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150 CHARACTERIZATIONS OF SQUARES

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Abstract. We prove 72 more sufficient conditions for when a quadrilateral is a square, which together with the 78 in the previous part [28] brings the total number of known compiled characterizations of squares to 150.

1. Introduction

We continue to collect and compile characterizations of squares, a work that began in [28], where the first named author proved 78 sufficient conditions for a convex quadrilateral to be a square. In this paper we add an additional 72 necessary and sufficient conditions, but we only prove the sufficient parts since most of the necessary conditions are well-known or easily proved properties of squares. This brings the number of known characterizations of squares to a total of 150, the largest published compilation for any type of quadrilateral we know of. We have no doubt, however, there are many more characterizations of squares, both such that are hiding in the literature and others waiting to be discovered in the future.

It is recommended that the reader study [28] before the present paper since it contains the basics. After the publication of that paper, the second named author of this paper contributed with 22 additional characterizations on the basis of duality, which together with 23 found by the first named author constitute the new conditions that were missed in [28]. We have also found 27 more conditions discovered by other mathematicians, as indicated before each theorem, making a total of 72 additions. (We actually prove 73 sufficient conditions, but Theorem 2.1 *(a)* is a more general condition than Theorem 2.1 *(c)* in [28], so it replaces that one.)

Following the same structure as in [28], the characterizations are grouped according to which type of quadrilateral we start with and impose additional restrictions on for it to be a square. At the end, we have compiled a chronological list with the earliest known publications for each of the characterizations in both the present paper and [28].

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On the second page of [28], we summarized several basic properties of squares that are reasonable well-known to motivate that we did not prove any of them. Here we add a few more that are of the same category:

- The diagonals lie on symmetry lines
- The bimedians lie on symmetry lines
- The bimedians are perpendicular
- The bimedians have equal length, equal to any side
- The bimedians are perpendicular bisectors to the sides

A *bimedian* is a line segment connecting the midpoints of two opposite sides in a quadrilateral. With *semibimedians* we mean the four parts into which the bimedians divide each other. It is well-known that the two bimedians bisect each other in all quadrilaterals (they are the diagonals in Varignon's parallelogram), so the semibimedians just mean half of each bimedian.

2. Rectangles

We begin by studying fourteen characterizations for when a rectangle is a square. The first is from [48, p. 2], but it has surely been known longer, and it **replaces** the less general Theorem 2.1 *(c)* in [28]. The next three are due to the second named author, and the following seven are due to the first named author. a on the left-hand side of the equality in *(b)* can be any of the four sides. Condition *(l)* is from [45, p. 267]. The penultimate condition is taken from Romanian Mathematical Olympiads in 2002 [16, p. 27] (Problem 4 for Grade 7 in the Second Round, proposed by Mircea Fianu) and the last one is from the 2022 Central American and Caribbean Mathematics Olympiad [4]. We cite the official proofs to those two competition problems and have corrected two typos from [16, p. 27].

A circle tangent to one side of a quadrilateral and the extension of the two adjacent sides (see Figure 12) is called an *escribed circle* in accordance with [22, p. 71]. If the diagonals of quadrilateral $ABCD$ intersect at P, we name the four triangles ABP, BCP, CDP, DAP the *quarter triangles.*

Theorem 2.1. *A rectangle $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ satisfies any one of:*

(a) it has a diagonal bisecting an angle
(b) $a = \frac{1}{4}(a + b + c + d)$
(c) it has two adjacent equal semibimedians
(d) the bimedian intersection is equidistant to the sides
(e) it has equal heights to two adjacent sides
(f) it has two adjacent escribed circles with equal radii
(g) it has two adjacent quarter triangles with equal circumradii
(h) it has two adjacent quarter triangles with equal inradii
(i) it has two adjacent quarter triangles with equal exradii
(j) it has shortest perimeter for a given area
(k) it has largest area for given diagonals
(l) it has shortest diagonals for a given area
(m) $DF + BE = AE$, *where E and F are points on BC and DC respectively such that* $\angle DAF = \angle FAE$

(n) *it has four internal circles forming a loop, each tangent to two other circles and two sides*

if and only if it's a square.

Proof. *(a)* In a rectangle, alternate angles $\angle BAC = \angle DCA$, and if also $\angle BCA = \angle DCA$, then triangle ABC is isosceles and so we have $AB = BC$ (see Figure 1). Then $ABCD$ is a square according to Theorem 2.1 *(a)* in [28].

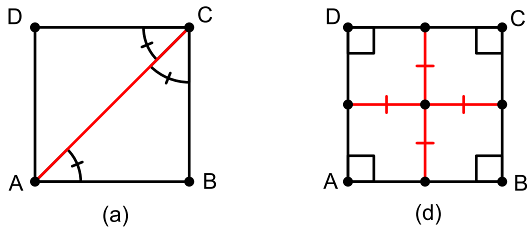

FIGURE 1. Given assumptions in *(a)* and *(d)*

(b) In a rectangle, $a = c$ and $b = c$, and together with $4a = a + b + c + d$, we get $a = b$, so the rectangle is a square according to Theorem 2.1 *(a)* in [28].

(c) The two bimedians are equal in length to two adjacent sides in a rectangle, so a rectangle satisfying this condition is a square according to Theorem 2.1 *(a)* in [28].

(d) When the bimedian intersection is equidistant to the sides, the rectangle is partitioned by the bimedians into four quadrilaterals with equal sides (four rhombi) and a right vertex angle each (see Figure 1), so the rectangle is a square according to Theorem 3.1 *(a)* in [28].

(e) We denote by h_a and h_b the heights to adjacent sides a and b respectively. The area of the rectangle is given by

$$K = ah_a = bh_b$$

so we get that $a = b$ is equivalent to $h_a = h_b$, and since the first equality characterizes squares according to Theorem 2.1 *(a)* in [28], then so does the second.

(f) The diameters of the escribed circles to a rectangle are equal to the lengths of the sides they are tangent to (the non-extended sides). Denoting two adjacent escribed radii by r_a and r_b directly yields that $r_a = r_b$ is equivalent to $a = b$, which is equivalent to a square according to Theorem 2.1 *(a)* in [28].

(g) Denoting two adjacent quarter triangle circumradii by R_1 and R_2 (red circles in Figure 2) and applying the extended law of sines, we get

$$R_1 = R_2 \quad \Leftrightarrow \quad \frac{a}{2\sin\theta} = \frac{b}{2\sin(\pi - \theta)} \quad \Leftrightarrow \quad a = b$$

where θ is one of the angles between the diagonals.

(h) Denoting the diagonal lengths by p and two adjacent quarter triangle inradii by r_1 and r_2 (green circles in Figure 2), we get by applying the

well-known formula for the triangle inradius that

$$r_1 = r_2 \quad \Leftrightarrow \quad \frac{2T_a}{\frac{1}{2}p + \frac{1}{2}p + a} = \frac{2T_b}{\frac{1}{2}p + \frac{1}{2}p + b} \quad \Leftrightarrow \quad a = b$$

since the two adjacent quarter triangle areas satisfy $T_a = T_b$ (due to equal base and height).

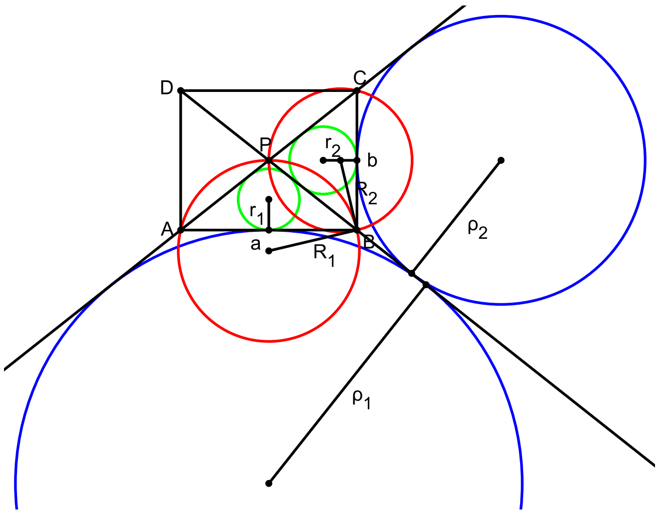

FIGURE 2. Quarter triangle circumradii, inradii, and exradii

(i) With the same notations as in *(h)* and by denoting the two adjacent quarter triangle exradii by ρ_1 and ρ_2 (blue circles in Figure 2) yields

$$\rho_1 = \rho_2 \quad \Leftrightarrow \quad \frac{2T_a}{\frac{1}{2}p + \frac{1}{2}p - a} = \frac{2T_b}{\frac{1}{2}p + \frac{1}{2}p - b} \quad \Leftrightarrow \quad a = b.$$

(j) In a rectangle with adjacent sides a and b and given area $K = ab$, we want to minimize the perimeter $L = 2a + 2b$. Applying the AM-GM inequality, we get

$$L = 2a + \frac{2K}{a} \geq 2\sqrt{2a \cdot \frac{2K}{a}} = 4\sqrt{K}$$

where equality holds if and only if $2a = \frac{2K}{a}$, that is, only when $K = a^2$, which is equivalent to $a = b$ according to $K = ab$.

(k) In a rectangle with adjacent sides a and b and given diagonals with lengths p satisfying $p^2 = a^2 + b^2$ by the Pythagorean theorem, we want to maximize the area $K = ab$. We get by using the AM-GM inequality that

$$K^2 = a^2b^2 = a^2(p^2 - a^2) \leq \frac{a^2 + (p^2 - a^2)}{2} = \frac{p^2}{2}$$

where equality holds if and only if $a^2 = p^2 - a^2$. Together with $p^2 = a^2 + b^2$, this is equivalent to $a = b$.

(l) In a rectangle with adjacent sides a and b and given area $K = ab$, we shall minimize the diagonal lengths p. We get

$$p^2 = a^2 + b^2 = a^2 + \left(\frac{K}{a}\right)^2 \geq 2\sqrt{a^2 \cdot \frac{K^2}{a^2}} = 2K$$

where equality holds if and only if $a^2 = \frac{K^2}{a^2}$, that is, only when $a = b$ according to $K = ab$.

(m) Let AF intersect the line BC in M and the perpendicular in A to AM intersect the line BC in N. Since $\angle DAF = \angle MAE$ it follows that AEM is an isosceles triangle with $AE = EM$ (see Figure 3). Triangle AEN is also isosceles since

$$\angle ENA = 90^\circ - \angle EMA = 90^\circ - \angle EAM = \angle EAN$$

implying that $EN = AE$. Because B is an interior point on segment EN, it follows that

$$DF + BE = AE = EN = EB + BN,$$

so $DF = BN$. It also holds that $\angle BAN = \angle DAF$. Therefor the right triangles DAF and BAN are congruent (AAS), so $AB = AD$. This proves that $ABCD$ is a square according to Theorem 2.1 *(a)* in [28].

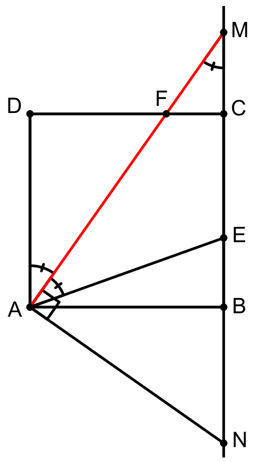

FIGURE 3. $DF + BE = AE$

(n) Let r_1, r_2, r_3, r_4 be consecutive radii, and the first two circles be tangent to side AB at points X and Y (see Figure 4). By the Pythagorean theorem,

$$XY = \sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2} = 2\sqrt{r_1 r_2}.$$

Side AB of the rectangle then has length

$$AB = r_1 + 2\sqrt{r_1 r_2} + r_2 = (\sqrt{r_1} + \sqrt{r_2})^2.$$

By symmetry, the opposite side has length

$$CD = (\sqrt{r_3} + \sqrt{r_4})^2.$$

Since $ABCD$ is a rectangle, $AB = CD$ and we get

$$\sqrt{r_1} + \sqrt{r_2} = \sqrt{r_3} + \sqrt{r_4}.$$

In the same way, the other pair of opposite sides yield

$$\sqrt{r_1} + \sqrt{r_4} = \sqrt{r_2} + \sqrt{r_3}.$$

Adding the last two equalities and simplifying, we obtain $r_1 = r_3$ and then $r_2 = r_4$ from any one of these two equations. Hence

$$AB = (\sqrt{r_1} + \sqrt{r_2})^2 = (\sqrt{r_3} + \sqrt{r_2})^2 = BC$$

so the rectangle is a square according to Theorem 2.1 *(a)* in [28]. □

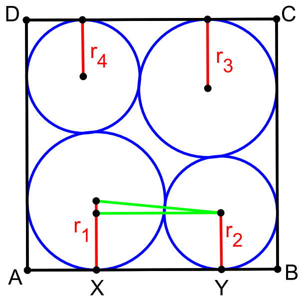

FIGURE 4. Four circles forming a tangent loop

3. Rhombi

Here we study nine conditions for when a rhombus is a square, all discovered by the second named author. If P is the diagonal intersection in a quadrilateral $ABCD$, we call $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$ the *semidiagonals* (not only in rhombi, but in all types of quadrilaterals). a' and $\angle A$ on the left-hand side of the equalities in *(a)* and *(b)* can be any of the four semidiagonals or angles respectively.

Theorem 3.1. *A rhombus $ABCD$ with semidiagonals a', b', c', d' satisfies any one of:*

(a) $a' = \frac{1}{4}(a' + b' + c' + d')$
(b) $\angle A = \frac{1}{4}(\angle A + \angle B + \angle C + \angle D)$
(c) it has concurrent perpendicular bisectors to the sides
(d) it has perpendicular bimedians
(e) a bimedian is perpendicular to a side
(f) its bimedian intersection is equidistant to the vertices
(g) the bimedians divide the Varignon parallelogram into four congruent triangles
(h) it has the largest area for a given perimeter
(i) it has the smallest area circumscribing a given circle

if and only if it's a square.

Proof. *(a)* In a rhombus, $a' = c'$ and $b' = d'$, and when it also holds that $4a' = a' + b' + c' + d'$, we get $a' = b'$. Then the rhombus is a square according to Theorem 3.1 *(c)* in [28].

(b) The angles in a rhombus satisfy $\angle A = \angle C$ and $\angle B = \angle D$. Then we get from $4\angle A = \angle A + \angle B + \angle C + \angle D$ that $\angle A = \angle B$, so the rhombus is a square according to Theorem 3.1 *(b)* in [28].

(c) Concurrent perpendicular bisectors means that it has a circumcircle, and a cyclic rhombus is a square according to Theorem 3.1 *(e)* in [28].

(d) The two angles between the bimedians are equal to the two different vertex angles in a rhombus (see Figure 5), and when they are both right angles, the rhombus has four right vertex angles. One is enough for it to be a square according to Theorem 3.1 *(a)* in [28].

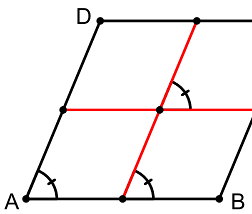

Figure 5. Bimedians in a rhombus

(e) The angle between a bimedian and a side in a rhombus is equal to a vertex angle (see Figure 5), so when a bimedian is perpendicular to a side, then the rhombus has a right vertex angle making it a square according to Theorem 3.1 *(a)* in [28].

(f) If the bimedians intersect at M, and M_a is the midpoint of side AB, then this condition implies that triangles AMM_a and BMM_a are congruent (SSS), see Figure 6. Hence $MM_a \perp AB$ and $ABCD$ is a square according to *(e)*.

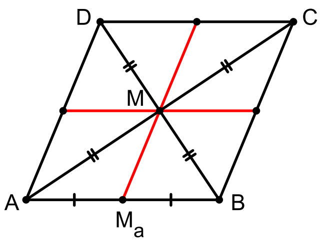

Figure 6. Bimedian intersection M

(g) Since the bimedians are the diagonals in the Varignon parallelogram, they bisect each other, so two adjacent triangles created by the bimedians are congruent if and only if the two angles between the bimedians are equal, i.e. they are each right angles. Then the rhombus is a square according to *(d)*.

(h) The area of a rhombus $ABCD$ with side a is $K = a^2 \sin A$ and the perimeter is $L = 4a$, so we have

$$K = \left(\frac{L}{4}\right)^2 \sin A \leq \left(\frac{L}{4}\right)^2$$

where equality holds if and only if A is a right angle. Only then is the rhombus a square according to Theorem 3.1 *(a)* in [28].

(i) In a rhombus with side a, height h, and inradius r (which is a given constant), we have $a \sin A = h$ where $h = 2r$, so the area is given by

$$K = a^2 \sin A = \frac{4r^2}{\sin A} \geq 4r^2$$

where equality holds if and only if A is a right angle. That is the only case when the rhombus a square according to Theorem 3.1 *(a)* in [28]. □

There are several dualities between the characterizations for rectangles and rhombi in this paper and in [28]. Some of them are summarized in Table 1. *A rectangle (rhombus) is a square if and only if any of the conditions in the left (right) column is satisfied*, where P is the diagonal intersection and M is the bimedian intersection.

Rectangle	**Rhombus**
$a = \frac{1}{4}(a+b+c+d)$	$\angle A = \frac{1}{4}(\angle A + \angle B + \angle C + \angle D)$
Two adjacent equal sides	Two adjacent equal vertex angles
Tangential	Cyclic
Concurrent angle bisectors	Concurrent perpendicular bisectors
P is equidistant to the sides	P is equidistant to the vertices
Perpendicular diagonals	Perpendicular bimedians
Equal bimedians	Equal diagonals
A diagonal bisects a vertex angle	A bimedian is perpendicular to a side
Two adjacent equal semibimedians	Two adjacent equal semidiagonals
M is equidistant to the sides	M is equidistant to the vertices

Table 1. Dual characterizations of squares

4. Parallelograms

Next we prove eight sufficient conditions for when a parallelogram is a square. Conditions *(a)* and *(e)* are due to the first named author, while *(b)*, *(c)* and *(d)* where discovered by the second named author. *(f)* was mentioned in [44, p. 340] and *(g)* was proposed as Problem A251 by Ho-Joo Lee in [39] and a short solution using complex numbers, that we cite, was given by Michel Bataille. To prove that *(h)* is a sufficient condition was Problem 4 on the first selection examination for the Junior Balkan Mathematical Olympiad in Rumania in 2002 [16, p. 58] (proposed by Dinu Şerbănescu; we cite this proof) and that it is both a necessary and sufficient condition was given as Problem 336 in September 2004 at The Mathematical Olympiads Correspondence Program (Olymon) in Canada [49].

Theorem 4.1. *A parallelogram $ABCD$ with diagonal intersection P satisfies any one of:*

(a) its bimedians have equal length and are perpendicular
(b) its bimedians have equal length and one is perpendicular to a side
(c) it has two adjacent equal semibimedians and perpendicular bimedians
(d) it is tangential and has equal diagonals
(e) it is tangential and has perpendicular bimedians
(f) it is harmonic
(g) triangle AMN is right-angled with $AM = AN$, where M and N are the midpoints of PD and BC respectively
(h) triangles ABC and AST are directly similar, where S and T are the midpoints of BP and CD respectively

if and only if it's a square.

Proof. *(a)* The bimedians in a parallelogram are parallel to the sides, so if they have equal length, then two adjacent sides have equal length (a rhombus). The angles between the bimedians are equal to the two different sorts of vertex angles in this rhombus, and if these angles are right, then it is a square according to Theorem 3.1 *(a)* in [28].

(b) As in *(a)*, it's a rhombus. The angle between a bimedian and a side is equal to a vertex angle, so if this is a right angle, then the rhombus is a square according to Theorem 3.1 *(a)* in [28].

(c) Since the bimedians bisect each other in all quadrilaterals, two adjacent equal semibimedians is equivalent to a rhombus, and perpendicular bimedians means that it is a square according to Theorem 3.1 *(c)* in this paper.

(d) Tangential implies it's a rhombus, and equal diagonals that it's a square according to Theorem 3.1 *(d)* in [28].

(e) Tangential implies it's a rhombus, and perpendicular bimedians that it's a square according to Theorem 3.1 *(c)*.

(f) A harmonic quadrilateral is defined to be a cyclic quadrilateral with sides satisfying $ac = bd$. When it is also a parallelogram, it must be a rectangle (Theorem 2.1 *(e)* in [31]), so $ac = bd$ implies that $a = b$, which yields a square according to Theorem 2.1 *(a)* in [28].

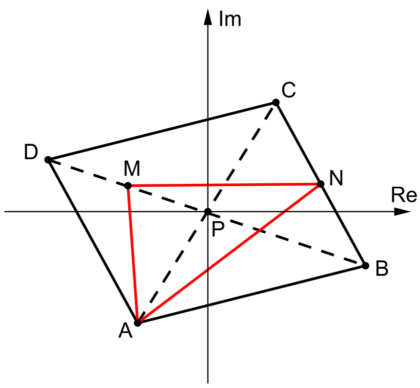

FIGURE 7. Parallelogram $ABCD$

(g) We place the parallelogram in a complex plane with origin at P (see Figure 7). Let vertices A and B be represented by complex numbers A and B respectively. Then the other two vertices C and D are $-A$ and $-B$ respectively, and M and N are $-\frac{B}{2}$ and $\frac{B-A}{2}$ respectively. That triangle AMN is right-angled with $AM = AN$ is equivalent to

$$A + \frac{B}{2} = i\left(B - \frac{A}{2}\right) \quad \Leftrightarrow \quad w\left(1 + \frac{i}{2}\right) = i - \frac{1}{2} \quad \Leftrightarrow \quad w = i$$

where we defined $w := \frac{A}{B}$. This is equivalent to $PA \perp PB$ and $PA = PB$, which in turn holds if and only if $ABCD$ is a square according to Theorem 4.1 *(e)* in [28].

(h) From the similarity of triangles AST and ABC (see Figure 8), we obtain

$$\frac{AS}{AB} = \frac{AT}{AC} \tag{1}$$

and

$$\angle SAT = \angle BAC \quad \text{or} \quad \angle BAS = \angle CAT. \tag{2}$$

The relations (1) and (2) imply that triangles BAS and CAT are similar. Thus

$$\frac{AS}{AT} = \frac{AB}{AC} = \frac{BS}{CT} \tag{3}$$

and $\angle ABS = \angle ACT$. Therefor $ABCD$ is a rectangle.

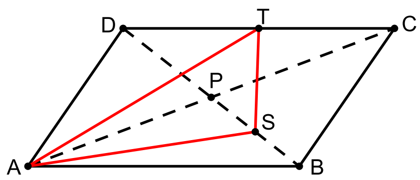

Figure 8. Similar triangles AST and ABC

Next we observe that $BS = \frac{1}{4}BD = \frac{1}{4}AC$ and $CT = \frac{1}{2}AB$. Then the last equality in (3) becomes $\frac{AB}{AC} = \frac{AC}{2AB}$, so we get

$$2AB^2 = AC^2 = AB^2 + BC^2.$$

Hence $AB = BC$, which proves that $ABCD$ is a square according to Theorem 2.1 *(a)* in [28]. □

5. Various trapezoids

In this section we study five characterizations regarding when different types of trapezoids are squares. The first four are due to the first named author and the fifth is from the old book [18, p. 53]. A *trisosceles trapezoid* is a trapezoid with three sides of equal length and a *bicentric trapezoid* is a trapezoid with both an incircle and a circumcircle, so it is both tangential and cyclic. An *isosceles trapezoid* $ABCD$ can be defined as a quadrilateral with two pairs of adjacent equal angles, for instance $\angle A = \angle B$ and $\angle C = \angle D$. A well-known property of these is that they have a pair of congruent sides called the *legs*; in this case $BC = DA$. The other pair of sides (AB and CD) are called the *bases*; they are always parallel.

Theorem 5.1. *A quadrilateral $ABCD$ satisfies any one of:*

(a) it's a trisosceles trapezoid with a right angle
(b) it's a trisosceles trapezoid with parallel legs
(c) it's a trisosceles trapezoid with equal bases
(d) it's a bicentric trapezoid with a right angle
(e) it's both an isosceles trapezoid and a kite

if and only if it's a square.

Proof. *(a)* We only consider the case when $BC = CD = DA$. It is easy to see that a trisosceles trapezoid is a special case of an isosceles trapezoid, so we have that $\angle A = \angle B$ and $\angle C = \angle D$. No matter which of the four vertex angles is a right angle, we directly get that all four angles are right angles. Hence we have a rectangle with three equal sides, and only two adjacent equal sides guarantees that the trapezoid is a square according to Theorem 2.1 *(a)* in [28].

(b) Parallel legs implies that any two adjacent angles are supplementary angles and thus all four angles are right angles since for instance $\angle C = \angle D$. Then it is a rectangle with three equal sides, so a square according to Theorem 2.1 *(a)* in [28].

(c) Equal bases implies all four sides are equal (a rhombus), and then a pair of adjacent equal angles ensures it's a square according to Theorem 3.1 *(b)* in [28].

(d) A bicentric trapezoid is a special case of an isosceles trapezoid, and with one right angle it's a rectangle. Any tangential rectangle is a square according to Theorem 2.1 *(d)* in [28].

(e) Suppose the isosceles trapezoid is defined by $\angle A = \angle B$ and $\angle C = \angle D$ with property $BC = DA$, and the kite as $DA = AB$ and $BC = CD$ (other cases are similar). It follows that $AB = BC = CD = DA$ so it's a rhombus, and with $\angle A = \angle B$ it's a square according to Theorem 3.1 *(b)* in [28]. □

We note that two special cases of *(e)* are that when a quadrilateral is both a rectangle and a kite, or both a rhombus and an isosceles trapezoid, then it must be a square.

6. Multitype quadrilaterals

Consider a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ and diagonal intersection P. Let $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$ be the semidiagonals. Before we state the next theorem, we summarize the definitions of the quadrilaterals that we shall study together with one useful characterization, since a few of them are not so well-known.

Quadrilateral	**Definition**	**Characterization**	**Ref.**
Cyclic	Has a circumcircle	$a'c' = b'd'$	[27]
Equidiagonal	Equal diagonals	$a' + c' = b' + d'$	[25]
Extangential	Has an excircle	$\lvert a - c\rvert = \lvert b - d\rvert$	[22]
Orthodiagonal	Perpendicular diagonals	$a^2 + c^2 = b^2 + d^2$	[21]
Semidiagonal	$a' + b' = c' + d'$		[13]
Tangential	Has an incircle	$a + c = b + d$	[22]
Trapezoid	A pair of parallel sides	$\angle A + \angle D = \angle B + \angle C$	[24]

Table 2. Some quadrilaterals

Semidiagonal quadrilateral is, as far as we know, a new type of quadrilateral that is introduced in [13]. Next we study six different combinations of four types of quadrilaterals that guarantee that we get a square. The first five is due to the second named author, while the sixth was a problem in the Hungarian Kömal magazine [36] in 2013, although formulated a bit differently, and we present our own proof of it.

Theorem 6.1. *A quadrilateral is at the same time any one of the following combinations:*

(a) tangential, extangential, cyclic, and trapezoid
(b) tangential, extangential, equidiagonal, and trapezoid
(c) tangential, extangential, cyclic, and equidiagonal
(d) tangential, extangential, equidiagonal, and semidiagonal
(e) tangential, extangential, cyclic, and semidiagonal
(f) tangential, orthodiagonal, cyclic, and trapezoid

if and only if it's a square.

Proof. *(a)* The characterization for extangential quadrilaterals $|a-c| = |b-d|$ can also be stated as $a+b=c+d$ or $a+d=b+c$. Together with the tangential characterization $a+c=b+d$ we get that a quadrilateral that is both extangential and tangential satisfies $a=b$ and $c=d$ or $a=d$ and $b=c$, so it is a kite. It's well-known that a cyclic trapezoid is an isosceles trapezoid. Then a quadrilateral satisfying all four conditions is a square according to Theorem 5.1 *(e)* in this paper.

(b) An equidiagonal trapezoid is an isosceles trapezoid (see Theorem 17 (iii) in [25]), so again we have a quadrilateral that is both a kite and an isosceles trapezoid, that is, a square according to Theorem 5.1 *(e)*.

(c) Theorem 17 (iv) in [25] states that a quadrilateral is both cyclic and equidiagonal if and only if it is an isosceles trapezoid. Then the conclusion follows from Theorem 5.1 *(e)*.

(d) Equidiagonal $a'+c'=b'+d'$ and semidiagonal $a'+b'=c'+d'$ implies that $a'=d'$ and $b'=c'$. This property is characteristic of an isosceles trapezoid, so once again we get a square according to Theorem 5.1 *(e)*.

(e) If we rewrite the condition for a semidiagonal to $a'=c'+d'-b'$ and substitute in the condition $a'c'=b'd'$ for a cyclic, we get $(c'+d')(b'-c')=0$ after simplification and factorization. Hence $b'=c'$ and thus $a'=d'$, implying an isosceles trapezoid, and for the fifth time all four conditions imply a square for the same reason.

(f) Squaring the tangential condition $a+c=b+d$ yields $a^2+c^2+2ac=b^2+d^2+2bd$, which by the orthodiagonal condition $a^2+c^2=b^2+d^2$ simplifies to $ac=bd$. Inserting $a=b+d-c$ and rewriting, we get $(d-c)(b-c)=0$. The solutions are $d=c$ and then $a=b$, or $b=c$ and thus $a=d$. Hence we get a kite. Since a cyclic trapezoid is an isosceles trapezoid, we get a square due to Theorem 5.1 *(e)*. □

7. Cyclic quadrilaterals

In the following three sections we study characterizations of squares that are related to different circles. First out is the circumcircle and quadrilaterals capable of having those are called cyclic. Conditions *(a)* and *(b)* in the next theorem are from [40] and the mathematical Olympiad Baltic Way in 2008 [38] respectively. *(c)* is from [46, pp. 232–233] (the inequality case is due to Nicuşor Minculete, but he neglected to state when equality holds), and *(d)* is from [6, p. 92]. *(e)* is from the 1979 Czechoslovakian Mathematical Olympiad [3] and it was given again at the 1998 Indian Mathematical

Olympiad [51]. The last condition is from Baltic Way in 1993 [37], but the proof we give is from [42].

Theorem 7.1. *A cyclic quadrilateral $ABCD$ with sides a, b, c, d, escribed radii r_a, r_b, r_c, r_d, and semiperimeter s, that is inscribed in a given circle with center O and radius R, satisfies any one of:*

(a) its diagonals are perpendicular diameters
(b) the expression $(ab+cd)(ac+bd)(ad+bc)$ has a maximal value
(c) it has area $K = r_a^2 + r_b^2 + r_c^2 + r_d^2$
(d) it has area $K = \left(\frac{s}{2}\right)^2$
(e) it has $R = 1$ and $abcd \geq 4$
(f) the central angles $\angle AOB$, $\angle BOC$, $\angle COD$, $\angle DOA$, taken in some order, are the same size as the angles of the quadrilateral $ABCD$

if and only if it's a square.

Proof. *(a)* If the diagonals are diameters, then the quadrilateral is a rectangle (a consequence of Thales' theorem), and with perpendicular diameters (diagonals), it's a square according to Theorem 2.1 *(b)* in [28].

(b) The product of the area K and the circumradius R in a cyclic quadrilateral is according to equation (1) in [28] given by

$$4KR = \sqrt{(ab+cd)(ac+bd)(ad+bc)}$$

where R is a constant when we have a given circle. Hence the product we seek to maximize is largest when the area of the quadrilateral is maximal. Using one of the formulas for the area of a convex quadrilateral (see [5, p. 15]),

$$K = \tfrac{1}{2}pq\sin\theta \leq \tfrac{1}{2}\cdot 2R\cdot 2R\cdot 1,$$

so maximum occurs when the diagonals p and q are perpendicular diameters ($\theta = \frac{\pi}{2}$). Then the quadrilateral is a square according to *(a)*.

(c) In the proof of Theorem 6.1 *(g)* in [31], we proved that the area of a cyclic quadrilateral satisfies $K \leq (r_a + r_c)(r_b + r_d)$ where equality holds if and only if it is a rectangle. We get

$$\begin{aligned} K &\leq (r_a + r_c)(r_b + r_d) \\ &\leq \tfrac{1}{2}[(r_a + r_c)^2 + (r_b + r_d)^2] \\ &\leq r_a^2 + r_b^2 + r_c^2 + r_d^2 \end{aligned}$$

where we applied the AM-GM inequality in the two forms $xy \leq \frac{1}{2}(x^2+y^2)$ and $(x+y)^2 \leq 2(x^2+y^2)$, both with equality only for $x = y$. Equality holds in the area calculation if and only if the escribed circles to the rectangle satisfy $r_a + r_c = r_b + r_d$, $r_a = r_c$ and $r_b + r_d$. These yield $r_a = r_b = r_c = r_d$, and according to Theorem 2.1 *(f)*, two adjacent equal escribed radii is enough to conclude that the rectangle is a square.

(d) The area of a cyclic quadrilateral is given by the well-known Brahmagupta's formula (see [5, p. 42])

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Then we get from the assumption that

$$\sqrt[4]{(s-a)(s-b)(s-c)(s-d)} = \frac{s}{2} = \frac{(s-a)+(s-b)+(s-c)+(s-d)}{4}$$

and this is the equality case of the AM-GM inequality. Therefore we have

$$s-a=s-b=s-c=s-d \quad \Leftrightarrow \quad a=b=c=d$$

which means the quadrilateral is a cyclic rhombus, that is, a square according to Theorem 3.1 *(e)* in [28].

(e) Applying Ptolemy's theorem (see [5, p. 35]) and the AM-GM inequality yields

$$4 \geq AC \cdot BD = ac + bd \geq 2\sqrt{abcd} \geq 2\sqrt{4} = 4$$

where we also used that neither diagonal can exceed the diameter. Hence equality must hold throughout, so AC and BD are both diameters, implying that $ABCD$ is a rectangle (with $a=c$ and $b=d$), and also that $ac=bd$ (from the AM-GM inequality), so we get $a=b$. Then the rectangle is a square according to Theorem 2.1 *(a)* in [28].

(f) Opposite vertex angles in a cyclic quadrilateral are supplementary, so a pair of angles from $\{\angle AOB, \angle BOC, \angle COD, \angle DOA\}$ add to 180°. Then there are two possibilities depending on if these two angles are adjacent or opposite.

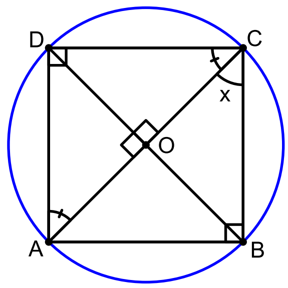

Figure 9. Case 1

Case 1 (adjacent). Suppose without loss of generality that $\angle BOA + \angle BOC = 180^\circ$, which means that O lies on AC. Then AC is a diameter and we have $\angle ABC = \angle CDA = 90^\circ$, implying that at least one of the central angles is 90° (see Figure 9). Suppose without loss of generality it is $\angle COD = 90^\circ$. Then $\angle AOD = 90^\circ$, so $CD = AD$ and $\angle OAD = \angle OCD = 45^\circ$. Now let $\angle BCO := x$, implying that $\angle BAC = 90^\circ - x$ and $BAD = \angle 135^\circ - x$ and $\angle BCD = 45^\circ + x$. We further have $\angle BOC = 180^\circ - 2x$ and $\angle BOA = 2x$. These equalities imply either $2x = 45^\circ + x$ or $2x = 135 - x$. In both cases we get $x = 45^\circ$. This amounts to $\angle BCD = \angle BAD = 90^\circ$, so $BC = CD = DA = AB$ since triangles BOC, COD, DOA, AOB are congruent (SAS). Hence all angles of $ABCD$ are 90° (a rectangle), but since the central angles have the same value, the diagonals are perpendicular and $ABCD$ is a square according to Theorem 2.1 *(b)* in [28].

Case 2 (opposite). Suppose without loss of generality that $\angle BOA + \angle COD = 180^\circ$, so $\angle BDA + \angle CAD = 90^\circ$. Then the diagonals are perpendicular, implying that $ABCD$ is an isosceles trapezoid (see Figure 10). Assume without loss of generality that $AB = CD$, so that BC and DA are parallel. Then $\angle ABC = \angle BCD$ and $\angle BAD = \angle CDA$. If all four vertex angles are equal, $ABCD$ is a rectangle and the proof concludes as in case 1.

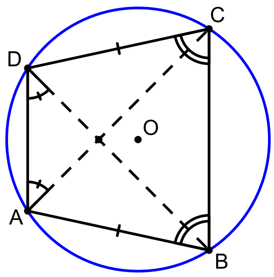

FIGURE 10. Case 2

Otherwise we get either $\angle BOA = \angle COD$ and $\angle BOC = \angle AOD$, resulting in all four vertex angles being $90°$ since $\angle BOA + \angle COD = 180°$, or we have $\angle BOA = \angle BOC$ and $\angle AOD = \angle COD$. Then $AB = BC$ and $DA = CD$, so $AB = BC = CD = DA$ (a rhombus) and since $ABCD$ is cyclic, it must be a square according to Theorem 3.1 *(e)* in [28]. $\square$

8. TANGENTIAL QUADRILATERALS

A tangential quadrilateral is a quadrilateral with an incircle and its center I is called the incenter. The distances from the vertices to the points where the incircle is tangent to the sides are called the *tangent length*, and these are denoted by e, f, g, h. The central angles between radii to the tangent points will be denoted by α, β, γ, δ (see Figure 11).

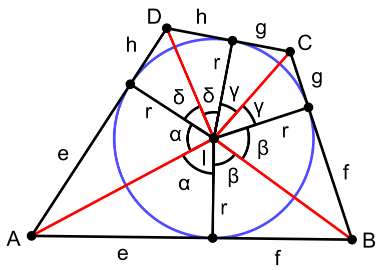

FIGURE 11. Central angles and tangent lengths

We have the following five conditions, all due to the first named author.

Theorem 8.1. *A tangential quadrilateral $ABCD$ with sides a, b, c, d, incenter I, inradius r, tangent length e, f, g, h, and central angles α, β, γ, δ satisfies any one of:*

(a) $\alpha = \beta = \gamma = \delta$
(b) $e = f = g = h$
(c) $AI = BI = CI = DI$
(d) $r = \frac{1}{2}\sqrt[4]{abcd}$
(e) $K = \frac{1}{2}(AI^2 + BI^2 + CI^2 + DI^2)$

if and only if it's a square.

Proof. *(a)* From $\alpha = \beta = \gamma = \delta$ we get that all eight triangles where these four different angles are included are congruent by ASA (see Figure 11), so the tangential quadrilateral is a rhombus with equal vertex angles. That is a square according to Theorem 3.1 (b) in [28].

(b) We have that $\alpha = \beta = \gamma = \delta$ implies $e = f = g = h$ due to ASA congruence (see Figure 11).

(c) We have that $\alpha = \beta = \gamma = \delta$ implies $AI = BI = CI = DI$ due to ASA congruence (see Figure 11).

(d) The area of a tangential quadrilateral is given by ([46, p. 132])

$$K = \sqrt{abcd}\sin\frac{A+C}{2}$$

and the inradius by the well-known formula $r = \frac{K}{s}$, so we get

$$r = \frac{2\sqrt{abcd}\sin\frac{A+C}{2}}{a+b+c+d} \leq \frac{2\sqrt{abcd}\sin\frac{A+C}{2}}{4\sqrt[4]{abcd}} \leq \frac{2\sqrt{abcd}}{4\sqrt[4]{abcd}} = \tfrac{1}{2}\sqrt[4]{abcd}$$

where we applied the AM-GM inequality to get the first inequality, so equality holds there if and only if the quadrilateral is a rhombus ($a = b = c = d$). In the second inequality we have equality if and only if the quadrilateral is cyclic ($\angle A + \angle C = \pi$), so in total equality holds only in a cyclic rhombus, that is, in a square according to Theorem 3.1 *(e)* in [28].

(e) Let K_c be the area of the quadrilateral with vertices at the points where the incircle is tangent to the sides. Then the tangential quadrilateral has area

$$K = K_c + \tfrac{1}{2}\left(e^2\sin A + f^2\sin B + g^2\sin C + h^2\sin D\right)$$

where

$$K_c = \tfrac{1}{2}r^2\left(\sin A + \sin B + \sin C + \sin D\right).$$

We get, by applying the Pythagorean theorem, that

$$\begin{aligned} K &= \tfrac{1}{2}\left((e^2+r^2)\sin A + (f^2+r^2)\sin B + (g^2+r^2)\sin C + (h^2+r^2)\sin D\right) \\ &= \tfrac{1}{2}\left(AI^2\sin A + BI^2\sin B + CI^2\sin C + DI^2\sin D\right) \\ &\leq \tfrac{1}{2}(AI^2 + BI^2 + CI^2 + DI^2) \end{aligned}$$

where equality holds if and only if $\angle A = \angle B = \angle C = \angle D = \frac{\pi}{2}$ (a rectangle). A rectangle with an incircle must be a square since its sides satisfy $a + c = b + d$ and $a = c$ and $b = d$, implying that $a = b = c = d$ (a rhombus), and the conclusion follows from Theorem 2.1 *(a)* in [28]. □

9. Bicentric quadrilaterals

A quadrilateral that is at the same time both cyclic and tangential is called bicentric. Here we study eight conditions that make a bicentric quadrilateral a square. Conditions *(a)* and *(b)* are from the 1995 Brazilian Mathematical Olympiad [20], which was to prove that if any two of O, I, and P coincide, then the quadrilateral is a square. The third possible coincide was included as Theorem 9.1 *(c)* in [28]. *(c)* is from [23], while *(e)* and *(f)* are from the recent paper [47]. Conditions *(g)* and *(h)* are different interpretations of the equality case of inequalities proposed by L. Carlitz and G. J. Griffith in 1972 according to [41, pp. 404–405], wherefrom we cite their proofs.

The radii of the four escribed circles are denoted by r_a, r_b, r_c, r_d and we call these the escribed radii, and the centers of these four circles constitute vertices of a quadrilateral called the *excenter quadrilateral* [29, p. 16], which we denote $I_aI_bI_cI_d$.

Theorem 9.1. *A bicentric quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, circumcenter O, incenter I, diagonal intersection P, circumradius R, inradius r, semiperimeter s, and escribed radii r_a, r_b, r_c, r_d satisfies any one of:*

(a) O *and* P *coincide*
(b) I *and* P *coincide*
(c) $\sum_{cyc} \sin\frac{A}{2}\cos\frac{B}{2} = 2$
(d) $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} = 4\sqrt{2r}$
(e) $r_a = r_b = r_c = r_d$
(f) the circumcenters of $ABCD$ *and* $I_aI_bI_cI_d$ *coincide*
(g) it has area $K = 2\sqrt{2}Rr$
(h) $s^2 = 8\sqrt{2}Rr$

if and only if it's a square.

Proof. *(a)* Applying Thales' theorem, we get that all four vertex angles are right, so $ABCD$ is a rectangle. Then $a = c$ and $b = d$. But it's also tangential, so $a + c = b + d$, implying that $a = b$. This proves that $ABCD$ is a square according to Theorem 2.1 *(a)* in [28].

(b) In a cyclic quadrilateral, $\angle DAC = \angle DBC$. But $\angle DAC = \angle CAB$ and $\angle ABD = \angle CBD$, so $\angle A = \angle B$. Then $\angle A = \angle B = 90°$, and in the same way, $\angle C = \angle D = 90°$ (a rectangle). A tangential rectangle is a square according to Theorem 2.1 *(d)* in [28].

(c) In [23], we derived that the expression

$$\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}$$

in a bicentric quadrilateral is equal to

$$\frac{(\sqrt{ab}+\sqrt{cd})(\sqrt{ad}+\sqrt{bc})}{\sqrt{(ab+cd)(ad+bc)}}.$$

The AM-GM inequality $2\sqrt{xy} \le x + y$ yields that $x + y + 2\sqrt{xy} \le 2(x + y)$, so $(\sqrt{x}+\sqrt{y})^2 \le 2(x + y)$, which is equivalent to

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x+y}} \le \sqrt{2}$$

where equality holds if and only if $x = y$ for positive numbers x and y. Applying this, we get that

$$\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2} \le (\sqrt{2})^2$$

where equality holds if and only if $ab = cd$ and $ad = bc$, that is, only when $a = c$ and $b = d$ (a parallelogram). A bicentric parallelogram is a square according to Theorem 4.1 *(g)* in [28].

(d) By the AM-GM inequality, we have

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}{4} \geq \sqrt[4]{\sqrt{abcd}} = \sqrt[4]{K} \geq \sqrt[4]{(2r)^2} = \sqrt{2r}$$

where equality in the first inequality holds only in a rhombus and in the second only in a square. We used that the area of a bicentric quadrilateral is given by

$$K = \sqrt{abcd} \qquad (4)$$

(see [5, p. 50]), and the inequality $K \geq 4r^2$ which was proved to hold in a tangential quadrilateral in [28, p. 25].

(e) We have according to Lemma 9.1 in [28] that

$$r_a = \frac{a}{c}r$$

and similar formulas hold for the other escribed radii. Hence

$$r_a = r_b = r_c = r_d \quad \Leftrightarrow \quad \frac{a}{c} = \frac{b}{d} = \frac{c}{a} = \frac{d}{b}$$

so we get $a = c$ and $b = d$ (a parallelogram). A bicentric parallelogram is a square according to Theorem 4.1 *(g)* in [28].

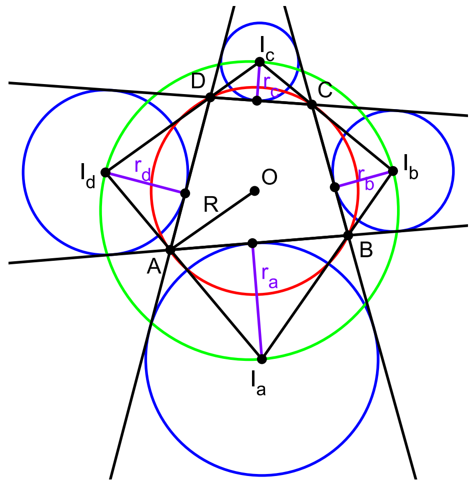

Figure 12. Excenter quadrilateral $I_aI_bI_cI_d$

(f) In [46, pp. 162–163], it's proved that the distance between the circumcenter O in a bicentric quadrilateral $ABCD$ and the center I_a in one of the escribed circles is given by the formula

$$OI_a^2 = R^2 + \left(\sqrt{4R^2 + r^2} - r\right) r_a.$$

Similar formulas hold for the three other distances OI_b, OI_c and OI_d. It's well-known that $I_aI_bI_cI_d$ is always a cyclic quadrilateral (see Figure 12; for a proof, see [29, pp. 16–17]). If O is also the circumcenter of $I_aI_bI_cI_d$, then

$$OI_a^2 = OI_b^2 = OI_c^2 = OI_d^2$$

and it's easy to see that this is equivalent to $r_a = r_b = r_c = r_d$. According to *(d)*, this holds if and only if $ABCD$ is a square.

(g) The diagonals in a cyclic quadrilateral are given by $AC = 2R\sin B$ and $BD = 2R\sin A$, so by Ptolemy's theorem,

$$ac + bd = AC \cdot BD = 4R^2 \sin A \sin B. \tag{5}$$

In a tangential quadrilateral, $\cot\frac{A}{2} = \frac{e}{r}$ and $\cot\frac{B}{2} = \frac{f}{r}$ (see Figure 11), so

$$a = e + f = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right),$$

and

$$b = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = r\left(\cot\frac{B}{2} + \tan\frac{A}{2}\right)$$

since $\angle C = \pi - \angle A$ in a cyclic quadrilateral. The other two sides are given by the similar formulas

$$c = r\left(\tan\frac{A}{2} + \cot\frac{B}{2}\right), \qquad d = r\left(\cot\frac{A}{2} + \tan\frac{B}{2}\right)$$

where we have also used $\angle D = \pi - \angle B$. A short calculation confirms that

$$\begin{aligned} ac + bd &= r^2\left(4 + \left(\cot\frac{A}{2} + \tan\frac{A}{2}\right)\left(\cot\frac{B}{2} + \tan\frac{B}{2}\right)\right) \\ &= 4r^2\left(1 + \frac{1}{\sin A \sin B}\right). \end{aligned} \tag{6}$$

Multiplying (5) and (6), we obtain

$$(ac + bd)^2 = 16R^2r^2(1 + \sin A \sin B).$$

The AM-GM inequality and (4) yields

$$ac + bd \geq 2\sqrt{abcd} = 2K.$$

Hence

$$4K^2 \leq (ac + bd)^2 = 16R^2r^2(1 + \sin A \sin B) \leq 16R^2r^2(1 + 1 \cdot 1) = 32R^2r^2$$

and we get

$$K^2 \leq 8R^2r^2$$

where equality holds if and only if $ac = bd$ and $\angle A = \angle B = \frac{\pi}{2}$, which in a bicentric quadrilateral is equivalent to a tangential rectangle, that is, a square according to Theorem 2.1 *(d)* in [28].

(h) Applying the AM-GM inequality, we get

$$ac + bd \leq \tfrac{1}{4}(a + c)^2 + \tfrac{1}{4}(b + d)^2 = \tfrac{1}{2}(a + c)^2 = \tfrac{1}{2}s^2 \tag{7}$$

where we used that $a + c = b + d$ holds in tangential quadrilaterals. Equality holds if and only if $a = c$ and $b = d$ (a parallelogram), and the only bicentric parallelogram is a square according to Theorem 4.1 *(g)* in [28].

Using the formulas for a and c from the proof of *(f)* yields

$$a + c = r\left(\tan\frac{A}{2} + \cot\frac{A}{2} + \tan\frac{B}{2} + \cot\frac{B}{2}\right) = r\left(\frac{2}{\sin A} + \frac{2}{\sin B}\right)$$

which attains its minimum value when $\angle A = \angle B = \frac{\pi}{2}$ (a square). From (7) and (6), we obtain

$$s^4 \geq 4(ac + bd)^2 = 64R^2r^2(1 + \sin A \sin B)$$

and since the minimum of $a+c$ occurs when $\angle A = \angle B = \frac{\pi}{2}$, it follows that

$$s^4 \geq 128R^2r^2$$

where equality holds only for a square. □

10. Convex quadrilaterals

In this section we study six characterizations that make a general convex quadrilateral a square. Both condition *(a)*, which is just another way of stating that *a quadrilateral is a square if and only if it is both a rhombus and a rectangle*, and *(c)* are from the old book [19, p. 124], and *(b)* is from the old book [10, p. 41]. *(d)* is a beautiful symmetric but not minimal condition (two equalities can be dropped) due to the second named author, while *(e)* is from the 2004 Indian Mathematical Olympiad [2], and *(f)* is from the 2002 Third selection examination for the Junior Balkan Mathematical Olympiad [16, pp. 79–80] (proposed by Mircea Fianu). We reproduce the proofs given at the two last cited sources.

Theorem 10.1. *A convex quadrilateral $ABCD$ with sides a, b, c, d, diagonals p, q, and semidiagonals a', b', c', d' satisfies any one of:*

- *(a) it has 4 equal sides and 4 equal angles*
- *(b) it has equal, perpendicular, and bisecting diagonals*
- *(c) each diagonal and each bimedian is a symmetry line*
- *(d) $a = b = c = d$ and $a' = b' = c' = d'$*
- *(e) $QA = QB = QC = QD$ and $\frac{LK}{LM} = \frac{CD}{CB}$ where K, L, M, N are the midpoints of AB, BC, CD, DA respectively and BD bisects KM at Q*
- *(f) the area of triangle XPY is constant, where P is the diagonal intersection, and X and Y are points on the sides of $ABCD$ such that angle XPY is equal to the angle between the diagonals*

if and only if it's a square.

Proof. *(a)* This is equivalent to the definition of squares we used in [28], since 4 equal angles and 4 right angles are equivalent due to the angles sum of a convex quadrilateral.

(b) Bisecting diagonals means it's a parallelogram, the addition of equal diagonals that it must be a rectangle, and also perpendicular diagonals that the rectangle is a square according to Theorem 2.1 *(b)* in [28].

(c) That each diagonal is a symmetry line is a characterization of rhombi and that each bimedian is a symmetry line is a characterization of rectangles, so this is just another way of stating *(a)*.

(d) $a = b = c = d$ implies a rhombus (by definition), and according to Theorem 3.1 *(c)* in [28], only two adjacent semidiagonals are needed to be equal for $ABCD$ to be a square.

(e) Triangles QKB and QMD are congruent (SAS), so $MD = KB$ and thus $AB = CD$. It also follows, by alternate angles, that AB and DC are parallel (see Figure 13). A pair of equal and parallel sides implies that $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in [30]. Triangles QMD and QMC are also congruent (SSS), so $\angle QMD = \angle QMC = 90°$ and

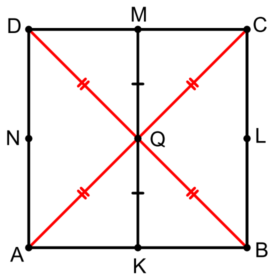

FIGURE 13. Midpoints K, L, M, N

by alternate angles, $\angle QKA = \angle QKB = 90°$. From these angle equalities and the fact that a bimedian in a parallelogram is parallel to a pair of opposite sides, it follows that $\angle A = \angle B = \angle C = \angle D = 90°$ (a rectangle). Finally, triangles LCM and LBK are congruent (SAS), so $LM = LK$, and then $CD = CB$, implying that $ABCD$ is a square according to Theorem 2.1 *(a)* in [28]

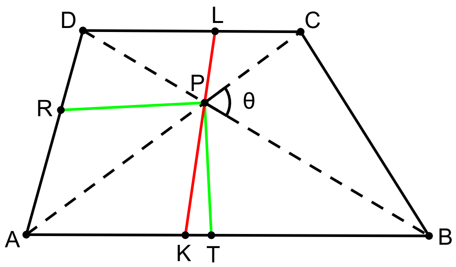

FIGURE 14. $\angle DPL = \angle BPK = \theta$

(f) Assume that $\angle APD := \theta \leq 90°$ is the angle between the diagonals. Then $[APD] = [BPC]$ by the assumption, where square brackets denote area of the included triangle. We get

$$\tfrac{1}{2}AP \cdot DP \sin\theta = \tfrac{1}{2}BP \cdot CP \sin\theta$$

so

$$\frac{AP}{CP} = \frac{BP}{DP}.$$

Together with $\angle DPC = \angle APB$, this implies that triangles DPC and APB are similar and CD is parallel to AB (see Figure 14). Next we draw a line segment KL through P with $K \in AB$ and $L \in CD$ such that $\angle DPL = \angle BPK = \theta$. Then triangles DPL and BPK are similar and have equal area, so they are congruent. We get that $DP = BP$, and in the same way, $CP = AP$. Therefor $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in [30]. We also get that $[BPC] = [BPK]$ and $[BPC] = [APB]$. Hence $A \equiv K$ and

$$\theta = \angle BPA = \angle BPK = \angle BPC = 90°.$$

This confirms that $ABCD$ is a rhombus.

Finally we consider the angle bisectors RP and TP to the angles between the diagonals, with $R \in DA$ and $T \in AB$. Then $\angle RPT = \angle APD = \theta = 90°$, so $[RPT] = [APD]$. Hence $[APT] = [DPR]$, and we obtain

$$[DPR] = [APR] = \tfrac{1}{2}[APD].$$

This proves that PR is a median in triangle APD, so $AP = DP$. Then the rhombus is a square according to Theorem 3.1 *(c)* in [28]. □

In [19, p. 124], Henrici conclude the short section on squares by stating that in a square, the median lines (what we call bimedians) are equal, and each is the perpendicular bisector of the other. This is true, but he adds that the converse also holds and that the proof is left to the reader. This is however *not* a true statement and here is why. The bimedians always bisect each other in all quadrilaterals, but if they are perpendicular and equal, then the quadrilateral has equal and perpendicular diagonals (see [25, p. 137]). Such quadrilaterals are not necessarily squares, but what we called *midsquare quadrilaterals* in [25, p. 137] (see Figure 15). It's only if their *diagonals* also bisect each other that they are squares (see [25, p. 138]). This was part *(b)* in the theorem we just proved.

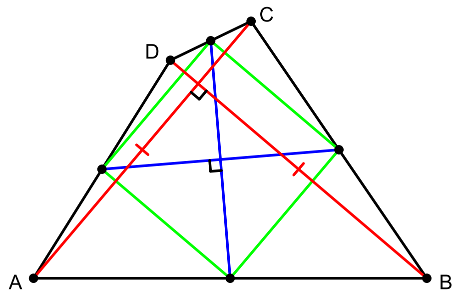

Figure 15. A midsquare quadrilateral $ABCD$

11. Area

In the last theorem we prove six area formulas that are sufficient conditions for when a convex quadrilateral is a square, where *(b)* is due to the second named author. Conditions *(c)*, *(d)*, and *(e)* were inspired from inequalities proved in [46, pp. 181–182, 185–186, 195], but the equality cases were not stated specifically there for the first two of them. Those inequalities are due to A. Pop in 1989 and Gerasimov in 1967. The last condition is the equality case of an inequality given as Problem 12033 in the American Mathematical Monthly in 2018, proposed by Dao Thanh Oai and Leonard Giugiuc [14]. We cite proofs from the last two sources here.

Theorem 11.1. *A convex quadrilateral with sides a, b, c, d, diagonals p, q, semidiagonals a', b', c', d', quarter triangle circumradii R_1, R_2, R_3, R_4, bimedians m, n, and distance v between the diagonal midpoints, has an area K satisfying any one of:*

(a) $K = a^2 = b^2 = c^2 = d^2$
(b) $K = \frac{1}{2}(a'^2 + b'^2 + c'^2 + d'^2)$
(c) $K = R_1^2 + R_2^2 + R_3^2 + R_4^2$
(d) $K = \frac{1}{2}(m^2 + n^2 + v^2)$
(e) $K = \frac{1}{4}\sqrt{(a+b)(b+c)(c+d)(d+a)}$
(f) $K = \frac{1}{8}(a^2 + b^2 + c^2 + d^2 + p^2 + q^2 - ab - cd + pq)$

if and only if it's a square.

Proof. *(a)* From $a^2 = b^2 = c^2 = d^2$ we get a rhombus, and since rhombii have area $K = a^2 \sin A$, we also get that the rhombus has a right vertex angle, so it's a square according to Theorem 3.1 *(a)* in [28].

(b) If we denote one of the angles between the diagonals by θ, we have that the area of a convex quadrilateral satisfies

$$\begin{aligned} K &= \tfrac{1}{2}\sin\theta(a'b' + b'c' + c'd' + d'a') \\ &= \tfrac{1}{2}\sin\theta(a' + c')(b' + d') \\ &\le \tfrac{1}{4}\sin\theta[(a' + c')^2 + (b' + d')^2] \\ &\le \tfrac{1}{2}\sin\theta(a'^2 + b'^2 + c'^2 + d'^2) \\ &\le \tfrac{1}{2}(a'^2 + b'^2 + c'^2 + d'^2) \end{aligned}$$

where we used the AM-GM inequality in the two forms $xy \le \frac{1}{2}(x^2+y^2)$ and $(x+y)^2 \le 2(x^2+y^2)$. We have equality throughout the area calculation if and only if $a' + c' = b' + d'$, $a' = c'$, $b' = d'$, and $\theta = 90°$, that is, only when the diagonals have equal lengths, bisect each other, and are perpendicular. This is equivalent to a square according to Theorem 10.1 *(b)*.

(c) Using the extended law of sines, we get

$$R_1^2 + R_2^2 + R_3^2 + R_4^2 = \frac{a^2 + b^2 + c^2 + d^2}{4\sin^2\theta} \ge \frac{a^2 + b^2 + c^2 + d^2}{4} \ge K$$

where θ is one of the angles between the diagonals. The last inequality was proved in the proof of Theorem 12.1 *(a)* in [28], where equality holds if and only if the quadrilateral is a square.

(d) In a convex quadrilateral, the bimedians and the diagonals are related according to the formula $2(m^2 + n^2) = p^2 + q^2$, and we also have

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2,$$

where the last equality is due to Euler (both relations were derived in [5, pp. 9–10]). We get

$$\begin{aligned} m^2 + n^2 + v^2 &= \frac{p^2 + q^2}{2} + \frac{a^2 + b^2 + c^2 + d^2 - p^2 - q^2}{4} \\ &= \frac{a^2 + b^2 + c^2 + d^2}{4} + \frac{p^2 + q^2}{4} \ge K + K = 2K \end{aligned}$$

where the two inequalities we used at the end were proved in [28, p. 33] and [25, p. 138]. Equality holds in the first if and only if the quadrilateral is a square, and in the second if and only if the quadrilateral has diagonals that are equal and perpendicular; altogether only in a square.

(e) The area of a convex quadrilateral is, according to Bretschneider's formula (see [5, pp. 18–19]), given by

$$
\begin{aligned}
K^2 &= (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+C}{2}\right)\\
&\le (s-a)(s-b)(s-c)(s-d) \qquad (8)
\end{aligned}
$$

where equality holds if and only if it is cyclic. Next we note that

$$
\begin{aligned}
2\sqrt{(s-c)(s-d)} &= \sqrt{(a+b)^2-(c-d)^2} \le a+b,\\
2\sqrt{(s-a)(s-d)} &= \sqrt{(b+c)^2-(a-d)^2} \le b+c,\\
2\sqrt{(s-a)(s-b)} &= \sqrt{(c+d)^2-(a-b)^2} \le c+d,\\
2\sqrt{(s-b)(s-c)} &= \sqrt{(d+a)^2-(b-c)^2} \le d+a
\end{aligned}
$$

where equality holds if and only if $c = d = a = b$. Multiplying these four and substituting into (8) yields

$$K^2 \le \tfrac{1}{16}(a+b)(b+c)(c+d)(d+a)$$

where equality holds only for a cyclic rhombus, that is, just for a square.

(f) By Ptolemy's inequality and the AM-GM inequality, we have

$$2pq \le 2(ac+bd) \le a^2+c^2+b^2+d^2$$

and $2pq \le p^2+q^2$. Since real algebraic squares are never negative,

$$0 \le (a-c)^2 + (b-d)^2 + (p-q)^2$$

where equality holds if and only if $a = c$, $b = d$, $p = q$, that is, only in rectangles (parallelograms with equal diagonals). Expanding, adding these three inequalities, rewriting, and dividing all terms by 2 yields

$$ac+bd-pq+4pq \le a^2+b^2+c^2+d^2+p^2+q^2.$$

The area of a convex quadrilateral satisfies

$$K = \tfrac{1}{2}pq\sin\theta \le \tfrac{1}{2}pq$$

so $8K \le 4pq$ with equality only for perpendicular diagonals. Via substitution, we get

$$ac+bd-pq+8K \le a^2+b^2+c^2+d^2+p^2+q^2$$

where equality holds if and only if the quadrilateral is a rectangle with perpendicular diagonals, so only in a square. Now we only have to solve for K to finish the derivation. □

The observant reader might have noticed by reading between the lines in the proof of *(d)* an interesting conclusion: if we know that the area of a convex quadrilateral is given by $K = \frac{1}{2}(m^2+n^2+v^2)$, then it is a square (and they have $v = 0$), but if we only know that the area is given by the formula $K = \frac{1}{2}(m^2+n^2)$, then we are limited to conclude that the diagonals are equal and perpendicular (a midsquare quadrilateral).

12. Most useful characterizations

Of the 150 characterizations of squares we have studied in this paper and in [28], which have been the most useful in the proofs of other characterizations, and how frequently were they used? This is accounted for in Table 3.

Characterization	**Number of proofs**
Rectangle with 2 adjacent equal sides	42
Rectangle with perpendicular diagonals	18
Rhombus with a right vertex angle	15
Rhombus that is cyclic	9
Rhombus with 2 adjacent equal vertex angles	9
Quad with 4 equal sides and 4 right angles	6
Both isosceles trapezoid and kite	6

Table 3. Most frequently used characterizations

Note that at the end of several proofs, it is possible to refer to at least two different characterizations as the reason for why the quadrilateral is a square, so the numbers in Table 3 are approximate. Nonetheless they are clear on which are the most useful characterizations. It is interesting that the definition of squares we chose in [28] is only on sixth place. Besides the seven characterizations in Table 3, twenty others were used between one and five times, for a total of forty-five times.

13. Chronological compilation

We conclude by summarizing all 150 characterizations of squares that have been collected in theorems in the present paper and in [28]. They are given in chronological order with respect to the oldest source for the *sufficient* condition that we know of, but several of them have surely been published earlier (like for instance those with number 35, 36, 41, 55 and 58; they are definitely elementary, but rarely included in math books).

The following notations are used: A quadrilateral $ABCD$, abbreviated Q or quad, has sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, diagonals $p = AC$, $q = BD$, area K, semiperimeter s, diagonal intersection P (in *italic*), and semidiagonals $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$. A cyclic quadrilateral has circumradius R (in *italic*) and circumcenter O, and a tangential quadrilateral has inradius r and incenter I. For a bicentric quadrilateral, all of these four latter terms apply. Regarding the meaning of other notations, please see the corresponding theorems in the two papers.

BQ is used as abbreviation for bicentric quadrilateral, CQ for cyclic quadrilateral, ETQ for extangential quadrilateral, IT for isosceles trapezoid, ODQ for orthodiagonal quadrilateral, P for parallelogram, Re for rectangle, Rh for rhombus, TQ for tangential quadrilateral, Δ^s for triangles, V's P for Varignon's parallelogram, and w for with.

BraMO, BriMO, CACMO, CheMO, IndMO, IreMO are used as abbreviations for the Brazilian, British, Central American and Caribbean, Czechoslovakian, Indian, and Irish Mathematical Olympiad respectively.

#	Year	Source	Short description	Ref.
1	-300	Euclid	Q with 4 equal sides and 4 right angles	[33]
2	1833	Young	Rhombus with a right vertex angle	[59]
3	1868	Wright	P w equal and perpendicular diagonals	[57]
4	1869	Chauvenet	Q w equal, perp. & bisecting diagonals	[10]
5	1871	Todhunter	P w largest area for a given perimeter	[53]
6	1879	Henrici	Q w each diag. & each bimed. a sym. line	[19]
7	1879	Henrici	Q with 4 equal sides and 4 equal angles	[19]
8	1879	Henrici	P w equal diag. & 1 diag. bisects an angle	[19]
9	1889	Dupuis	Rectangle with 2 adjacent equal sides	[15]
10	1889	Dupuis	Rectangle w largest area for given perim.	[15]
11	1893	Smith	P that's both a rectangle and a rhombus	[56]
12	1896	Halsted	Q that's both a kite & isosceles trapezoid	[18]
13	1898	Hadamard	ODQ with equal and bisecting diagonals	[17]
14	1899	Beman	Rectangle with perpendicular diagonals	[8]
15	1948	Tóth	Bicentric Q with $R = \sqrt{2}r$	[54]
16	1950	Schoenberg	Q with $K = \frac{1}{2}(w^2 + x^2 + y^2 + z^2)$	[52]
17	1961	Kazarinoff	Q with largest area for a given perimeter	[34]
18	1961	Kazarinoff	Cyclic Q w largest area for a given perim.	[34]
19	1962	Skopec	Q with $K = \frac{1}{4}(a^2 + b^2 + c^2 + d^2)$	[9]
20	1972	Klamkin	Bicentric Q with $8pq = (a + b + c + d)^2$	[35]
21	1972	Carlitz	Bicentric Q with $K = 2\sqrt{2}Rr$	[41]
22	1972	Carlitz	Bicentric Q with $s^2 = 8\sqrt{2}Rr$	[41]
23	1974	Jacobs	P w 1 right angle & 2 adj. equal sides	[56]
24	1976	Ivanova	Tangential Q with $s = 4r$	[41]
25	1977	Andreescu	Cyclic Q with $K = (s/2)^2$	[6]
26	1979	[CheMO]	Cyclic Q with $R = 1$ and $abcd \geq 4$	[3]
27	1981	Niven	Q with $K = \frac{1}{16}(a + b + c + d)^2$	[43]
28	1986	Meyers	Cyclic Q where diag. are perp. diameters	[40]
29	1993	[Baltic Way]	$\angle AOB$, $\angle BOC$, $\angle COD$, $\angle DOA$ eq. CQ	[37]
30	1995	[BraMO]	BQ with O and I coinciding	[20]
31	1995	[BraMO]	BQ with O and P coinciding	[20]
32	1995	[BraMO]	BQ with I and P coinciding	[20]
33	2000	Lee	P w AMN right-angled and $AM = AN$	[39]
34	2000	[IreMO]	Cyclic Q with $K = (abcd)^{3/4}/(R\sqrt{2})$	[1]
35	2002	Posamentier	Rectangle w a diag. bisecting an angle	[48]
36	2002	Posamentier	Rhombus with equal diagonals	[48]
37	2002	Fianu	Rectangle with $DF + BE = AE$	[16]
38	2002	Fianu	The area of triangle XPY is constant	[16]
39	2002	Serbanescu	Parallelogram w ABC and AST similar	[16]
40	2004	[IndMO]	$QA = QB = QC = QD$ and $\frac{LK}{LM} = \frac{CD}{CB}$	[2]
41	2005	Yadav	Rhombus that is cyclic	[58]
42	2005	Tydd	Rectangle with equal bimedians	[55]
43	2006	Andreescu	Re w largest area inscr. in a given circle	[7]
44	2006	Andreescu	Cyclic Q with $K = 2R^2$	[7]
45	2006	Andreescu	Q inscribed in given circle w largest area	[7]
46	2008	Usiskin	Q with rotational symmetry of order 4	[56]
47	2008	[Baltic Way]	CQ w $(ab + cd)(ac + bd)(ad + bc)$ max	[38]
48	2012	Josefsson	Bicentric Q with $\sum_{cyc} \sin\frac{A}{2}\cos\frac{B}{2} = 2$	[23]
49	2013	Pop	Q w $K = \frac{1}{4}\sqrt{(a + b)(b + c)(c + d)(d + a)}$	[46]
50	2013	[Kömal]	TQ, orthodiagonal, cyclic, trapezoid	[36]
51	2014	Josefsson	Quad with $K = \frac{1}{2}(a^2 + c^2) = \frac{1}{2}(b^2 + d^2)$	[25]
52	2015	Josefsson	Quad with $K = \frac{1}{8}\left((a + c)^2 + (b + d)^2\right)$	[26]
53	2018	Pamfilos	Harmonic parallelogram	[44]
54	2018	Dao	$K = \frac{1}{8}(pq - ab - cd + p^2 + q^2 + \sum_{cyc} a^2)$	[14]
55	2020	Alsina	Rectangle that is tangential	[5]

56	2020	Alsina	Q w $K = \frac{1}{2}\sqrt[3]{(ab+cd)(ac+bd)(ad+bc)}$	[5]
57	2022	[CACMO]	Rectangle w 4 interior tangent circles	[4]
58	2022	Dalcín	Rhombus with 2 adjacent equal angles	[11]
59	2022	Dalcín	Rhombus w 2 adj. equal semidiagonals	[11]
60	2022	Dalcín	Q w $\angle A = \angle B, \angle C = \angle D$ and $a = b = c$	[11]
61	2022	Dalcín	Q w $\angle A = \angle B = \angle D$ and $a = b, c = d$	[11]
62	2022	Dalcín	Q w $\angle A = \angle B, \angle C = \angle D$, $a = b, c = d$	[11]
63	2022	Dalcín	Q with $a = d, b = c$ and $a' = d', b' = c'$	[11]
64	2022	Dalcín	Q with $\angle A = \angle C = \angle D$ and $a = b = c$	[11]
65	2022	Dalcín	Q with $a = b = c$ and $a' = b' = c'$	[11]
66	2022	Dalcín	Q with $a = b = c$ and $a' = c' = d'$	[11]
67	2022	Dalcín	Q with $a = d$ and $a' = b' = c' = d'$	[11]
68	2022	Dalcín	Q with $b = c = d$ and $a' = d', b' = c'$	[11]
69	2023	Dalcín	P w perpendicular diagonals and $a' = b'$	[12]
70	2023	Dalcín	Q w $\angle A = \angle B, \angle C = \angle D, a = c$ and $p \perp q$	[12]
71	2023	Dalcín	Q w $\angle A = \angle B, \angle C = \angle D, a' = c'$, $p \perp q$	[12]
72	2023	Dalcín	Q w $a' = b', c' = d', a = c$ and $p \perp q$	[12]
73	2023	Dalcín	IT w $a' = b', c' = d'$ and $\angle A = \angle C$	[12]
74	2023	Dalcín	ODQ with $a = b = d$ and $\angle A = \angle B$	[12]
75	2023	Dalcín	ODQ with $a = b = d$ and $a' = b'$	[12]
76	2023	Dalcín	ODQ with $a = d$ and $a' = b' = c'$	[12]
77	2023	Dalcín	ODQ with $\angle A = \angle B = \angle C$ and $b = d$	[12]
78	2023	Dalcín	ODQ with $\angle A = \angle B = \angle C$ and $a = d$	[12]
79	2023	Dalcín	ODQ with $d' = a' = b'$ and $\angle A = \angle B$	[12]
80	2023	Dalcín	ODQ with $a = c$ and $a' = b' = c'$	[12]
81	2023	Josefsson	Rectangle w concurrent angle bisectors	[28]
82	2023	Josefsson	Re w diagonals divide it into 4 congr. Δ^s	[28]
83	2023	Josefsson	Rectangle w diag. intersect. eq.di. to sides	[28]
84	2023	Josefsson	Rhombus with $AE = BE$	[28]
85	2023	Josefsson	Rhombus w diag. intersect. eq.di. to vert.	[28]
86	2023	Josefsson	P that is cyclic with perp. diagonals	[28]
87	2023	Josefsson	P that is both cyclic and tangential	[28]
88	2023	Josefsson	Q w $a' = b', c' = d', b = c$ and $\angle A = \angle D$	[28]
89	2023	Josefsson	Q w $\angle C = \angle D, b = c = d$ and $p \perp q$	[28]
90	2023	Josefsson	Q w $\angle A = \angle B, \angle C = \angle D, b = c$, $a' = d'$	[28]
91	2023	Josefsson	Kite with equal and bisecting diagonals	[28]
92	2023	Josefsson	Kite w $AB = CD$ & a right vertex angle	[28]
93	2023	Josefsson	Kite w $AB \parallel CD$ & a right vertex angle	[28]
94	2023	Josefsson	Q w $a = b, c = d$ and $\angle B = \angle D = 90°$	[28]
95	2023	Josefsson	Tangential Q with $K = 4r^2$	[28]
96	2023	Josefsson	Q circumscr. given circle w smallest area	[28]
97	2023	Josefsson	TQ w largest area for a given perimeter	[28]
98	2023	Josefsson	Bicentric Q with $\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} = 4$	[28]
99	2023	Josefsson	Bicentric Q with $r_a + r_b + r_c + r_d = 4r$	[28]
100	2023	Josefsson	Bicentric Q with $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + \frac{1}{r_d} = \frac{4}{r}$	[28]
101	2023	Josefsson	BQ w largest area for a given perimeter	[28]
102	2023	Josefsson	Q w $K = \frac{1}{6}(ab + ac + ad + bc + bd + cd)$	[28]
103	2023	Pop	BQ w cir.cent. of $ABCD$ & $I_aI_bI_cI_d$ coinc.	[47]
104	2023	Pop	Bicentric Q with $r_a = r_b = r_c = r_d$	[47]
105	2024	Pamfilos	Re w shortest diagonals for given area	[45]
106	2025	Dalcín	Rectangle with $a = \frac{1}{4}(a + b + c + d)$	[32]
107	2025	Dalcín	Re w 2 adjacent equal semibimedians	[32]
108	2025	Dalcín	Re w bimedian intersec. eq.dist. to sides	[32]
109	2025	Josefsson	Re w shortest perimeter for given area	[32]
110	2025	Josefsson	Re w largest area for given diagonals	[32]
111	2025	Josefsson	Re w equal heights to two adjacent sides	[32]
112	2025	Josefsson	Re w two adj. equal escribed circle radii	[32]

113	2025	Josefsson	Re w 2 adj. quarter Δ^s w eq. circumradii	[32]
114	2025	Josefsson	Re w 2 adj. quarter Δ^s w equal inradii	[32]
115	2025	Josefsson	Re w 2 adj. quarter Δ^s w equal exradii	[32]
116	2025	Dalcín	Rhombus with $a' = \frac{1}{4}(a' + b' + c' + d')$	[32]
117	2025	Dalcín	Rh w $\angle A = \frac{1}{4}(\angle A + \angle B + \angle C + \angle D)$	[32]
118	2025	Dalcín	Rhombus with concurrent perp. bisectors	[32]
119	2025	Dalcín	Rhombus with perpendicular bimedians	[32]
120	2025	Dalcín	Rhombus w a bimedian perp. to a side	[32]
121	2025	Dalcín	Rhombus w bimed. intersec. eq.di. to vert.	[32]
122	2025	Dalcín	Rh w bimed. div. V's P into 4 congr. Δ^s	[32]
123	2025	Dalcín	Rhombus w max area for a given perim.	[32]
124	2025	Dalcín	Rhombus w min area circ.scr. given circle	[32]
125	2025	Josefsson	P w equal and perpendicular bimedians	[32]
126	2025	Dalcín	P w equal bimed. & one is perp. to a side	[32]
127	2025	Dalcín	P w 2 adj. eq. bimed. parts & perp. bimed.	[32]
128	2025	Dalcín	P that is tangential and has equal diag.	[32]
129	2025	Josefsson	P that is tangential and has perp. bimed.	[32]
130	2025	Josefsson	Trisosceles trapezoid with a right angle	[32]
131	2025	Josefsson	Trisosceles trapezoid with parallel legs	[32]
132	2025	Josefsson	Trisosceles trapezoid with equal bases	[32]
133	2025	Josefsson	Bicentric trapezoid with a right angle	[32]
134	2025	Dalcín	TQ, extangential, cyclic, trapezoid	[32]
135	2025	Dalcín	TQ, extangential, equidiagonal, trapezoid	[32]
136	2025	Dalcín	TQ, extangential, cyclic, equidiagonal	[32]
137	2025	Dalcín	TQ, ETQ, equidiagonal, semidiagonal	[32]
138	2025	Dalcín	TQ, extangential, cyclic, semidiagonal	[32]
139	2025	Josefsson	Cyclic Q with $K = r_a^2 + r_b^2 + r_c^2 + r_d^2$	[32]
140	2025	Josefsson	Tangential Q with $\alpha = \beta = \gamma = \delta$	[32]
141	2025	Josefsson	Tangential Q with $e = f = g = h$	[32]
142	2025	Josefsson	Tangential Q with $AI = BI = CI = DI$	[32]
143	2025	Josefsson	Tangential Q with $r = \frac{1}{2}\sqrt[4]{abcd}$	[32]
144	2025	Josefsson	TQ w $K = \frac{1}{2}(AI^2 + BI^2 + CI^2 + DI^2)$	[32]
145	2025	Josefsson	BQ with $\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} = 4\sqrt{2r}$	[32]
146	2025	Dalcín	Q w $a = b = c = d$ & $a' = b' = c' = d'$	[32]
147	2025	Josefsson	Q with $K = a^2 = b^2 = c^2 = d^2$	[32]
148	2025	Dalcín	Q with $K = \frac{1}{2}(a'^2 + b'^2 + c'^2 + d'^2)$	[32]
149	2025	Josefsson	Q with $K = R_1^2 + R_2^2 + R_3^2 + R_4^2$	[32]
150	2025	Josefsson	Q with $K = \frac{1}{2}(m^2 + n^2 + v^2)$	[32]

Table 4. Characterizations of squares

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