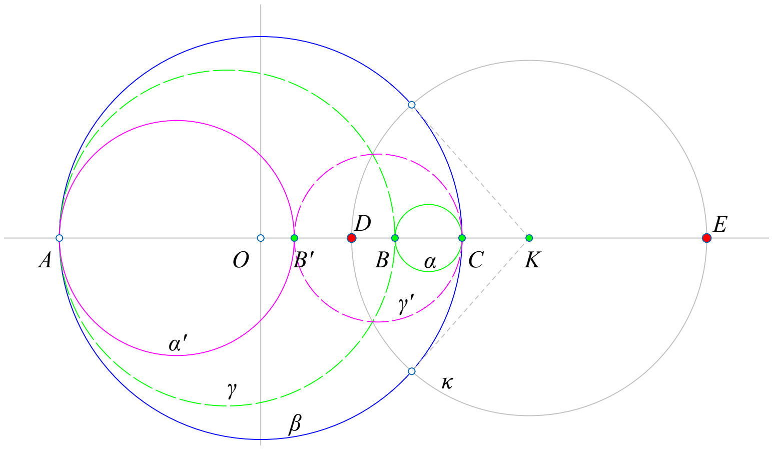

FIGURE 3. Inversion interchanging the arbeli $\{\alpha\beta\gamma, \alpha'\beta\gamma'\}$

points (B, B'). Its diametral points $\{D, E\}$ are the *common harmonics* of the couples of points $\{(A, C), (B, B')\}$. Points $\{D, E\}$, per definition, are simultaneously harmonic conjugate to (A, C) and (B, B') ([16, I, p.350]). All these inversions generate a group $\mathcal{G}$ of transformations that leaves invariant every circle β' through the points $\{A, C\}$.

In [15] we discuss homographies of lines and conics and show, in particular, that an inversion relative to a circle κ induces a homography on every circle β orthogonal to κ. It follows that the group $\mathcal{G}$ can be represented by a group of homographies on β or on any other circle β' passing through $\{A, C\}$. Since these inversions leave invariant each of the two arcs of the circles β' defined by the points $\{A, C\}$, they define also homographies acting on the segment AC and leaving it invariant. The following, easily deducible, formulas give a description of the action of this group and the restriction of this action on the segment AC.

The inversion on the circle $\kappa(K(k, 0))$ orthogonal to $\beta(O, r)$ is described through:

$$(1) \qquad \left\{ \begin{aligned} x' &= \frac{k(x^2+y^2) - (r^2+k^2)x + kr^2}{x^2+y^2-2kx+k^2} \\ y' &= \frac{(k^2-r^2)y}{x^2+y^2-2kx+k^2} \end{aligned} \right\} \quad \text{with} \quad k^2 > r^2 .$$

The composition of two such inversions on circles $\{\kappa(K(k, 0)), \kappa'(K'(k', 0))\}$ is :

$$(2) \qquad \left\{ \begin{aligned} x'' &= \frac{k''(x^2+y^2) + (r^2+k''^2)x + k''r^2}{x^2+y^2+2k''x+k''^2} \\ y'' &= \frac{(k''^2-r^2)y}{x^2+y^2+2k''x+k''^2} \end{aligned} \right\} \quad \text{with} \quad k'' = \frac{r^2-kk'}{k'-k} .$$

We see that this is also the composition of the reflection $(x, y) \mapsto (-x, y)$ on the y-axis followed by the inversion on the circle $\kappa''(K''(k'', 0))$. The reflection can be seen as a limiting case of inversion for $K \to \infty$.

It follows that the restriction of these transformations on AC is expressed through the homographies:

$$(3) \qquad \left\{ \begin{array}{c} x' = \dfrac{kx - r^2}{x-k}, \\ x'' = \dfrac{k''x + r^2}{x+k''}\ , \end{array} \right\}$$

the second kind being again the composition of the reflection $x \mapsto -x$ followed by a homography of the first kind. In terms of the group $PGL(2, \mathbb{R})$ these generate the subgroup consisting of classes of non-zero multiples of matrices of the form

$$(4) \quad \left\{ \begin{pmatrix} k & -r^2 \\ 1 & -k \end{pmatrix}, \begin{pmatrix} k & r^2 \\ 1 & k \end{pmatrix} \quad \text{with} \quad k^2 > r^2 \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

representing the group $\mathcal{G}$ of permutations of the family of arbeli on the chord AC as a subgroup of $PGL(2, \mathbb{R})$. We formulate this as a theorem:

Theorem 2.1. *The group of transformations permuting the arbeli over the chord AC is represented as a subgroup of $PGL(2, \mathbb{R})$ through the non-zero multiples of the matrices (4).*

3. The incircles

Returning to figure 2, the circle κ is orthogonal to all the circles $\{\sigma\}$ passing through the points $\{A, C\}$, which consequently remain invariant under the corresponding inversion f_κ. An interesting such case is the circle σ passing through the contact points $\{P_0, Q_0\}$ of the incircle ξ_0 with the two equal circles of the symmetric arbelos (see Figure 4).

Since the inversion f_κ leaves σ invariant and interchanges the incircles of the symmetric and the general arbelos, it maps these contact points to the corresponding contact points of the incircle of the general arbelos ABC. It follows that this circle σ carries the contact points $\{P, Q\}$ of the incircle of every arbelos of the family.

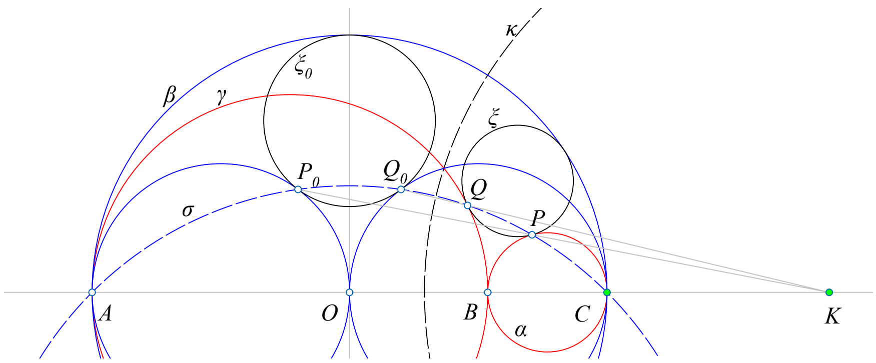

Figure 4. The inversion f_κ interchanging also the corresponding incircles

Figure 5 shows the configuration of the symmetric arbelos on the chord AC. An angle chasing argument, indicated in the figure, shows that the angles $\widehat{P_0AD} = \widehat{P_0CA}$. This implies that the circle σ has its center S'

on β and its radius is $r\sqrt{2}$. In addition it is easily seen that the ratio of the orthogonal sides of the right triangle AOJ is $AO/OJ = 2$, and $\{|EE'| = |IO| = 2r/3.\}$ We have proved the theorem.

Theorem 3.1. *With the notation adopted so far, the incircles of the family of arbeli with common chord AC touch the two inner circles at points $\{P, Q\}$ lying on the circle σ with center S' and radius $r\sqrt{2}$.*

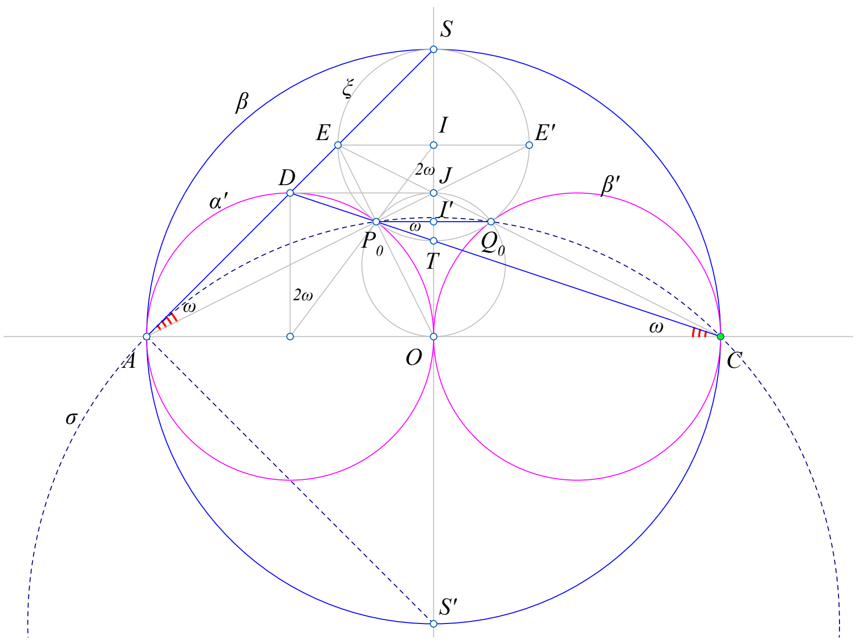

FIGURE 5. The circle σ carrying the contacts $\{P, Q\}$ of the incircles

The following easily to prove lemma, whose proof results from the fact that the center of an inversion interchanging two circles is also a similarity center of the circles, leads to a property known to Archimedes.

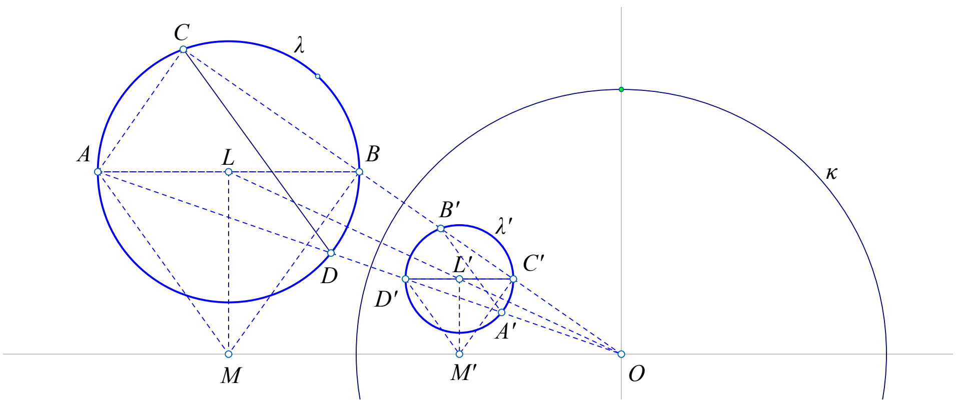

FIGURE 6. Inversion property $\frac{|AB|}{|LM|} = \frac{|D'C'|}{|L'M'|}$

Lemma 3.1. *The inversion f_κ w.r.t. to the circle $\kappa(O,R)$ maps the circle $\lambda(L)$ to the circle $\lambda'(L')$. We consider also a line OM and two diameters $\{AB, D'C'\}$ parallel to this line and the distances $\{LM, L'M'\}$ of the centers from this line. Then, the triangles $\{ABM, D'C'M'\}$ are similar and the ratios are equal: $\frac{|AB|}{|LM|} = \frac{|D'C'|}{|L'M'|}$ (see Figure 6).*

Returning to figure 5, where we noticed that the diameter of the incircle $|EE'| = 2r/3$ of the symmetric arbelos is equal to the distance $|IO|$ of its center from the chord AC of the arbelos, we deduce the analogous property for the general arbelos:

Corollary 3.1. *The diameter of the incircle of an arbelos is equal to the distance of its center from the chord.*

Figure 7 indicates this by the square on the diameter $P'Q'$ of the incircle ξ, whose opposite side is on the chord AC. The figure suggests also another property:

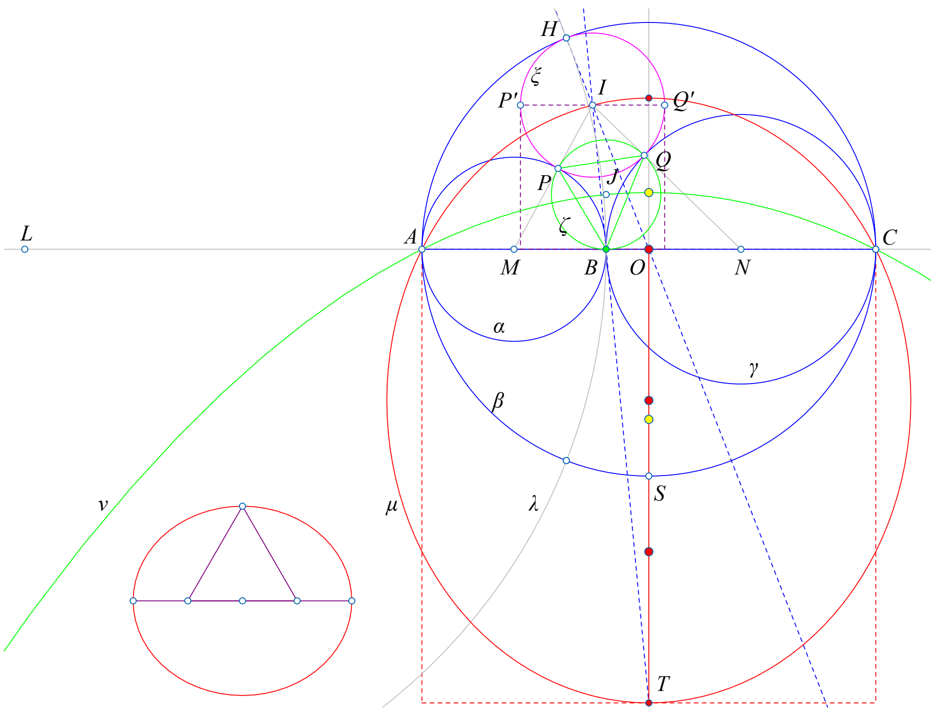

FIGURE 7. Loci $\{\mu, \nu\}$ of centers of incircles and Bankoff circles

Theorem 3.2. *The geometric locus of the centers I of the incircles ξ of the arbeli of the family is an ellipse with center on the y-axis, one focus at the origin O, and eccentricity $1/2$.*

Proof. The proof results by considering I as intersection of two lines $\{I = OH \cap BT\}$. Point H is the contact point of β and ξ and T is the point $(0, -2r)$. That line BT passes through I results from a triangle property relating the *Gergonne* center of a triangle and the symmedian point of its incircle. It is namely well known and easy to prove that the Gergonne center of a triangle coincides with the symmedian point of its intouch triangle, i.e. the triangle of contacts with its incircle [8].

In figure 7 the triangle under consideration is the one of the centers of the three circles IMN. Its intouch triangle is BQP consisting of the contact points of the three circles.

Because the contact point P is similarity center of the circles $\{\alpha, \xi\}$ the triangles $\{ABP, Q'P'P\}$ are similar and $\{P', P, B\}$ are collinear. Analogously the points $\{B, Q, Q'\}$ are collinear and the diameter $P'Q'$ of ξ is antiparallel to PQ. Since I is the middle of $P'Q'$ line BI is a symmedian of the triangle BQP, hence also a Gergonne Cevian of the triangle IMN and passes through T. These, together with some other characteristic properties of the triangles $\{IMN, BQP\}$ have been discussed in [14, Theorem 6.1].

If $(b, 0)$ are the coordinates of B, then those of H can be calculated using the fact that the circle $\lambda(L)$ orthogonal to β and AC and passing through $B(b, 0)$ passes also through the contact point H of $\{\beta, \xi\}$. Point H is on the polar of L relative to β and its coordinates $H(x_H, y_H)$ are

$$x_H = \frac{2br^2}{b^2 + r^2} \quad , \quad y_H = \frac{r(r^2 - b^2)}{b^2 + r^2} \ .$$

Using this, the lines OH and BT can be calculated and their intersection is found to be:

$$x_I = \frac{4br^2}{3r^2 + b^2} \quad , \quad y_I = \frac{2r(r^2 - b^2)}{3r^2 + b^2} \ . \tag{5}$$

Eliminating b from these equations we find that the coordinates (x, y) of I satisfy the equation of an ellipse μ :

$$\frac{x^2}{\frac{4r^2}{3}} + \frac{(y + 2r/3)^2}{\left(\frac{4r}{3}\right)^2} = 1 \ . \tag{6}$$

The eccentricity of this ellipse is $1/2$ and it is characterized by its connection with the equilateral triangle, having its focals at two vertices of the equilateral and passing through the third vertex, as it is seen in the small addition of figure 7 picturing a similar to μ ellipse. Another characteristic of this ellipse is that the five points: the two vertices, the two focals and the center, divide its major diameter in four congruent segments.

Notice that combining corollary 3.1 with formula (5) we obtain the radius of the incircle in the form

$$r_\xi = \frac{r r_\alpha r_\gamma}{r^2 - r_\alpha r_\gamma} \ . \tag{7}$$

We notice also that the homothety with center H and ratio that of the radii of the circles $\{\beta, \xi\}$: $r/r_\xi = (r^2 - r_\alpha r_\gamma)/(r_\alpha r_\gamma)$ transforms the incircle ξ to β and the circles $\{\alpha, \beta, \gamma\}$ to the circles $\{\alpha', \beta', \gamma'\}$ defining an arbelos with incircle β (see Figure 8). This fact can be used to transfer properties of families of arbeli with common incircle, studied in [14] to families of arbeli over a fixed chord.

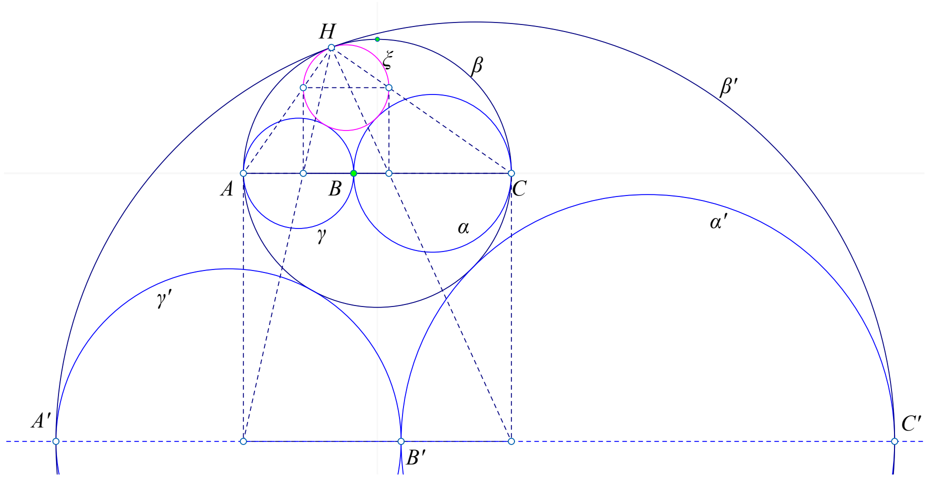

FIGURE 8. Reduction to arbeli with common incircle

4. THE HYPERBOLIC ASPECT

There is an intimate relation of the arbelos with the non-euclidean or hyperbolic geometry of the plane, as this is represented in the Poincaré model of the upper half plane. In this model namely, the arbelos represents a *"triply asymptotic"* (or *"trebly asymptotic"* after Coxeter [4, p.301]) triangle, i.e. a triangle with its vertices $\{A, B, C\}$ on the *"horizon"* of the hyperbolic

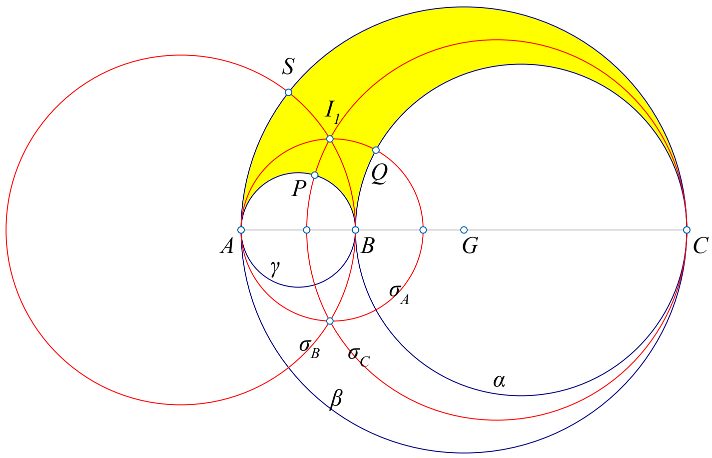

FIGURE 9. The *"altitudes"* of the asymptotic triangle

plane, represented by the union of the x-axis and the point at infinity of the y-axis. Hyperbolic geometry of the plane, and properties of hyperbolic triangles in particular, have been extensively studied. Abundant references in the bibliography can be found in the imp‘ressive PhD thesis by Barbu [3]. The group described in **§** 2 is simply the subgroup of hyperbolic isometries preserving the big (half) circle β of the arbelos and its interior, which, from the hyperbolic geometry viewpoint, is a line and a half-plane defined by that line. Also the incircle of the arbelos coincides with the incircle of the hyperbolic asymptotic triangle.

Here we make some remarks regarding this coincidence. Since all triply asymptotic triangles of the hyperbolic plane are isometric (via hyperbolic congruences) the hyperbolic radius of the incircle is the same constant. Figure 9 shows a preliminary step for its calculation. The three circles $\{\sigma_A, \sigma_B, \sigma_C\}$ are the three *"altitudes"* of the hyperbolic asymptotic triangle. They are represented by circles passing through the vertices $\{A, B, C\}$ and cutting orthogonally the opposite sides. It is easily checked that the σ_A-inversion, which is a hyperbolic isometry (hyperbolic reflection in σ_A) preserves β and interchanges $\{\beta, \gamma\}$, as well as $\{\sigma_B, \sigma_C\}$. Analogous properties hold for $\{\sigma_B, \sigma_C\}$. This implies also that the inversions in these three circles preserve the incircle ξ of the arbelos, which consequently is orthogonal to them and intersects them at its contact points $\{P, Q, S\}$ with the sides of the arbelos (see Figure 10). It follows, that the circles $\{\sigma_A, \sigma_B, \sigma_C\}$ belong to the pencil

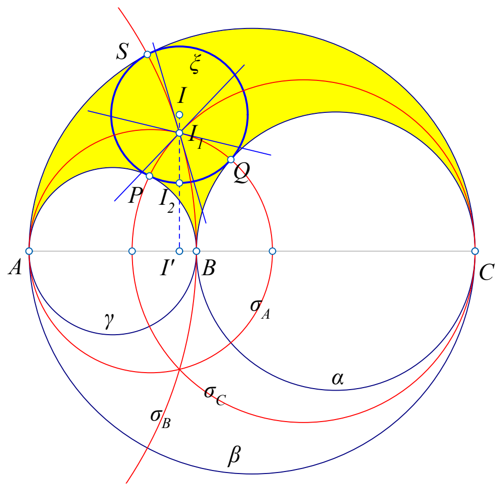

FIGURE 10. The hyperbolic radius I_1I_2 of the asymptotic triangle

$\mathcal{P}$ of intersecting type, containing all the circles orthogonal to the x-axis and the circle ξ. All these circles intersect at a point I_1 and its reflection in the x-axis, the line II_1 being orthogonal to the x-axis. These properties identify ξ with the hyperbolic incircle of the asymptotic triangle and show that I_1 is its center. We have proved most of the following theorem.

Theorem 4.1. *The arbelos' incircle coincides with the incircle of the corresponding asymptotic hyperbolic triangle having the hyperbolic center I_1 and the hyperbolic radius I_1I_2 of hyperbolic length $k \cdot \ln(\sqrt{3})$, where k is a constant depending on the unit-length of the hyperbolic model.*

Proof. The claim about the radius follows from the definition of the hyperbolic distance in this model, which in our case is expressed by the formula $k \cdot \ln(|I'I_1|/|I'I_2|) = k \cdot (\ln(|I'I_1|) - \ln(|I'I_2|))$. But the euclidean length $|I'I_2| = r_\xi$ is the radius of ξ, and $|I'I_1|$ can be computed as the radius of the circle $\sigma(I', |I'I_1|)$ belonging to the pencil $\mathcal{P}$. For this circle holds:

$$|I'I_1|^2 + r_\xi^2 = II'^2 = 4r_\xi^2 \quad \Rightarrow \quad |I'I_1|^2 = 3r_\xi^2 \quad \Rightarrow$$
$$\ln(|I'I_1|) - \ln(|I'I_2|) = \ln(\sqrt{3}r_\xi) - \ln(r_\xi) = \ln(\sqrt{3}) \ .$$

Figure 10 suggests also the equality of the angles between the circles $\{\sigma_A, \sigma_B, \sigma_C\}$. This follows from the fact, that the inversion in each of these circles interchanges the two others, and inversions are conformal mappings. We notice also that I_1 is simultaneously the hyperbolic incenter and the hyperbolic orthocenter of the asymptotic triangle, the analogous property in the Euclidean geometry characterizing the equilateral triangle.

5. The Bankoff circles

The *Bankoff* circle of the arbelos ABC ([2], [18], [12]) is the circle $\zeta = (BPQ)$ passing through the contact points of the three circles: the two small sides $\{\alpha, \gamma\}$ of the arbelos and its incircle ξ (see Figure 7). Its properties have been studied also in [14] and show that it is the inverse of the incircle relative to the circle $\sigma(S, |SA|)$. Using this and formulas (5) we obtain the equation of the locus of the centers J of the Bankoff circles:

$$x^2 + 4ry - r^2 \ = \ 0 \ . \tag{8}$$

This is a parabola ν with focus at $(0, -3r/4)$ and vertex at $(0, r/4)$ (see Figure 7). Writing the equation of the ellipse μ in the equivalent form $\mu : 4x^2 + 3y^2 + 4ry - 4r^2 = 0$, we see easily that ν belongs to the pencil of conics generated by $\{\mu, \beta\}$:

$$\nu \ = \ \mu \ - \ 3\beta \ . \tag{9}$$

We notice that this conic pencil contains the couple of parallel lines represented through equation $y(y - 4r) = 0$. The radius of the Bankoff circle ζ and its congruent Archimedean twins is found to be

$$r_\zeta \ = \ \frac{r_\alpha r_\gamma}{r} \ . \tag{10}$$

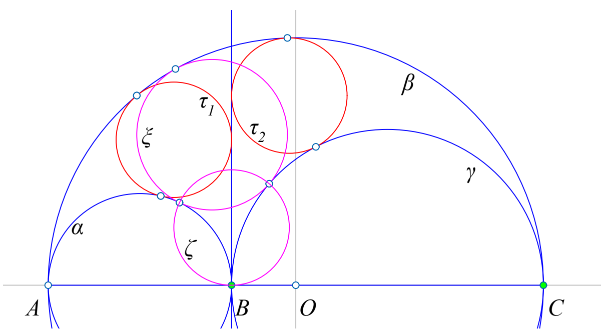

Figure 11. The Archimedean twins $\{\tau_1, \tau_2\}$

6. The Archimedean twins

The Archimedean twins ([18], [1]) are two congruent circles $\{\tau_1, \tau_2\}$ tangent respectively to two *sides* of the arbelos and also tangent to the common tangent of the small sides $\{\alpha, \gamma\}$ (see Figure 11). It is known ([2]) that they are congruent to the Bankoff circle ζ, hence their radius is given by formula (10). Circles related somehow to the Arbelos and being congruent to the Bankoff circle are called *"Archimedean"*. There are several ways to construct such circles. A nice example is given by Okumura [13]. In the following we show that the geometric locus of the centers of circle τ_1 is an arc of a curve of degree 4 (a quartic). For this we use the following simple lemmata.

Lemma 6.1. *Circle $\gamma(O, r_\gamma)$ is tangent internally to circle $\beta(D, r)$ at a point A. Consider a Cartesian coordinate system centered at O and D on its x-axis (see Figure 12). For a point P moving on γ construct the corresponding circle $\tau(K)$ tangent at P to γ and also tangent to β at point Q. Then, K describes an ellipse and the diameter d of τ, as a function of the coordinate x of $P(x, y)$ moving on γ, is given by a homographic relation* [15].

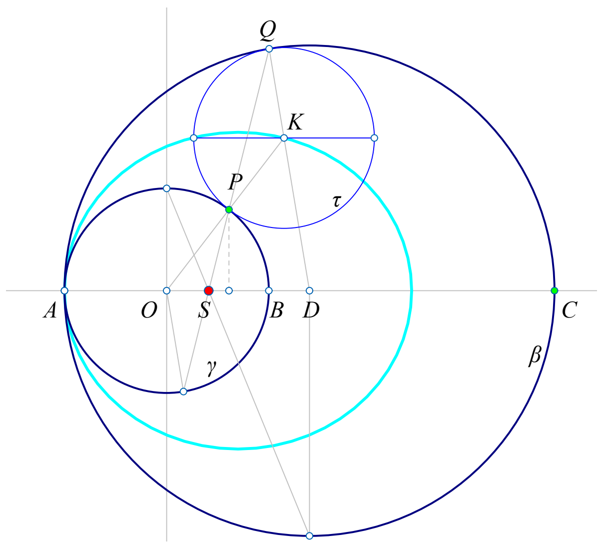

Figure 12. Circle τ tangent to two tangents circles $\{\gamma, \beta\}$

Proof. Line PQ passes through the similarity center S of $\{\gamma, \beta\}$. Using polar coordinates ([10, I, p.306]) w.r.t. O, we see that the center K of τ describes an ellipse with focals at $\{O, D\}$ and

$$d = 2r_\gamma(r - r_\gamma)\frac{r_\gamma + x}{r_\gamma(r_\gamma + r) - (r - r_\gamma)x} . \tag{11}$$

Lemma 6.2. *With the notation of lemma 6.1, considering the special case of P making the circle τ one of the twins, the tangent η of the circles $\{\gamma, \tau\}$ at P passes through C (see Figure 13).*

Proof. The circle $\lambda(A, |AB|)$ is orthogonal to τ. Hence the λ-inversion leaves τ invariant and maps the circle β to the parallel CS' of BX. The

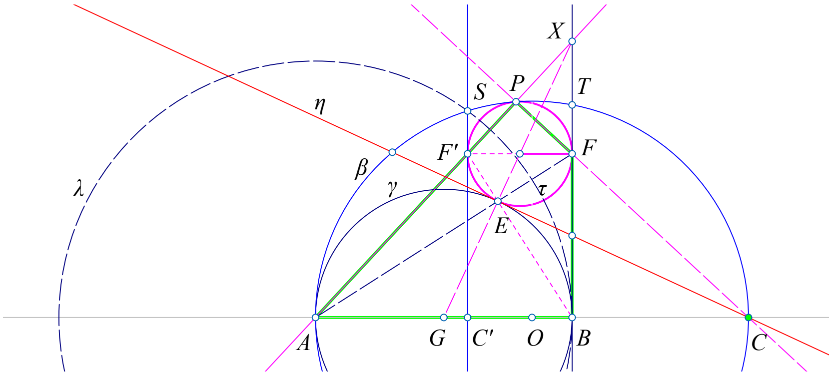

FIGURE 13. The common tangent of $\{\gamma, \tau\}$ passes through C

contact point P of $\{\beta, \tau\}$ maps to the τ-diametral F'. Thus, it is defined the cyclic quadrangle $ABFP$. Sides $\{PF, AB\}$ intersect at C and the equation $|CP||CF| = |CA||CB|$ shows that C is on the radical axis of the circles $\{\kappa, \gamma\}$ which is line η.

Theorem 6.1. *The geometric locus of the center K of the incircle $\tau(K, r_\tau)$ of the curvilinear triangle ABT is an algebraic curve of degree 4 (a quartic) (see Figure 14).*

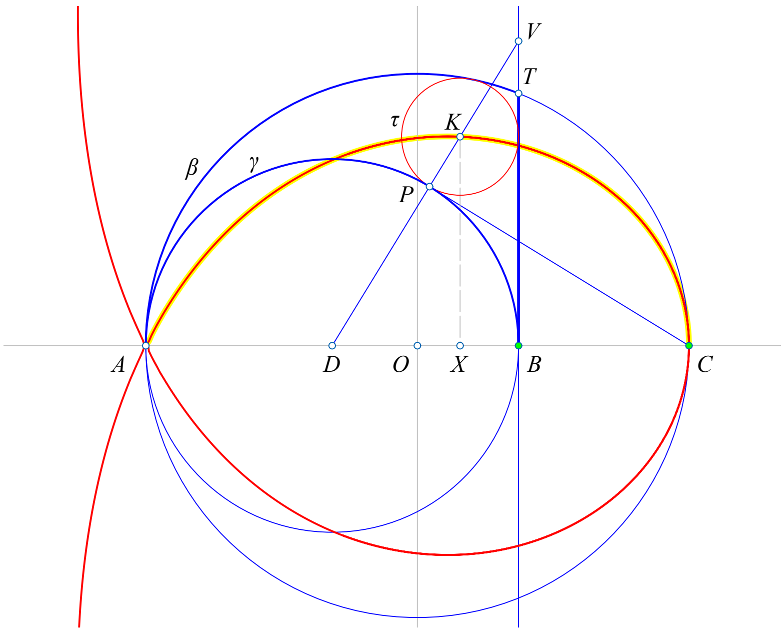

FIGURE 14. The quartic (arc) described by the center K

Proof. The center of the circle $\beta(O, r)$ with diameter AC is at the origin of coordinates and with $A(a, 0)$. The variable circle $\gamma(D, r_\gamma)$ has diameter AB with $B(b, 0)$. The triangles $\{DBV, DPC\}$ are congruent and, since

$|DV| = |DC|$, $K(x,y)$ has

$$|DK| = \frac{r_\gamma + r_\tau}{r_\gamma}|DP| \quad \text{and} \quad \frac{|DX|}{|DK|} = \frac{|DX|}{r_\gamma + r_\tau} = \frac{|DB|}{|DV|} = \frac{|DB|}{|DC|} = \frac{r_\gamma}{2r - r_\gamma} \; .$$

Using formula (11) we see that

$$DX = \frac{r_\gamma^2}{2r - r_\gamma} \quad \text{and} \quad K - D = \frac{2r - r_\gamma}{r}(P - D) \; .$$

From this, expressing $D = (A+B)/2$ and $r_\gamma = |B - A|/2$ and eliminating the coordinate b of $B(b,0)$, we find that $K(x,y)$ satisfies the quartic equation

$$y^4 + 4a(5x - 7a)y^2 + 16a(x+a)(x-a)^2 \; = \; 0 \; . \tag{12}$$

Obviously, the center of the other twin circle describes the symmetric relative to the y-axis of the arc shown in the figure.

7. The right strophoid

Figure 15 displays a configuration in which we consider all possible arbeli not on a chord AC of constant length, but arbeli having the chord VO constant. We'll see in a moment how these relate to arbeli on a constant chord. VO is the common chord of two equal circles $\{\lambda(L), \lambda'(L')\}$ inter-

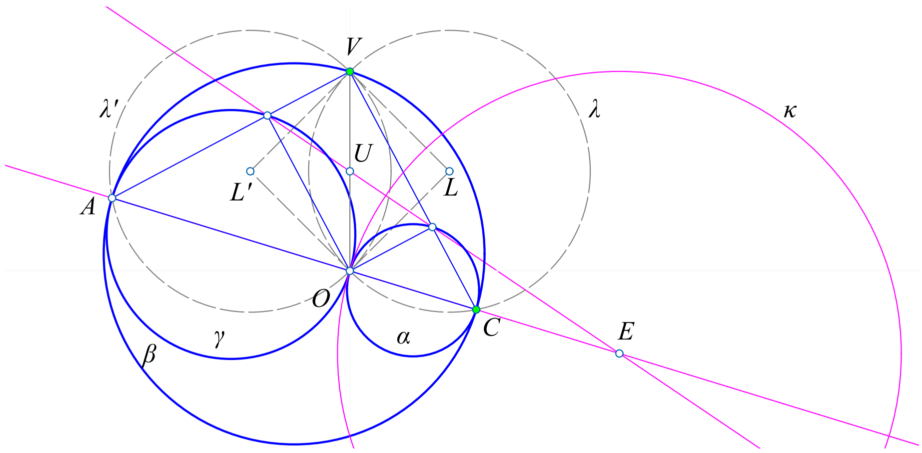

Figure 15. Arbeli with constant chord VO

secting orthogonally. The chord AC passes through O and varies with its endpoints on these circles. Point E is the *"inverting center"* of the circle κ interchanging the two circles $\{\alpha, \gamma\}$ of the arbelos. Obviously, varying the chord AC we obtain all possible ratios OA/OC and through them all possible arbeli, up to similarity.

Theorem 7.1. *Referring to figure 15, as the chord through O of the arbelos AOC varies, the inverting center E describes a right strophoid.*

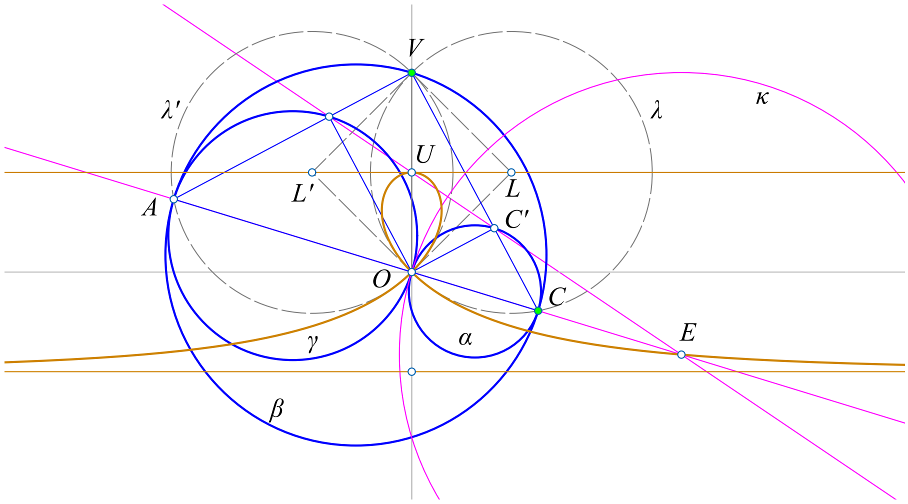

FIGURE 16. Center of the inverting circle describes a right strophoid

Proof. We adopt the Cartesian coordinate system with origin O and y-axis along OV with $U(0,d)$ and $V(0,2d)$ implying:

$$C = d(1+c\sqrt{2}\ ,\ 1+s\sqrt{2}) \quad \text{with} \quad c=\cos(\phi),\ s=\sin(\phi).$$

$$C' = \frac{d}{\sqrt{2}}(c-s, c+s+\sqrt{2}) \quad \text{defining the lines} \quad OC\ ,\ UC' \quad \text{intersecting at}$$

$$(x,y) \quad = \quad \frac{d(s-c)}{\sqrt{2}(s\sqrt{2}+1)}\left(c\sqrt{2}+1\ ,\ s\sqrt{2}+1\right)\ .$$

Eliminating $\{c,s\}$ from these equations we obtain the equation of the right strophoid

$$y(x^2+y^2)+d(x^2-y^2)=0\ . \tag{13}$$

It is easily checked that the projections $\{A',C'\}$ of O on the two lines $\{VA,VC\}$ define another diameter of the circle κ on the fixed diameter OV (see Figure 17). Also the line OX orthogonal to AC at O bisects the right angle $\widehat{A'OC'}$ and its intersection O' with line UE is also on the strophoid. Since the lines $\{OO',OE\}$ are bisectors of the angle $\widehat{A'OC'}$, $(A'C';O'E)=-1$ is a harmonic quadruple and the ratio of the diameters $A'O'/O'C'$ is equal to $A'O/C'E$ which is equal to the ratio of diameters AO/OC. This implies that the arbeli AOC and $A'O'C'$ are similar, the last one having a chord of constant length.

The aforementioned harmonic quadruple implies also that the circles $\{\kappa,\delta\}$ on diameters $\{A'C',O'E\}$ are orthogonal and produce a direct link of the strophoid with the arbeli $\{A'O'C'\}$. In fact, the inverting circle δ of the arbelos $A'O'C'$ belongs to the pencil of circles tangent to the y-axis at the origin and the points $\{E,O'\}$ of the strophoid belong to the diameters of these circles through the center U of κ.

This shows that the right strophoid can be generated using a tangent pencil of circles. For this take a point U on the common tangent of its

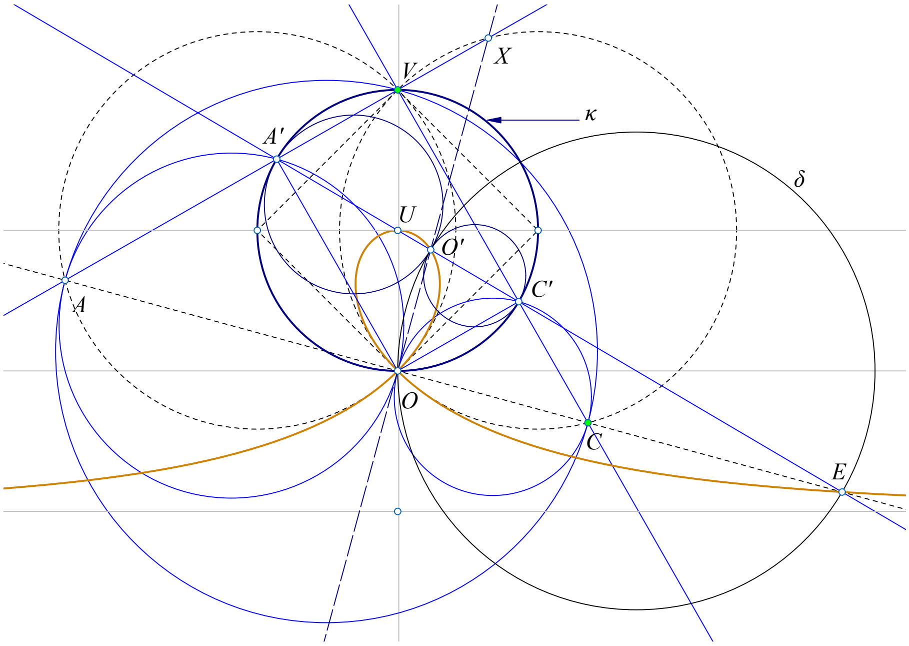

FIGURE 17. Arbeli on chords $\{A'O'C'\}$ of constant length

members (radical axis of the pencil). The diametral points of the member-circles defined by lines through U describe a right strophoid ([9, p.96], [7]). This is a special case of a general geometric generation of all possible circular cubics exposed in [11].

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