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Congruence Theorems for triangles and convex quadrilaterals involving heights

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Abstract. We investigate all Congruence Theorems for triangles involving heights, and possibly also sides and angles. Moreover, we prove that a convex quadrilateral, as long as it is not a parallelogram, is determined up to congruence by its heights. Finally, we prove two Congruence Theorems involving heights that hold for all convex n-gons.

1. Introduction

Two subsets of the Euclidean plane $\mathbb{R}^2$ are *congruent* if one can be transformed to the other via some sequence of translations, rotations, and reflections. *Congruence Theorems* provide criteria for subsets of $\mathbb{R}^2$ to be congruent. As an example, the Side-Side-Side (SSS) Congruence Theorem says that two triangles with all the same side lengths must be congruent. See [5] for a general introduction and discussion on Congruence Theorems.

Congruence Theorems for triangles are well known: they are taught in high schools and, historically speaking, go back all the way to Euclid [8]. Such Congruence Theorems typically involve lengths of sides and measures of angles. Congruence Theorems for convex quadrilaterals have been studied using the classical approach for triangles (see [7]) and also with a more modern point of view (see [2]). Most recently, in [6], Perucca and Torti proved Congruence Theorems for convex n-gons, where n can be arbitrarily large, with the given information being lengths of certain sides and diagonals and the measures of certain angles. See also [1] about so-called generic Congruence Theorems for polygons and polyhedra in higher dimensions. For an axiomatic introduction to Congruence Theorems for polygons, we refer the reader to [3], which is also interesting from a didactic standpoint.

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For triangles, *heights* are defined as follows. Given a triangle in $\mathbb{R}^2$, let A be a vertex and let ℓ_A be the extension of the opposite side of the triangle to an infinite line in $\mathbb{R}^2$. The unique line segment L_A stemming from A that hits ℓ_A orthogonally is called the altitude of A, and the length h_A of L_A is called the height of A. Observe that, if the angle at one of the other vertices is obtuse, then L_A is not contained inside the triangle. The definitions of altitude and height can easily be generalized for polygons (see section 3). Altitudes and heights of triangles have been studied since the 3rd century BC; for example, it is attributed to Archimedes that the three (possibly extended) altitudes of a triangle intersect at a single point (the orthocenter). It might then come as a surprise that there does not appear to be any systematic treatment of Congruence Theorems for convex polygons or even triangles involving heights.

In this paper, we study Congruence Theorems involving heights for triangles, convex quadrilaterals, and convex n-gons. For triangles, we obtain all possible Congruence Theorems in which the given information includes a non-empty subset of the heights and (possibly) the side lengths and angles, and we show in the process which extra Congruence Theorems apply to acute triangles. We show by examples that for every result that is specialized to acute triangles, the acuteness assumption is necessary (see, for instance, Examples 2.1 and 2.2). As well, for every case in which it is not obvious that a Congruence Theorem doesn't hold, we have a counterexample.

For convex quadrilaterals and beyond, we mostly give sufficient criteria for congruence, but no complete classification. It seems reasonable that one could find all possible Congruence Theorems involving heights for convex quadrilaterals. The methods presented in this paper (in particular in the proof of Theorem 1.5) should be of use in carrying out that task, and we opted for leaving convex quadrilaterals for a future investigation.

Before stating our theorems formally, we specify the standard notation that we will use throughout the paper. For a triangle, we label the vertices as A, B, and C, and we call the triangle ABC. We denote the corresponding angles by α, β and γ, and the heights by h_A, h_B and h_C. For the (unoriented) side connecting vertices A to B, we write AB, and for the length we write $\overline{AB}$. For a triangle $A'B'C'$ we use the corresponding notation; for example, α' is the angle at A'.

Firstly, all heights determine a triangle.

Theorem 1.1. *A triangle is determined up to congruence if we know its three heights.*

Next, we consider knowing one side and two heights, or two sides and one height, and then knowing two angles and one height, or one angle and two heights.

Theorem 1.2. *A triangle ABC is determined up to congruence if we know*

(*i*) $\overline{AB}$, h_A *and* h_B *and the triangle is acute;*
(*ii*) $\overline{AB}$, $\overline{BC}$ *and* h_B*;*
(*iii*) $\overline{AB}$, h_A *and* h_C *and the triangle is acute;*
(*iv*) $\overline{AB}$, $\overline{BC}$ *and* h_C.

Theorem 1.3. *A triangle ABC is determined up to congruence if we know*

(*i*) *α, β and any one of h_A, h_B, and h_C;*
(*ii*) *α, h_B and h_C;*
(*iii*) *α, h_A, h_B and the triangle is acute.*

Our last result for triangles concerns the case of knowing one angle, one side, and one height.

Theorem 1.4. *A triangle ABC is determined up to congruence if we know*
(*i*) *α, $\overline{AB}$, and h_A and the triangle is acute;*
(*ii*) *α, $\overline{BC}$ and h_A;*
(*iii*) *α, $\overline{BC}$, and h_B and the triangle is acute;*
(*iv*) *α, $\overline{CA}$ and h_B.*

Our Examples 2.1, 2.2, and 2.3 for triangles show explicitly that dropping the acuteness assumption leads to a number of extra pathologies and a lack of rigidity. With this in mind, for $n \geq 4$, we restrict ourselves to convex n-gons.

For convex quadrilaterals, we show that knowing all the heights is not enough to specify the congruence class. Indeed, it is possible to construct infinitely many non-congruent parallelograms with the same heights (see Example 3.1). However, we prove that, for convex quadrilaterals, parallelism is the only obstruction.

Theorem 1.5. *A convex quadrilateral that is not a parallelogram is determined up to congruence by its heights.*

Finally, we present two results that apply to convex n-gons, where $n \geq 3$ is arbitrary.

Theorem 1.6. *A convex polygon is determined up to congruence if we know all heights toward two neighboring sides and the angle between those sides.*

Theorem 1.7. *A convex polygon is determined up to congruence by the lenghts of its sides and its heights.*

There are plenty of directions for future research: as we indicated above, it seems within reach to provide a complete list of Congruence Theorems for convex quadrilaterals. The case of pentagons might be manageable too, and would help point to more general phenomena. One could also try for Congruence Theorems for non-convex quadrilaterals and beyond. As well, one could instead ask for Similarity Theorems (two subsets of $\mathbb{R}^2$ are similar if they are related by scaling), rather than Congruence Theorems. Another intriguing idea is to study Congruence Theorems involving heights or other quantities for triangles or convex quadrilaterals in the hyperbolic plane. Finally, one could study further shapes beyond polygons, allowing for some curvature away from vertices, or even working with polytopes in higher dimensional Euclidean spaces (research in the higher dimensional setting is rather limited at this point, see [1]).

Aside from being independently useful and interesting, the results above and the examples within the main paper can be turned into exercises for motivated students and problems for mathematical competitions. Pursuing some portion of the directions above could be an accessible research project for undergraduates. School students and math enthusiasts can understand the statements and the research questions presented in this article; we hope that people will find these Congruence Theorems thought-provoking, and will feel inspired to try to come up with their own Congruence Theorems.

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2. Congruence theorems for triangles involving heights

Throughout this section, we keep all of the notation from the introduction.

In the work below, we will use the Angle-Side-Angle (ASA) and Side-Angle-Side Congruence Theorems. The ASA Congruence Theorem says that, if ABC and $A'B'C'$ satisfy $\alpha = \alpha'$, $\beta = \beta'$, and $\overline{AB} = \overline{A'B'}$, then ABC is congruent to $A'B'C'$. The Side-Angle-Side Congruence Theorem says that if $\overline{AB} = \overline{A'B'}$, $\overline{CA} = \overline{C'A'}$, and $\alpha = \alpha'$, then ABC is congruent to $A'B'C'$.

We also need the following definition and notation: for a vertex A, the foot F_A is the unique intersection point of L_A and ℓ_A. We will often make use of the observation that for a triangle ABC, the three points A, B, and F_A form a right triangle. If the angle β is acute, then the angle of ABF_A at B is β, and otherwise it is $\pi - \beta$.

2.1. Three heights.

Proof of Theorem 1.1. Suppose we have two triangles ABC and $A'B'C'$ such that $h_A = h_{A'}$, $h_B = h_{B'}$ and $h_C = h_{C'}$. The area Area(ABC) of ABC is equal to all three of $\frac{1}{2}h_A\overline{BC}$, $\frac{1}{2}h_B\overline{CA}$ and $\frac{1}{2}h_C\overline{AB}$, and similar for $A'B'C'$. Let $\lambda > 0$ be the constant such that Area$(ABC) = \lambda$Area$(A'B'C')$. We deduce from the equality of the heights that the triangles are similar, with proportionality constant λ. The equality $h_{A'} = h_A = \lambda h_{A'}$ forces $\lambda = 1$ and moreover shows that ABC and $A'B'C'$ are congruent. □

2.2. One side and two heights, or two sides and one height.

Proof of Theorem 1.2. Toward (i), note that, as β is acute, it is determined by the formula $\sin\beta = h_A/\overline{AB}$. Assuming (i), α is determined by the analogous formula, and the Congruence Theorem follows from the ASA congruence theorem.

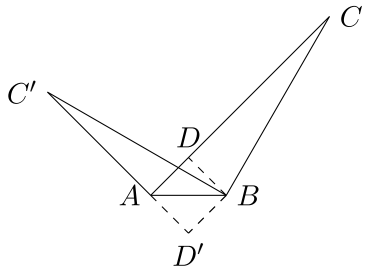
For (ii), by Pythagoras' Theorem we can compute $\overline{F_BA}$ and $\overline{CF_B}$, which we then sum to get $\overline{CA}$. Knowing all side lengths, we can apply the Side-Side-Side Congruence Theorem.

For (iii), we can compute β via the formula $\sin\beta = h_A/\overline{AB}$, using acuteness as in (i). Through the formula $\sin\beta = h_C/\overline{BC}$, we obtain $\overline{BC}$. We then apply the SAS Congruence theorem using $\overline{AB}$, $\overline{BC}$, and β.

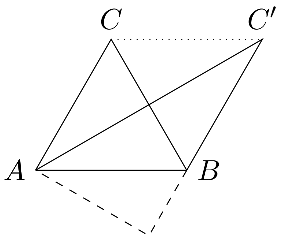
To see (iv), by Pythagoras' Theorem, h_C and $\overline{BC}$ determine $\overline{F_CB}$. Then $\overline{AF_C} = \overline{AB} - \overline{F_CB}$, and $\overline{AF_C}$ and h_C determine $\overline{AC}$, by Pythagoras again. We conclude via the SSS Congruence Theorem. □

Example 2.1. If we drop the acuteness assumption, a triangle ABC is not necessarily determined up to congruence if we know $\overline{AB}$, $\overline{BC}$ and h_B.

Consider a triangle ABC such that $\alpha = 45°$ and $\beta = 120°$. Relabel F_B as D and call D' the symmetric of D at the line AB. That is, if ℓ is the line stemming from D that hits AB orthogonally, then D' is the unique point on ℓ and on the other side of AB with the same distance to AB as D. Consider a triangle $A'B'C'$ such that $A = A'$, $B = B'$, and $\alpha' = 135°$. We clearly have $\overline{AB} = \overline{A'B'}$. Moreover, $h_B = \overline{BD} = \overline{BD'} = h_{B'}$. Finally, we have $\overline{BC} > \overline{AB}$ (one may compute $\overline{BC} = \overline{AB}(\sqrt{3}+1)$) so we can move C' without altering $A', B' and \alpha'$ while keeping that $\overline{B'C'} = \overline{BC}$ (one may compute that $h_{C'} = \frac{\sqrt{3}+1}{2}$).



Example 2.2. Similarly, a triangle ABC is not necessarily determined up to congruence if we know $\overline{AB}$, $\overline{BC}$, h_A and h_C. For example, consider an equilateral triangle ABC and a triangle $A'B'C'$ such that $A' = A$, $B' = B$ (in particular, $\overline{AB} = \overline{A'B'}$) and $\gamma' = 120°$. Moreover, we may suppose that CC' is parallel to AB, ensuring that $h_C = h_{C'}$. By construction, CA and $B'C'$ are parallel and hence $\overline{BC} = \overline{CA} = \overline{B'C'}$. Finally, we have $h_A = h_{A'}$ because the lines BC and $B'C'$ are symmetric with respect to the line connecting AB with the intersection point of BC and $C'A'$.



2.3. **Two angles and one height, or one angle and two heights.**

Proof of Theorem 1.3. For (i), we show that we can always compute $\overline{AB}$, and hence we can apply the ASA Congruence Theorem. If we know h_A or h_B, then we can find $\overline{AB}$ via the formulas $h_A = \overline{AB}\sin\beta$ or $h_B = \overline{AB}\sin\alpha$ respectively. If we know h_C, we split into subcases: α and β are acute, or one of them is obtuse. We record that, independently, $\overline{AF_C} = h_C \sin\alpha$ and $\overline{F_CB} = h_C \sin\beta$ are known. In the first situation, $\overline{AB} = \overline{AF_C} + \overline{F_CB}$. In the latter, we assume without loss of generality that α is obtuse. Then $\overline{AB} = \overline{F_CB} - \overline{AF_C}$. As stated above, knowing $\overline{AB}$ completes the proof.

To prove (ii), we use $\sin\alpha = h_B/\overline{AB}$ and $\sin\alpha = h_C/\overline{AC}$ to get $\overline{AB}$ and $\overline{AC}$ respectively, and with these quantities in hand we use the SAS Congruence Theorem.

To see (iii), we obtain $\overline{AB}$ via $\overline{AB} = h_B \sin\alpha$, and then $\sin\beta$ via $\sin\beta = h_A/\overline{AB}$. Since ABC is acute, β is uniquely determined by $\sin\beta$. With α, β and $\overline{AB}$, we apply the ASA Congruence Theorem. □

A triangle ABC is not determined up to congruence if we know α, h_A and h_B. Indeed, see Example 2.3 below, in which it will be shown that even $\overline{AB}$, α, h_A and h_B are not enough.

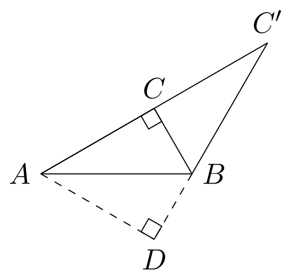
2.4. **One angle, one side, and one height.**

Proof of Theorem 1.4. Beginning with (i), h_A and $\overline{AB}$ determine $\sin\beta$ via $\sin\beta = h_A/\overline{AB}$, and, as we've argued before, acuteness then implies that β is uniquely determined. We then go by the ASA Congruence Theorem. Item (iii) is totally analogous: $\sin\gamma = h_B/\overline{BC}$ and the acute assumption yields that γ is determined, and hence we can apply ASA again.

The proof of (ii) is more interesting. If we know $\overline{BC}$ and h_A, we can move A the line ℓ parallel to BC and containing A without changing those quantities. The angle α is maximal when A is on the perpendicular bisector of BC and the angle strictly decreases the more we move A away from the perpendicular bisector (this could be seen as a consequence of the Inscribed Angle Theorem). So if we fix α, then A can be either the point A_0 that maximizes α, or one of exactly two points on ℓ that are of equal distance from A_0. Thus, up up to reflecting in the line from A_0 to $\overline{BC}$ that hits $\overline{BC}$ orthgonally (that is, up to performing a certain congruence-preserving transformation), the triangle ABC is determined.

For (iv), we compute $\overline{AB} = h_B/\sin\alpha$ and we apply the SAS Congruence Theorem with $\overline{CA}$, $\overline{AB}$, and α. □

Example 2.3. A triangle ABC is not necessarily determined up to congruence if we know α, $\overline{AB}$, and h_A. Note that adding the information of h_B won't help, since, by $h_B = \overline{AB}\sin\alpha$, knowing α and $\overline{AB}$ is equivalent to knowing α and h_B. Consider a triangle ABC such that $\alpha = 30°$ and $\beta = 60°$ and call D the symmetric of C at the line AB (defined as in Example 2.1). Then consider the triangle $A'B'C'$ such that $A = A'$, $B = B'$, $\alpha' = \alpha$ and $\beta' = 120°$. We clearly have $\overline{AB} = \overline{A'B'}$. Moreover, $h_A = \overline{CA} = \overline{DA} = h_{A'}$.



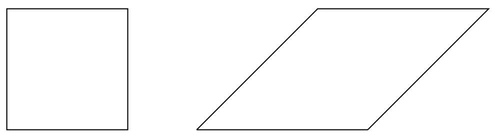
Similarly, knowing α, $\overline{BC}$, and h_B is not sufficient to determine the triangle up to congruence. Similar to above, the formula $h_B = \overline{AB}\sin\alpha$ implies that adding $\overline{AB}$ won't change anything. We know by the theorem above that there is a unique acute triangle ABC carrying this data. One can construct a unique triangle $A'B'C'$ such that $A' = A$, $B' = B$, and $\beta' = \pi - \beta$, which is easily seen to have $h_{B'} = h_B$ and $\overline{B'C'} = BC$.

3. Congruence theorems for quadrilaterals and convex polygons

Heights for polygons are defined analogously to heights for triangles. For polygons, we label the vertices $A_1, A_2, \ldots, A_n$ and write the polygon as $A_1 \ldots A_n$, side lengths as $\overline{A_i A_{i+1}}$ (with $A_{n+1} = A_1$), etc. If $A_1 \ldots A_n$ is a polygon, we can define the height of a vertex A_i relative to a side $A_j A_{j+1}$, for all $j \neq i$. Indeed, let $\ell_{A_j A_{j+1}}$ be the extension of $A_j A_{j+1}$ to an infinite line in $\mathbb{R}^2$. Then the height (of A_i relative to $A_j A_{j+1}$) is the length of the unique line segment $L_{A_i, A_j A_{j+1}}$ stemming from A_i that hits $\ell_{A_j A_{j+1}}$ orthogonally. By analogy with the triangle case, $L_{A_i, A_j A_{j+1}}$ is called the altitude at A_i relative to $A_j A_{j+1}$, and the point at which A_i strikes $A_j A_{j+1}$ is called the foot and denoted $F_{A_i, A_j A_{j+1}}$.

We begin our investigation on Congruence Theorems involving heights with the case of convex quadrilaterals. We point out immediately that the direct analog of Theorem 1.1 cannot hold.

Example 3.1. For a convex quadrilateral, knowing all heights is not sufficient to determine it up to congruence. Consider for example the square with all side lengths equal to 1 and the parallelogram with angles 45° and 135° and such that the distance between the opposite vertical sides is 1 and the distance between opposite horizontal sides is 1 (or, the lengths of the two horizontal sides is $\sqrt{2}$). For both convex quadrilaterals, all heights are equal to 1.



More generally, suppose that two sides of a convex quadrilateral are parallel. Without loss of generality, they are the sides AB and CD. Then, the heights from A to CD, from B to CD, from C to AB and from D to AB are all equal, and in particular equal to the distance between the two sides.

As a consequence of the discussion in the example above, if we have a parallelogram and we know all heights, then we only know the distances between the opposite sides. And there is, up to congruence, an infinite family of parallelograms with prescribed distances of the opposite sides, precisely one of them being a rectangle.

As stated in the introduction, Theorem 1.5 shows that, for convex quadrilaterals, parallelograms present the only case where knowing all the heights is not sufficient.

Proof of Theorem 1.5. Consider a convex quadrilateral $ABCD$, where the vertices are listed in cyclic order, and suppose we know all of its heights. Necessarily, at least one of the angles is less than or equal to $\pi/2$. Up to congruence, we can assume that angle is at the vertex D, and we call this angle δ; this choice restricts the number of cases that we need to consider. Furthermore, we can choose coordinates such that A is the origin $(0,0)$, the point B is on the positive x-axis, $B = (x_B, 0)$, and the quadrilateral $ABCD$ is in the upper half-plane. We write $C = (x_C, y_C)$ and $D = (x_D, y_D)$ and remark that y_C and y_D are the known heights from C and D respectively toward the side AB. Our only unknowns are x_B, x_C, and x_D.

We call θ the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that the altitude $L_{A,BC}$ from A to the side BC forms with the positive direction of the x-axis. To ease notation, we write h for the corresponding height $h_{A,BC}$. Since A, the foot $F_{A,BC}$ and B are the vertices of a right triangle,

$$x_B = \frac{h}{\cos\theta}$$

(here, if $\theta = 0$, we think of a line as a degenerate triangle). Next, by connecting the altitude $L_{C,AB}$ to the line AB, we get the foot $F_{C,AB}$, which together with B and C form a right triangle with an angle of $|\theta|$ at C. Examining this triangle, we find that if $\beta \leq \frac{\pi}{2}$, which we note corresponds to $\theta \geq 0$, then the side that lies on the x-axis has length $x_B - x_C$, and we have $x_B - x_C = y_C \tan\theta$, and moreover that

$$x_C = \frac{h}{\cos\theta} - y_C \tan\theta\,.$$

If instead $\beta > \pi/2$, then $\theta < 0$ and the x-axis side has length $x_C - x_B$, but we end up with the same formula by noting that

$$x_C - x_B = y_C \tan|\theta| = y_C \tan(-\theta) = -y_C \tan\theta.$$

As above, for notations sake we set $h' = h_{D,BC}$, the height from D toward the side BC. Consider the points A and D and their altitudes relative to BC, as well as the horizontal segments from A and D respectively to the side BC. In both cases, the horizontal segments and altitudes are the sides of a right triangle. Since the two right triangles share an acute angle, namely the angles at A and D, they are similar. The similarity constant is $\frac{h'}{h}$. Next, we draw the triangle with vertices B, (x_B, y_D), and the point at which the horizontal line from D hits BC, which has an angle of $|\theta|$ at B. Since we assumed $\delta \leq \frac{\pi}{2}$, the side adjacent to the angle $|\theta|$ has length y_D, and hence the opposite side has length $y_D \tan|\theta|$. Moreover, the point on the line BC with y-coordinate y_D has x-coordinate $x_{D,BC} = x_B - y_D \tan\theta$. To see this, for $\beta \leq \pi/2$, this point has x-coordinate $x_{D,BC} = x_B - y_D \tan|\theta| = x_B - y_D \tan\theta$, and for $\beta > \pi/2$ this point has x coordinate $x_{D,BC} = x_B + y_D \tan|\theta| = x_B + y_D \tan(-\theta) = x_B - y_D \tan\theta$. We point out in passing that if instead $\delta > \pi/2$, then the length of this opposite side would be $y_C - y_D$. Now, x_D is $x_{D,BC}$ minus the length of the horizontal line in the triangle of D, which, using the similarity constant, we see to be $\frac{h'}{h}x_B$. That is,

$$x_D = (x_B - y_D \tan\theta) - \frac{h'}{h}x_B = \frac{h-h'}{\cos\theta} - y_D \tan\theta,.$$

The formulas above show that the values x_B, x_C, and x_D, which are equivalent to the data of the points A, B, C and D, are determined by the known heights and the angle θ. To prove the theorem, we show that distinct values of θ lead to the same values for all heights only if the sides DA and BC are parallel (equivalently, $h = h'$) and the sides AB and CD are parallel (equivalently, $y_C = y_D$).

For α the internal angle of $ABCD$ at A, we have $\sin\alpha = y_D/\overline{AD}$. Moreover, relabelling $h'' = h_{B,DA}$, we have $h'' = x_B \sin\alpha$. We deduce that

$$(h'')^2 = \frac{x_B^2 y_D^2}{(\overline{AD})^2} = \frac{x_B^2 y_D^2}{x_D^2 + y_D^2}.$$

Substituting our expressions for x_B and x_D above, we get

$$(h'')^2 = \frac{\frac{h^2}{\cos^2\theta} y_D^2}{\frac{(h-h')^2}{\cos^2\theta} + y_D^2(1+\tan^2\theta) - 2(h-h')y_D \frac{\sin\theta}{\cos^2\theta}} = \frac{h^2 y_D^2}{(h-h')^2 + y_D^2 - 2(h-h')y_D \sin\theta}.$$

Thus, if θ_1 and θ_2 give the same value of h'', then we can rearrange the above formula to get

$$(h-h')y_D \sin\theta_1 = (h-h')y_D \sin\theta_2.$$

Since h, h', and y_D are known, the only way for θ_1 and θ_2 to be distinct is if $h = h'$.

To probe the sides AB and CD, note that the line through C and D is described by the equation

$$(y_D - y_C)x - (x_D - x_C)y + (y_C x_D - y_D x_C) = 0.$$

Observing that the height $h'''' = h_{A,CD}$ is the distance from this line to the origin, we compute, keeping in mind that $h = h'$,

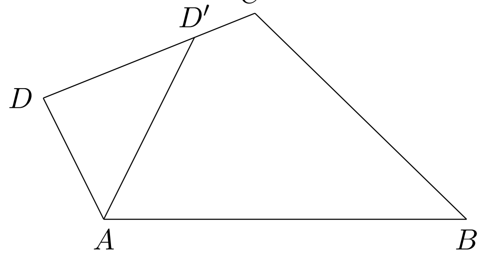
$$(h'''')^2 = \frac{(y_C x_D - y_D x_C)^2}{(y_D - y_C)^2 + (x_D - x_C)^2} = \frac{y_D^2 h^2}{(y_D - y_C)^2 + h^2 - 2h(y_C - y_D)\sin\theta}.$$

Similar to above, different values of θ give rise to the same height h'''' only if $y_C - y_D = 0$. □

Remark 3.1. Notice that we only made use of 6 out of 8 heights. In particular, we did not need to know the values $h_{B,CD}$ and $h_{C,DA}$.

In view of the remark above, we point out that, generally speaking, up to congruence, we can choose the coordinate of A to be $(0,0)$ and B to be on the x-axis, so this problem should have 5 degrees of freedom. However, in general, 5 heights are not enough to determine the polygon up to congruence, as the following example shows.

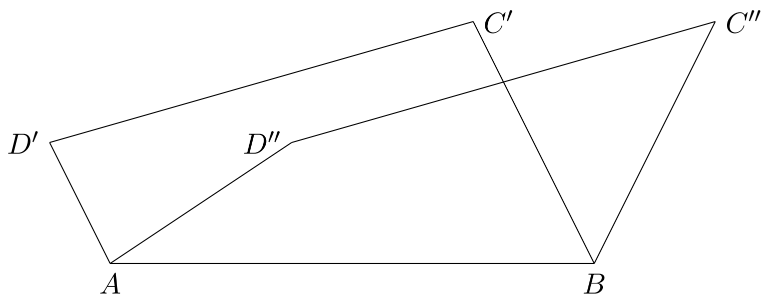
Example 3.2. We consider non-congruent convex quadrilaterals $ABCD$ and $ABCD'$, cyclically ordered, such that the points C, D and D' are aligned and such that the heights $h_{B,DA}$ and $h_{B,D'A}$ are the same. The two convex quadrilaterals share 5 heights but are not congruent. We can also construct such quadrilaterals so that no two sides are parallel:



Proof of Theorem 1.6. Let AB and BC be the neighboring sides. By translating, we can assume $B = (0, 0)$. By considering the height from A (respectively, C) relative to BC (respectively, AB) we can determine the length and angle of the line connecting B to A (respectively, C). Indeed, the relevant altitude is part of the right triangle with points A (respectively C), the relevant foot, and B, around which the angle is known. Since we know the position of B, the information of the length and angle determines A and C. For any other point A_i, using the known height to AB, we can determine a line parallel to AB on which A_i lies. Using the known height to BC, A_i lies on another known line parallel to BC. These two lines intersect in one point, which gives us the location of A_i. □

Proof of Theorem 1.7. Fix a convex n-gon and assume that we know the lengths of its sides and its heights. To determine the n-gon up to congruence, we will in fact make use of only $2n$ (appropriately selected) heights. Call the vertices A,B,C,D in cyclic order. Up to congruence we can fix the side AB and the half-plane with respect to AB where the the n-gon lies. By considering the point C, B and the foot of C relative to AB, we get a right triangle. Recalling that we know the length of BC, we see that there are at most two possibilities for C, depending on whether the angle β at B is less than $\pi/2$, or at least $\pi/2$. Fixing the choice for C determines the choice for D. Indeed, consider the line in $\mathbb{R}^2$ containing BC. Since the polygon is convex, we know which side D is on with respect to this line. Then D is determined: knowing $\overline{CD}$, the angle θ between CD and the altitude from D to BC determines D, and we can compute θ via $\cos\theta = h/\overline{CD}$. All together, we have at most two possibilities for the location of the two points C, D.

We claim that in fact there is just one possibility, so that C and D are determined. Iterating the reasoning over all sides of the polygon in cyclic order will show that we know the whole n-gon up to congruence.

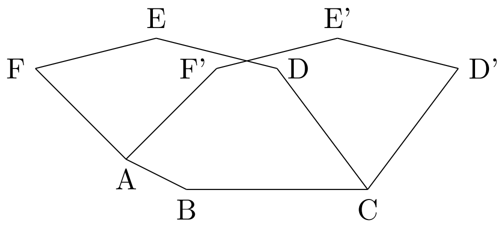


To prove the claim, we suppose that there are two distinct possibilities for C, D, and we call the two choices C', D' and C'', D'' respectively. Note that, by our assumptions, the lengths of $C'D'$ and $C''D''$ are equal. The angle β at B is either less than $\pi/2$, which corresponds to C being to the left of B, or at least $\pi/2$, which corresponds to C being to the right. In the two cases, we get different values for the height from B relative to CD, which is our sought contradiction. □

Example 3.3. A convex hexagon is not determined up to congruence if we know all side lengths and all 4 heights toward one same side. Indeed, we consider two non-congruent convex hexagons $ABCDEF$ and $ABCD'E'F'$, whose vertices (in Cartesian coordinates) are as follows:

$$A = (-2, 1) \quad B = (0, 0) \quad C = (6, 0)$$

$$D = (3, 4) \quad E = (-1, 5) \quad F = (-5, 4) \qquad D' = (9, 4) \quad E' = (5, 5) \quad F' = (1, 4)\,.$$



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