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ON THE STEINER CIRCUMELLIPSE OF A TRIANGLE

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Abstract. In this work, we study some properties of the Steiner circumellipse of a triangle, providing, in particular, two methods to find the tangent to it at a given point using tools from Triangle Geometry.

In addition, a method is introduced to consider any given ellipse as the Steiner circumellipse of a triangle, which, together with the previous results, provides a new construction for the tangent to an ellipse at a point.

1. Introduction

Given a triangle ABC with centroid G, the *Steiner circumellipse* is the circumconic with center G. It is named after Jakob Steiner (1796–1863), a Swiss mathematician who made significant contributions to Geometry (for example, the Chasles-Steiner theorem on the projective definition of conics is well-known).

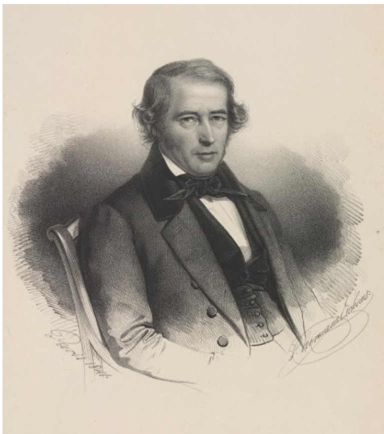

Figure 1. Jakob Steiner (1796–1863), around 1841.

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In Triangle Geometry, the Steiner circumellipse has been extensively studied (see, e.g., [4]). In this work, we present some interesting properties of this ellipse.

The determination of the tangent to an ellipse at a point is a well-studied classical problem. Here, we introduce a new construction based on determining the tangent to the Steiner circumellipse of a triangle. To this end, we pose the natural question: given any ellipse, can we determine a triangle such that this ellipse is the Steiner circumellipse of that triangle?

2. Some properties of the Steiner circumellipse

For an introduction to barycentric coordinates, see [2]. For a more in-depth and detailed study of Triangle Geometry and barycentric coordinates, refer to [3] and [4].

We recall that the Steiner circumellipse has the equation in barycentric coordinates given by $yz + zx + xy = 0$.

The Steiner inellipse of the triangle ABC is the inconic whose perspector is the centroid G of ABC. It can be verified that the Steiner circumellipse is the result of applying a homothety with center G and ratio 2 to the Steiner inellipse (see Figure 2).

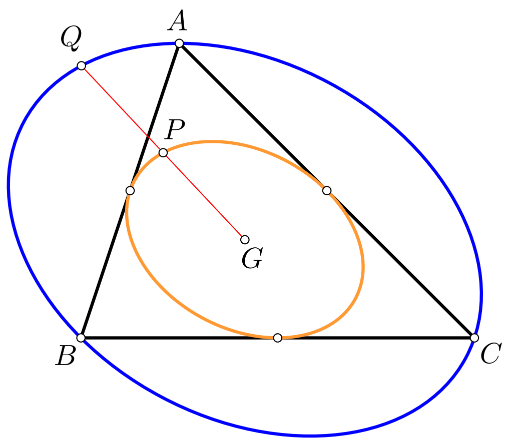

Figure 2. The Steiner circumellipse and inellipse are homothetic.

The following result provides a relationship between inconics and circumconics. In particular, it relates the Steiner inellipse and circumellipse of a triangle.

Proposition 1. *The dual conic of the circumconic with perspector at a given point P is the inconic with perspector at the isotomic conjugate $P^\bullet$ of P.*

In particular, the dual conic of the Steiner circumellipse is the Steiner inellipse.

Proof. The first part of the statement originally appears in section 10.6.4 of [4]. We include a proof here to complete the details.

If $P = (p : q : r)$, the circumconic $\mathcal{C}_P$ with perspector P has equation $pyz + qzx + rxy = 0$, whose associated matrix, up to scalar multiples, is

$$M = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix}.$$

The matrix of the dual conic $\mathcal{C}^{\#}$ of $\mathcal{C}_P$ is, up to scalar multiples, the adjugate matrix of M:

$$M^{\#} = \begin{pmatrix} -p^2 & pq & pr \\ pq & -q^2 & qr \\ pr & qr & -r^2 \end{pmatrix}.$$

Thus, the equation of $\mathcal{C}^{\#}$ will be

$$-p^2x^2 - q^2y^2 - r^2z^2 + 2pqxy + 2prxz + 2qryz = 0.$$

On the other hand, the equation of the inconic with perspector at the isotomic conjugate $P^\bullet = (\frac{1}{p} : \frac{1}{q} : \frac{1}{r})$ is

$$p^2x^2 + q^2y^2 + r^2z^2 - 2qryz - 2rpzx - 2pqxy = 0,$$

and we see that it coincides with $\mathcal{C}^{\#}$.

The last part then immediately follows from the fact that $G^\bullet = G$.

We recall an important property of the Steiner circumellipse:

Proposition 2. *The Steiner circumellipse of a triangle is the image under isotomic conjugation of the points at infinity.*

Proof. If $J = (x : y : z)$ is a point at infinity, we have $x + y + z = 0$. The isotomic conjugate of J is the point $J^\bullet = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$, which satisfies

$$\frac{1}{y} \cdot \frac{1}{z} + \frac{1}{z} \cdot \frac{1}{x} + \frac{1}{x} \cdot \frac{1}{y} = \frac{x + y + z}{xyz} = 0,$$

so that $J^\bullet$ lies on the Steiner circumellipse.

Next, we study in the following result the circumconics with perspector on the Steiner circumellipse, which additionally provides a characterization of the Steiner circumellipse through the notion of the complement of a point:

Theorem 1. *Let P be a point on the Steiner circumellipse $\mathcal{S}$ of the reference triangle ABC.*

(a) *The circumconic $\mathcal{C}_P$ with perspector P is a hyperbola whose points at infinity are the isotomic conjugates of the intersection points Q_1, Q_2 of $\mathcal{S}$ with the trilinear polar $\mathrm{pt}_{P^\bullet}$ of the isotomic conjugate $P^\bullet$ of P (see Figure 3).*

(b) *The envelope of the lines $\mathrm{pt}_{P^\bullet}$ as P varies on $\mathcal{S}$ is the Steiner inellipse of ABC. In fact, the point of tangency of $\mathrm{pt}_{P^\bullet}$ with the Steiner inellipse of ABC is the complement P' of P.*

(c) *The centroid of triangle PQ_1Q_2 coincides with the centroid G of ABC, so the complement P' of P lies on $\mathrm{pt}_{P^\bullet}$, and thus its isotomic conjugate $R = (P')^\bullet$ will belong to the hyperbola $\mathcal{C}_P$.*

This last property characterizes the Steiner circumellipse. That is, $P \in \mathcal{S}$ if and only if its complement P' lies on $\mathrm{pt}_{P^\bullet}$.

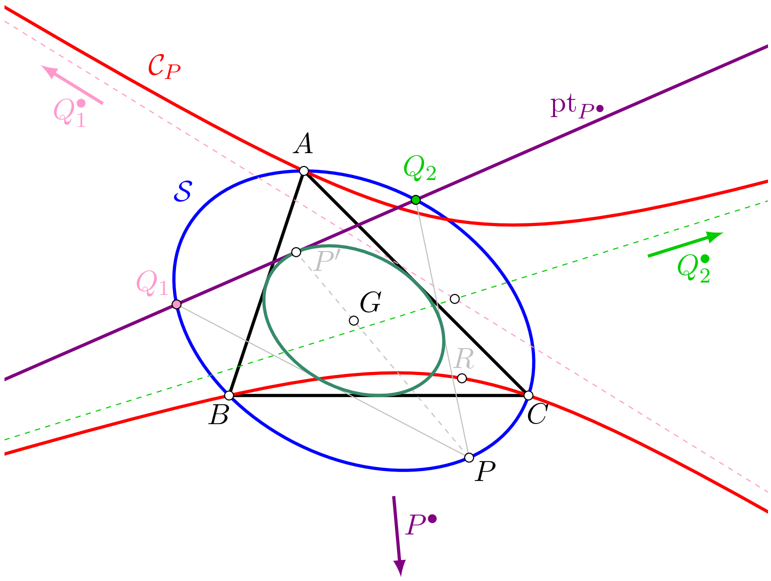

FIGURE 3. Properties of the circumconic with perspector on the Steiner circumellipse.

Proof. (a) Let us consider an arbitrary point $P = (-t : 1+t : t+t^2)$ on $\mathcal{S}$. Therefore, the equation of the circumconic $\mathcal{C}_P$ with perspector P will be

$$\mathcal{C}_P : \ -tyz + (1+t)zx + (t+t^2)xy = 0.$$

The discriminant of $\mathcal{C}_P$ is

$$\Delta = \frac{(1+t+t^2)^2}{4} > 0,$$

so $\mathcal{C}_P$ is always a hyperbola.

We know that the isotomic conjugate $P^\bullet$ of P is a point at infinity, and we have that the intersection of $\mathcal{S} : \ yz+zx+xy = 0$ with the trilinear polar of $P^\bullet$, $\mathrm{pt}_{P^\bullet} : \ -tx + (1+t)y + (t+t^2)z = 0$, consists of the points

$$Q_1 = (1+t : t+t^2 : -t), \quad Q_2 = (t+t^2 : -t : 1+t),$$

so that, since $\mathcal{S}$ is the transform under isotomic conjugation of the line at infinity, its isotomic conjugates will be the points at infinity of $\mathcal{C}_P$:

$$J_1 = Q_1^\bullet = (t : 1 : -1-t), \quad J_2 = Q_2^\bullet = (1 : -1-t : t).$$

(b) The envelope of the lines

$$\mathrm{pt}_{P^\bullet} : \ -tx + (1+t)y + (t+t^2)z = 0$$

is the conic given by the equation

$$(-1 \cdot x + 1 \cdot y + 1 \cdot z)^2 - 4(0 \cdot x + 1 \cdot y + 0 \cdot z) \cdot (0 \cdot x + 0 \cdot y + 1 \cdot z) = 0,$$

which, upon simplification, is equivalent to

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0,$$

which is precisely the equation of the Steiner inellipse of ABC.

It can be verified that the complement P' of P lies on the Steiner inellipse of ABC, and that the tangent line at this point (which is the polar of P' with respect to this conic) coincides with the line $\mathrm{pt}_{P\bullet}$.

(c) Since the sum of the coordinates of P, Q_1, and Q_2 is the same (equal to $1+t+t^2$), the centroid of triangle PQ_1Q_2 is the point

$$P + Q_1 + Q_2 = (1+t+t^2 : 1+t+t^2 : 1+t+t^2) = (1:1:1) = G.$$

Finally, if $X = (u:v:w)$, its complement $X' = (v+w : w+u : u+v)$ will lie on $\mathrm{pt}_{X\bullet} : ux+vy+wz = 0$ if and only if $vw+wu+uv = 0$, which is equivalent to saying that $X \in \mathcal{S}$.

3. Every ellipse is the Steiner circumellipse of a triangle

To show that any ellipse is the Steiner circumellipse of a triangle, we need to answer the following

Problem 1. *Given an arbitrary ellipse $\mathcal{E}$, to determine three points A, B, C on $\mathcal{E}$ such that $\mathcal{E}$ is the Steiner circumellipse of triangle ABC.*

We provide a solution to Problem 1 by finding the construction of a triangle ABC such that $\mathcal{E}$ is the Steiner circumellipse of ABC.

The following construction allows us to consider any ellipse as the Steiner circumellipse of some triangle:

Construction 1. i) *Suppose we are given an ellipse $\mathcal{E}$. First, let us find the center G of $\mathcal{E}$. Here we recall a graphical method to determine it. We take an arbitrary chord c and draw a second chord d parallel to c. By connecting the midpoints of c and d, we draw the chord m. Then the center G of $\mathcal{E}$ is the midpoint of the chord m (see Figure 4).*

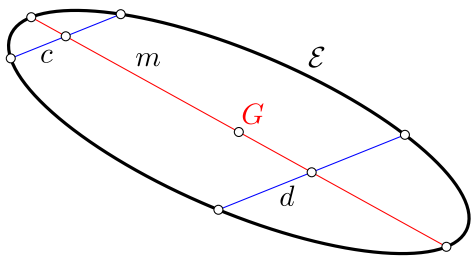

Figure 4. Construction of the center of an ellipse.

ii) *Suppose we know how to find the tangent at a given point A on $\mathcal{E}$. This is not restrictive, as we can take a point P outside $\mathcal{E}$, find the polar r_P of P with respect to $\mathcal{E}$, and take as point A one of the intersection points of the polar r_P with $\mathcal{E}$; the line PA is then the tangent to $\mathcal{E}$ at A.*

iii) *We find the point M that divides AG in the ratio $3 : -1$ (equivalently, the image of A under the homothety centered at G with a ratio of $-1/2$).*

iv) *We draw the line parallel to the tangent to $\mathcal{E}$ at A through point M, and the intersection of this parallel line with $\mathcal{E}$ will give us the other vertices B and C (see Figure 5).*

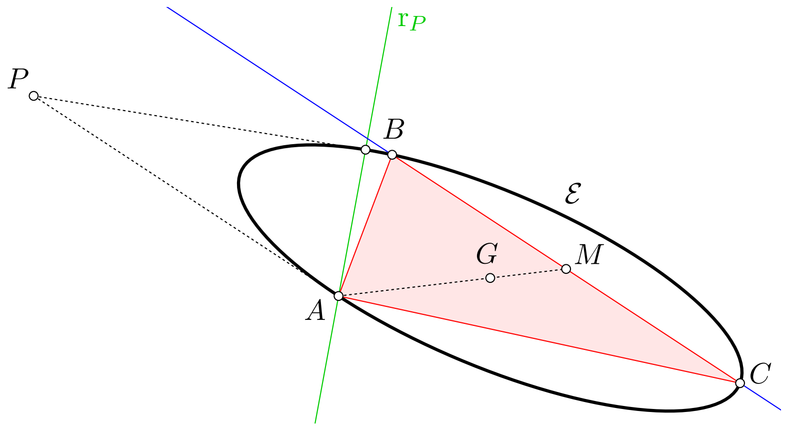

Figure 5. Finding a triangle whose Steiner circumellipse is $\mathcal{E}$.

4. Tangent to the Steiner circumellipse

Our starting point is the following problem, proposed by Francisco Javier García Capitán on February 18, 2016, on Facebook (see [1]). We provide a solution here using barycentric coordinates.

Problem 2. *Let ABC be a triangle and $\mathcal{S}$ its Steiner circumellipse. If P is on $\mathcal{S}$ and $A'B'C'$ is the cevian triangle of P, the centroids G_a, G_b, G_c of the triangles $AB'C'$, $BC'A'$, and $CA'B'$ are collinear, and the line they determine is the tangent to $\mathcal{S}$ at P (see Figure 6).*

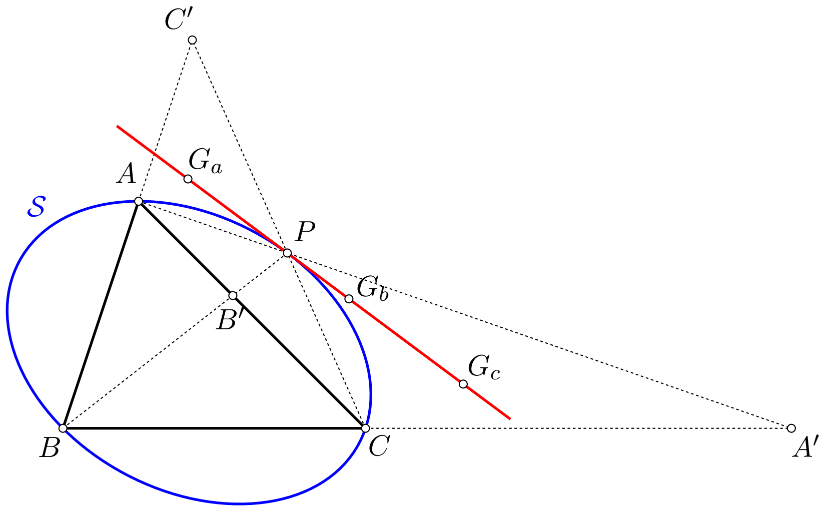

Figure 6. The tangent to $\mathcal{S}$ at P is the line $G_aG_bG_c$.

Proof. Let $P = (u : v : w) \in \mathcal{S}$, such that the vertices of the cevian triangle of P are $A' = (0 : v : w)$, $B' = (u : 0 : w)$, and $C' = (u : v : 0)$. The tangent to $\mathcal{S}$ at P is given by the polar of P with respect to $\mathcal{S}$, which is

$$(u, v, w)\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (v + w)x + (u + w)y + (v + u)z = 0.$$

To find G_a, we write $A = ((u+v)(u+w) : 0 : 0)$, $B' = ((u+v)u : 0 : (u+v)w)$, and $C' = ((u + w)u : (u + w)v : 0)$ (so that the sum of the coordinates of the three points is the same). Thus, we have

$$G_a = A + B' + C' = (3u^2 + 2u(v + w) + vw : (u + w)v : (u + v)w).$$

By cyclic transformation, we have

$$G_b = ((v + w)u : 3v^2 + 2v(w + u) + wu : (u + v)w),$$
$$G_c = ((v + w)u : (u + w)v : 3w^2 + 2w(u + v) + uv).$$

To see that G_a, G_b, and G_c are collinear, we compute the determinant formed by their coordinates:

$$\begin{vmatrix} 3u^2 + 2u(v + w) + vw & (u + w)v & (u + v)w \\ (v + w)u & 3v^2 + 2v(w + u) + wu & (u + v)w \\ (v + w)u & (u + w)v & 3w^2 + 2w(u + v) + uv \end{vmatrix}$$
$$= 6(u + v)(v + w)(w + u)(u + v + w)(uv + vw + wu) = 0,$$

which is zero because the last factor vanishes (since P is on $\mathcal{S}$).

To show that the line determined by G_a, G_b, and G_c coincides with the tangent to $\mathcal{S}$ at P, it suffices to check that G_a satisfies the equation of the polar of P with respect to $\mathcal{S}$. Substituting x, y, and z with the coordinates of G_a in this equation and simplifying yields the expression

$$2(2u + v + w)(uv + vw + wu) = 0,$$

thus completing the proof.

Now a natural question arises: how the point at infinity of the tangent to $\mathcal{S}$ at P can be expressed in terms of P? We answer this by the following theorem that also provides another way to find the tangent to $\mathcal{S}$ at a point:

Theorem 2. *The point at infinity of the tangent to $\mathcal{S}$ at a point P is the isotomic conjugate of the second intersection point of $\mathcal{S}$ with the line passing through P in the direction of the isotomic conjugate $P^\bullet$ of P.*

Proof. Since $P \in \mathcal{S}$, its isotomic conjugate $J = P^\bullet$ is a point at infinity. Let Q be the second intersection point of the line PJ with $\mathcal{S}$. It must be verified that the isotomic conjugate $Q^\bullet$ is the point at infinity of the tangent to $\mathcal{S}$ at P (see Figure 7).

Indeed, if we take an arbitrary point $P = (-t : 1 + t : t + t^2)$ on $\mathcal{S}$, we have $J = P^\bullet = (1 + t : -t : -1)$, and the second intersection point of PJ with $\mathcal{S}$ is given by the point

$$Q = (t(2 + t)(1 + 2t) : -(-1 + t)(1 + t)(1 + 2t) : (-1 + t)t(1 + t)(2 + t)).$$

The isotomic conjugate of Q is

$$Q^\bullet = (1 - t^2 : t(2 + t) : -1 - 2t),$$

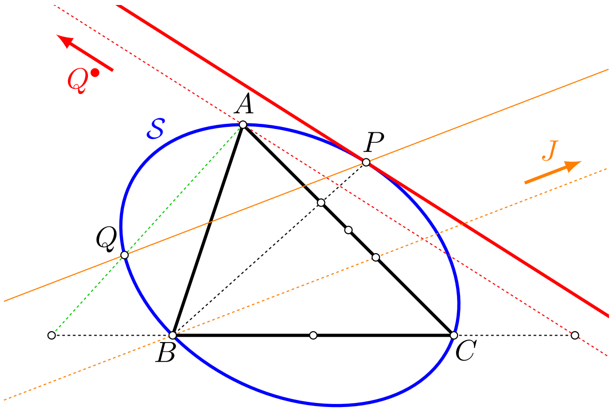

FIGURE 7. Identifying the point at infinity of the tangent to $\mathcal{S}$ at P.

and it can be verified that it is indeed the point at infinity of the tangent to $\mathcal{S}$ at P.

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