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The Relationship between a Convex Quadrilateral's Semi-Excircles and Diagonals

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Abstract. In this article, the author generalizes a result about the ratios of the areas of the pairs of triangles created by the diagonals of a tangential quadrilateral to a more general result for all convex quadrilaterals. The result is a theorem establishing a relationship between the areas of the triangles formed by a diagonal of a convex quadrilateral and the lengths of the segments formed by the points of tangency with semi-excircles.

1. Introduction

In his exploration of a Nagel line for a tangential quadrilateral (i.e., a quadrilateral that has an incircle tangent to all four sides), Myakishev [4] references a result found in the notes of Yiu [5]. While proving another property for tangential quadrilaterals, Yiu proves the following result.

Theorem 1.1. *Let $ABCD$ be a tangential quadrilateral, and let W, X, Y, and Z be the points of tangency of the incircle with sides AD, AB, BC, and CD, respectively. Let P be the intersection of the diagonals. Then*

$$\frac{\Delta BCD}{\Delta ABD} = \frac{PC}{PA} = \frac{YC}{WA}$$

and

$$\frac{\Delta ACD}{\Delta ABC} = \frac{PD}{PB} = \frac{ZD}{XB}.$$

Yiu [5] provides a short proof relying on the fact that the chords formed by the opposite points of tangency with the incircle intersect each other at the intersection of the diagonals (see Figure 1).

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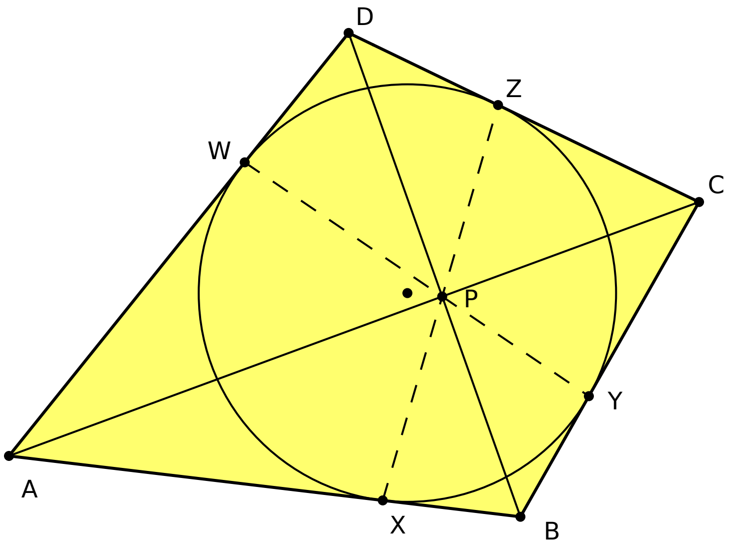

Figure 1

2. A Theorem for Convex Quadrilaterals

While exploring convex quadrilaterals and their semi-excircles, the author discovered a relationship similar to the one in Theorem 1.1 that holds for all convex quadrilaterals. Since the term excircle is typically reserved for a circle that is tangent to all four of the lines containing the sides of a quadrilateral, we will use the term semi-excircle coined by Gemawati et al [1] for a circle that is externally tangent to the side of a quadrilateral and the extensions of the two adjacent sides. While only certain quadrilaterals have an excircle [2], all convex quadrilaterals have four semi-excircles.

Although Myakishev [4] never explicity mentions semi-excircles, he uses the four isotomic conjugates of the points of tangency with the incircle of a tangential quadrilateral to define a Nagel point. In fact, the isotomic conjugate of a point of tangency of the side of a tangential quadrilateral with the incircle is the point of tangency with that side's semi-excircle for the same reason that a triangle's Nagel point is the isotomic conjugate of its Gergonne point [3]. Given that fact, Theorem 1.1 can be viewed as a special case of a more general theorem for convex quadrilaterals that we will state and prove after proving a few lemmas.

Lemma 2.1. *Let $ABCD$ be a convex quadrilateral. Let the diagonals intersect at point P, and let the four semi-excircles intersect sides AD, AB, BC, and CD at points I, J, K, and L, respectively, then*

$$\frac{AI \cdot DL \cdot CK \cdot BJ}{AJ \cdot DI \cdot CL \cdot BK} = 1.$$

Proof. Let O_1, O_2, O_3, and O_4 be the centers of the semi-excircles off sides AD, AB, BC, and CD, respectively. Let r_i be the radius of the semi-excircle with center O_i. Label the other points of tangency of the semi-excircles with the extended sides T_1 through T_8 as in Figure 2. Note that the vertices of the quadrilateral, the centers of the semi-excircles, and the 12 points of tangency form four pairs of similar kites. From these similar kites, we have the following proportions:

$$\frac{AI}{AJ} = \frac{r_1}{r_2}, \frac{DL}{DI} = \frac{r_4}{r_1}, \frac{CK}{CL} = \frac{r_3}{r_4}, \text{ and } \frac{BJ}{BK} = \frac{r_2}{r_3}.$$

The lemma follows directly from these proportions.

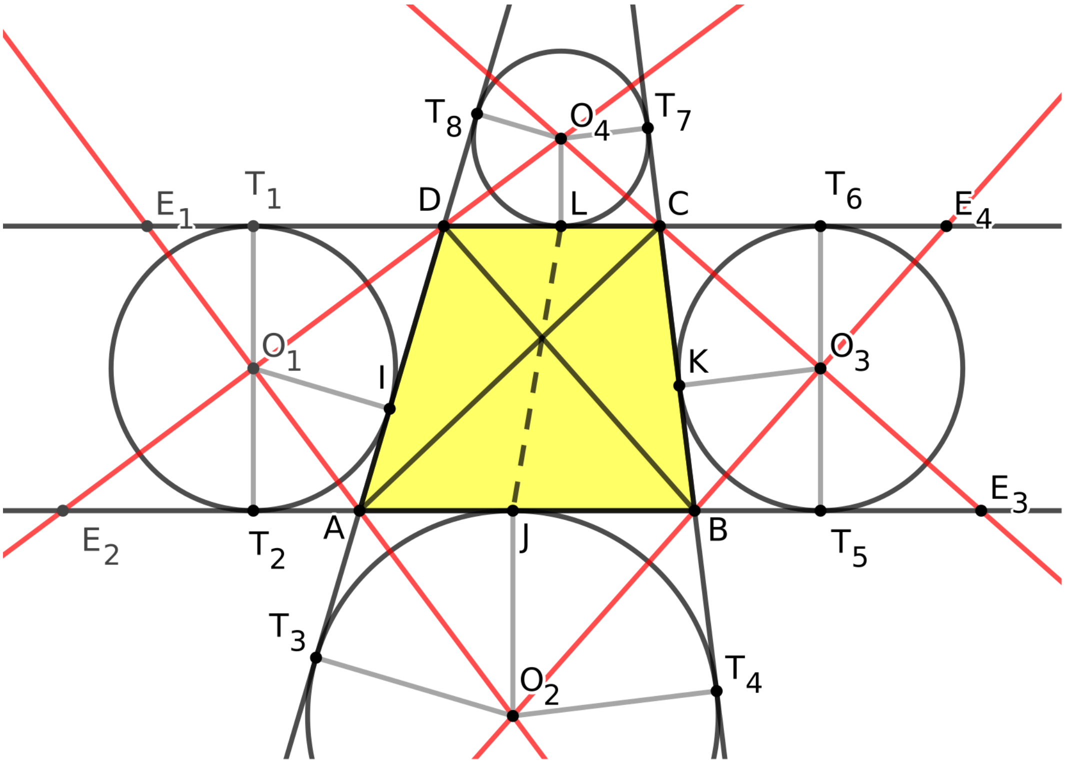

Figure 2

Lemma 2.2. *Let ABCD be a trapezoid with* $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$*. Let the four semi-excircles intersect sides AD, AB, BC, and CD at points I, J, K, and L, respectively, then*

$$\frac{CL}{DL} = \frac{AJ}{BJ} = \frac{AI}{BK}$$

Proof. Label the centers of the semi-excircles, the points of tangency, and the radii of the semi-excircles as in the proof of Lemma 2.1 and Figure 2. Additionally, label the intersections of the exterior angle bisectors at A, B, C, and D with the opposite parallel sides as E_1, E_2, E_3, and E_4, respectively.

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\angle E_1DO_1 \cong \angle AE_2O_1$. Using the angle bisectors, $\angle E_1DO_1 \cong \angle IDO_1$ and $\angle O_1AD \cong \angle O_1AE_2$. So, $\Delta AO_1D \sim \Delta AO_1E_2$. Therefore, $\angle AO_1D$ and $\angle AO_1E_2$ are right angles, being congruent and supplementary. By a similar argument, ΔBO_3C is a right triangle. Thus, we have the following sets of similar triangles:

$$\Delta AJO_2 \sim \Delta AIO_1 \sim \Delta O_1ID \sim \Delta O_4LD$$

and

$$\Delta BJO_2 \sim \Delta BKO_3 \sim \Delta O_3KC \sim \Delta O_4LC$$

Therefore,

$$\frac{r_2}{AJ} = \frac{DL}{r_4} \text{ and } \frac{BJ}{r_2} = \frac{r_4}{CL},$$

which implies

$$\frac{CL}{DL} = \frac{AJ}{BJ}.$$

As noted in the proof of Lemma 2.1,

$$\frac{AI}{AJ} = \frac{r_1}{r_2} \text{ and } \frac{BJ}{BK} = \frac{r_2}{r_3}.$$

Since $ABCD$ is a trapezoid, $r_1 = r_3$ and thus,

$$\frac{AJ}{BJ} = \frac{AI}{BK}.$$

Lemma 2.3. *Let $ABCD$ be a trapezoid with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Let the diagonals intersect at point P, and let the four semi-excircles intersect sides AD, AB, BC, and CD at points I, J, K, and L, respectively, then*

$$\frac{|\Delta ACD|}{|\Delta ABC|} = \frac{DP}{BP} = \frac{AI \cdot DL}{AJ \cdot BK} = \frac{CL \cdot DI}{BJ \cdot CK}$$

and

$$\frac{|\Delta BCD|}{|\Delta ABD|} = \frac{CP}{AP} = \frac{DL \cdot CK}{DI \cdot AJ} = \frac{BK \cdot CL}{AI \cdot BJ}.$$

Proof. The last equality in each of these sets follows from Lemma 2.1 for any convex quadrilateral. The equality of the ratio of the areas and the ratio of the segments of the diagonal is well known and straightforward to show with similar triangles for any convex quadrilateral.

Let R be the foot of the altitude from D to $\overline{AC}$, and let S be the foot of the altitude from B to $\overline{AC}$. Then, triangles ΔDRP and ΔBSP are similar with the congruent vertical angles and congruent right angles. Thus,

$$\frac{|\Delta ACD|}{|\Delta ABC|} = \frac{\frac{1}{2}AC \cdot DR}{\frac{1}{2}AC \cdot BS} = \frac{DR}{BS} = \frac{DP}{BP}.$$

By a similar argument,

$$\frac{|\Delta BCD|}{|\Delta ABD|} = \frac{CP}{AP}.$$

To prove the remaining equality in each of the two sets, label the centers of the semi-excircles, points of tangency, radii of the semi-excircles, and other intersections as in the proof of Lemma 2.2 and shown in Figure 2.

We need to show that the segment LJ is concurrent with the diagonals AC and BD. Let LJ intersect AC and BD at points X and Y, respectively. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\Delta CLX \sim \Delta AJX$ and $\Delta DLY \sim \Delta BJY$. Thus,

$$\frac{CL}{AJ} = \frac{LX}{JX} \text{ and } \frac{DL}{BJ} = \frac{LY}{JY}.$$

So, by Lemma 2.2,

$$\frac{LX}{JX} = \frac{LY}{JY},$$

and it must be that $X = Y = P$.

Also by Lemma 2.2,

$$\frac{AI}{AJ} = \frac{BK}{BJ}.$$

So,

$$\frac{AI}{AJ \cdot BK} = \frac{1}{BJ} \implies \frac{AI \cdot DL}{AJ \cdot BK} = \frac{DL}{BJ}.$$

Since $\Delta DLP \sim \Delta BJP$,

$$\frac{DL}{BJ} = \frac{DP}{BP},$$

and we have the desired result,

$$\frac{DP}{BP} = \frac{AI \cdot DL}{AJ \cdot BK}.$$

Similarly, since $\Delta CLP \sim \Delta AJP$ implies that

$$\frac{CP}{AP} = \frac{CL}{AJ},$$

we have

$$\begin{aligned} \frac{AI}{AJ} = \frac{BK}{BJ} &\implies \frac{1}{AJ} = \frac{BK}{AI \cdot BJ} \\ &\implies \frac{CL}{AJ} = \frac{BK \cdot CL}{AI \cdot BJ} \\ &\implies \frac{CP}{AP} = \frac{BK \cdot CL}{AI \cdot BJ}. \end{aligned}$$

With Lemma 2.3 handling the trapezoid case, we now generalize the result to all convex quadrilaterals and provide an analytic proof for the non-trapezoid cases.

Theorem 2.1. *Let $ABCD$ be a convex quadrilateral. Let the diagonals intersect at point P, and let the four semi-excircles intersect sides AD, AB, BC, and CD at points I, J, K, and L, respectively, then*

$$\frac{|\Delta ACD|}{|\Delta ABC|} = \frac{DP}{BP} = \frac{AI \cdot DL}{AJ \cdot BK} = \frac{CL \cdot DI}{BJ \cdot CK}$$

and

$$\frac{|\Delta BCD|}{|\Delta ABD|} = \frac{CP}{AP} = \frac{DL \cdot CK}{DI \cdot AJ} = \frac{BK \cdot CL}{AI \cdot BJ}.$$

Proof. The first and last equality in each of these sets has already been addressed in the proof of Lemma 2.3.

Assign coordinates so that $A = (0,0)$, $B = (1,0)$, $C = (c,d)$, and $D = (a,b)$ with b and d positive. Then, using vectors,

$$\frac{|\Delta ADC|}{|\Delta ABC|} = \frac{bc - ad}{d} \text{ and } \frac{|\Delta BDC|}{|\Delta ABD|} = \frac{bc - ad + d - b}{b}.$$

The exterior angle bisector at A has equation

$$y = \frac{b}{a - AD}x,$$

and the exterior angle bisector at B has equation

$$y = \frac{d}{BC + c - 1}x - \frac{d}{BC + c - 1}.$$

Note that these bisectors cannot be vertical. AD is a hypotenuse of a right triangle with leg $|a|$, and BC is a hypotenuse of a right triangle with leg $|c - 1|$. So $a - AD < 0$ and $BC + c - 1 > 0$.

Equating these bisectors, we find that the semi-excenter off side AB has coordinates

$$\left(\frac{d(a - AD)}{d(a - AD) - b(c - 1 + BC)}, \frac{db}{d(a - AD) - b(c - 1 + BC)}\right)$$

Thus,

$$J = \left(\frac{d(a - AD)}{d(a - AD) - b(c - 1 + BC)}, 0\right),$$

and

$$AJ = \frac{d(a - AD)}{d(a - AD) - b(c - 1 + BC)}.$$

The exterior angle bisector at D has equation

$$y = \frac{(d - b + b\frac{CD}{AD})x + bc - ad}{c - a + a\frac{CD}{AD}}$$

unless $AD(c - a) + aCD = 0$, in which case, the bisector is $x = a$.

In each case, the intersection of the exterior angle bisector at D with the exterior angle bisector at A gives the semi-excenter off side AD as

$$\left(\frac{(a - AD)(bc - ad)}{-d(a - AD) + b(c + CD - AD)}, \frac{b(bc - ad)}{-d(a - AD) + b(c + CD - AD)}\right)$$

and the point of tangency with the x-axis as

$$\left(\frac{(a - AD)(bc - ad)}{-d(a - AD) + b(c + CD - AD)}, 0\right).$$

Note that

$$\frac{(a - AD)(bc - ad)}{-d(a - AD) + b(c + CD - AD)} - a = \frac{b(AD(c - a) + aCD)}{-d(a - AD) + b(c + CD - AD)},$$

which is 0 if and only if $AD(c - a) + aCD = 0$.

Using the common tangents from A, it follows that

$$AI = \frac{(a - AD)(ad - bc)}{-d(a - AD) + b(c + CD - AD)}.$$

Considering possible division by zero, $-d(a - AD) + b(c + CD - AD) = 0$ only when $bc - ad = 0$, $b = d$, or $b = 0$. For a convex quadrilateral, b cannot be zero, nor can $bc - ad$ since the later is twice the area of ΔACD. If $b = d$, we have the trapezoid case which is covered by Lemma 2.3.

The exterior angle bisector at C has equation

$$y = \frac{(BC(d-b) - dCD)x + d(-aBC + CD) + bcBC}{BC(c-a) + CD(1-c)}$$

unless $BC(c-a) + CD(1-c) = 0$, in which case it is $x = c$.

In each case, intersecting this line with the exterior angle bisector at B gives a semi-excenter off BC with x- cooridnate

$$x = \frac{-ad(c+CB) + bc(c-1+CB) + d(c+CD)}{d(1-a+CD-CB) - b(1-c-CB)}.$$

Note that

$$\frac{-ad(c+CB) + bc(c-1+CB) + d(c+CD)}{d(1-a+CD-CB) - b(1-c-CB)} - c = \frac{d(BC(c-a) + CD(1-c))}{d(1-a-BC+CD) - b(1-c+BC)},$$

and this is zero only when $BC(c-a) + CD(1-c) = 0$.

So the point of tangency with the x-axis is

$$\left(\frac{-ad(c+CB) + bc(c-1+CB) + d(c+CD)}{d(1-a+CD-CB) - b(1-c-CB)}, 0\right).$$

It follows that

$$\begin{aligned} BK &= \frac{-ad(c+CB) + bc(c-1+CB) + d(c+CD)}{d(1-a+CD-CB) - b(1-c-CB)} - 1 \\ &= \frac{(c-1+CB)(d(1-a) + b(c-1))}{d(1-a+CD-CB) + b(c-1+CB)}. \end{aligned}$$

Again, we must consider possible division by zero. The denominator $d(1-a+CD-BC) - b(1-c-BC)$ is zero only if $bc - ad + d - b = 0$, $d = 0$, or $d = b$. For the convex quadrilateral, $d \neq 0$ and $bc - ad + d - b \neq 0$ since the later is twice the area of ΔBCD. As with the exterior angle bisector at D, the case where $b = d$ is handled by Lemma 2.3.

Intersecting the exterior angle bisectors at C and D, we find the semi-excenter off side CD to be (x_1, y_1) where

$$x_1 = \frac{cAD\,(ad - bc + b - d) + aBC\,(ad - bc) - adCD}{AD\,(ad - bc + b - d) + (BC - CD)(ad - bc) - bCD}$$

and

$$y_1 = \frac{dAD\,(ad - bc + b - d) + bBC(ad - bc) - bdCD}{AD\,(ad - bc + b - d) + (BC - CD)(ad - bc) - bCD}.$$

Note that if the exterior angle bisector at C is vertical, it is $x = c$.

$$x_1 - c = \frac{(BC(c-a) + CD(1-c))(ad - bc)}{AD(ad - bc + b - d) + (BC - CD)(ad - bc) - b(CD)},$$

which is 0 when the exterior angle bisector at C is vertical.

If the exterior angle bisector at D is vertical, it is $x = a$.

$$x_1 - a = \frac{(AD(c-a) + aCD)(ad - bc + b - d)}{AD(ad - bc + b - d) + (BC - CD)(ad - bc) - b(CD)},$$

which is 0 when the exterior angle bisector at D is vertical.

When is the denominator 0? It arises from the differences in the slopes of the exterior angle bisectors at C and D.

$$\begin{aligned}-CD\,(AD(ad-bc+b-d)+(BC-CD)(ad-bc)-b(CD)) = \\ (AD(d-b)+bCD)\,(BC(c-a)+CD(1-c)) - \\ (BC(d-b)-dCD)\,(AD(c-a)+aCD)\end{aligned}$$

Note that this expression will only be zero if the slopes of the two exterior angle bisectors are the same, which is impossible, or if one is horizontal and the other is vertical. Although the signs of the variables will vary, both $AD(d-b)+bCD=0$ and $AD(c-a)+aCD=0$ imply $bc-ad=0$ or $bc+ad=2ab$. But, we know $bc-ad>0$. So, the exterior angle bisector at C is only horizontal or vertical if $bc+ad=2ab$. Similarly, $BC(d-b)-dCD=0$ and $BC(c-a)+CD(1-c)=0$ both imply $bc-ad+d-b=0$ or $bc+ad=2cd+b-d$. Since $bc-ad+d-b>0$, the exterior angle bisector at D can only be horizontal or vertical if $bc+ad=2cd+b-d$.

If $bc+ad=2ab$ and $bc+ad=2cd+b-d$, either $b=d$ or $b=\frac{d(2c-1)}{2(c-1)}$. In the later case, the slope of $\overleftrightarrow{AD}$ and the slope of $\overleftrightarrow{CD}$ are both $\frac{d}{c-1}$. Thus, the denominator of x_1 can only vanish if the convex quadrilateral is a trapezoid.

Therefore, the formulas for x_1 and x_2 hold as long as the quadrilateral is not a trapezoid, and the trapezoid case is proven with Lemma 2.3.

The perpendicular line to CD through $(x_1.y_1)$ is

$$y=\frac{a-c}{d-b}(x-x_1)+y_1,$$

which intersects CD at (x_2,y_2) where the coordinates are

$$x_2=\frac{cAD\cdot CD(u+t)+auBC\cdot CD+tu^2-ad(a-c)^2-bct^2}{CD(u(AD+BC-CD)+tAD-bCD)}$$

and

$$y_2=\frac{d-b}{c-a}(x_2-a)+b,$$

where $u=ad-bc$ and $t=b-d$.

Computing and simplifying $(a-x_2)^2+(b-y_2)^2$ gives

$$DL^2=\frac{(u+t)^2(CD\cdot AD-a^2+ac-bt)^2}{(u(AD+BC-CD)+tAD-bCD)^2}.$$

Therefore,

$$DL=\frac{(ad-bc+b-d)(CD\cdot AD-a^2+ac-b(b-d))}{(ad-bc)(AD+BC-CD)+(b-d)AD-bCD}.$$

Now that we have coordinates, we need to show that

$$\frac{AI\cdot DL}{AJ\cdot BK}=\frac{bc-ad}{d}.$$

$$\left(\frac{AI \cdot DL}{AJ \cdot BK}\right)\left(\frac{d}{bc - ad}\right) = \frac{(-AD \cdot CD + b^2 - bd + a(a-c))(bm + d(a - AD))(bm + d(a + n - 1))}{m(b(-c + AD - CD) + d(a - AD))(b((-c+1)AD - nc - CD) + d((a-1)AD + an))}$$

where $m = 1 - c - BC$ and $n = BC - CD$.

Thus, we need to show that this fraction is equal to 1. This is a tedious exercise that can be verified easily with a CAS.

Expanding the numerator and denominator while replacing AD^2, BC^2, and CD^2, with $a^2 + b^2$, $(c-1)^2 + d^2$, and $(a-c)^2 + (b-d)^2$, respectively shows that they are both equal to the following:

$$\begin{aligned}
&BC(2a^2b^2 - 2a^2bd - 2ab^2c + 2abcd + 2b^4 - 4b^3d + 2b^2d^2) - 2b^4c^2 + a^2c^2d^2 \\
&+ 3b^3c^2d + 2ab^2c + 2ab^2c^3 - 2a^2b^2c^2 + 3ab^2d^2 - 4ab^2c^2 - a^2cd^2 - a^3cd^2 + \\
&AD \cdot BC(a^2d^2 - abcd - acd^2 + bc^2d) + AD(a^2cd^2 - abc^2d + abd^3 - ac^2d^2 \\
&- ad^4 - b^2cd^2 + bc^3d + bcd^3) - 2a^2b^2 + AD \cdot BC \cdot CD(-2abd + ad^2 + 2b^2c \\
&- bcd - 2b^2 + 2bd) - abcd^3 - abc^3d - 2b^4 + a^2d^4 + CD(-a^2cd^2 + abc^2d \\
&- abd^3 + b^2cd^2 + a^2d^2 - abcd) - a^2bcd - 2ab^3d + 4a^2b^2c - 2a^3bd - 3ab^2cd^2 \\
&+ 2ab^3cd - a^2bc^2d - 2b^2d^2 + 4b^3d + AD \cdot CD(-2abcd + acd^2 + 2b^2c^2 - bc^2d \\
&+ 2abd - ad^2 - 4b^2c + 3bcd + 2b^2 - 2bd) + 3abc^2d - 2abcd + a^3d^2 + 2a^3bcd \\
&+ BC(2a^3bd - a^3d^2 - 2a^2b^2c - a^2bcd + a^2cd^2 + 2ab^3d + 2ab^2c^2 - 3ab^2d^2 \\
&- abc^2d + abd^3 - 2b^4c + 3b^3cd - b^2cd^2) + 2a^2bd - 7b^3cd - abd^3 + 3b^2cd^2 \\
&+ 4b^4c + AD(-a^2d^2 + abcd + acd^2 - bc^2d) + BC \cdot CD(-a^2d^2 + abcd)
\end{aligned}$$

Therefore,

$$\frac{|\Delta ADC|}{|\Delta ABC|} = \frac{AI \cdot DL}{AJ \cdot BK}.$$

Since the choice to place $A = (0,0)$ and $B = (1,0)$ was arbitrary, we have proven the relationship between the areas of the pair of triangles formed by one diagonal and the lengths of the segments formed by the points of tangency of the semi-excircles. Thus, it is also true that

$$\frac{|\Delta BCD|}{|\Delta ABD|} = \frac{CP}{AP} = \frac{DL \cdot CK}{DI \cdot AJ} = \frac{BK \cdot CL}{AI \cdot BJ}.$$

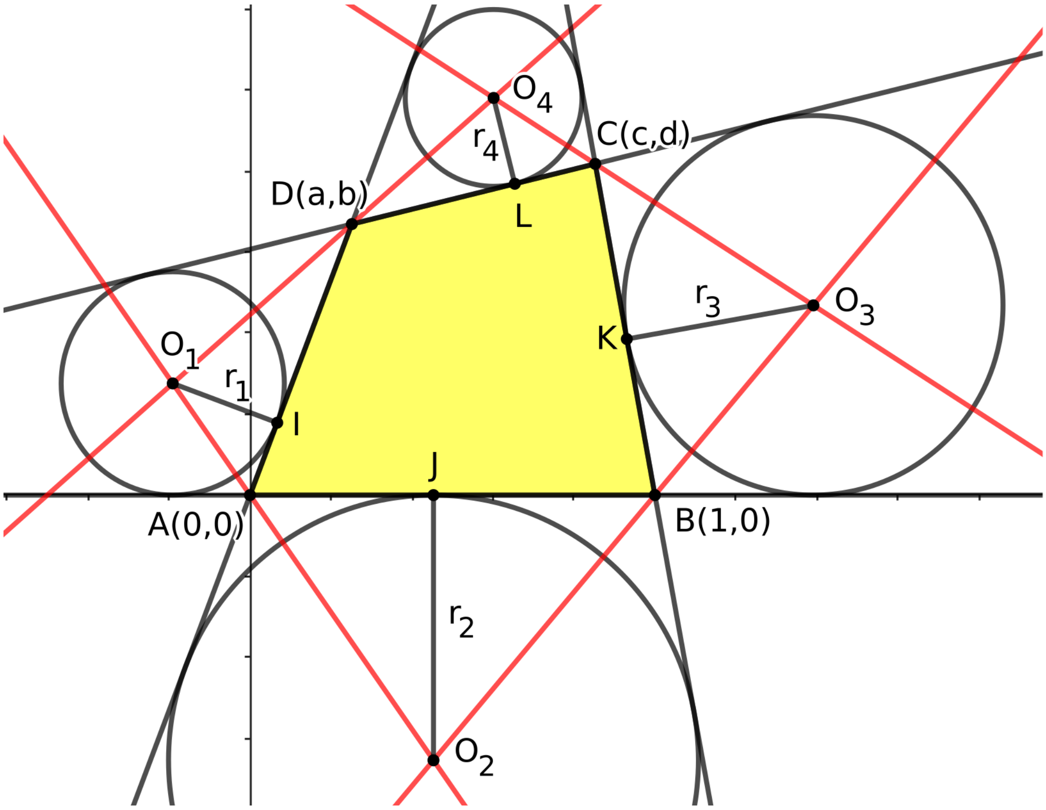

Figure 3

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