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GOLDEN SECTIONS IN CONTEMPORARY MUSIC AND IN THE EUCLIDEAN LINE SEGMENT

DIMITRIS M. CHRISTODOULOU and ATHANASIOS ZERVAS

Abstract. An equal temperament is a musical tuning system that approximates just intervals within an octave in order to suppress some beat acoustic frequencies known to be dissonant to the human ear. Division of an octave into n segments always results in n frequencies that form a geometric sequence with common ratio $r_n = 2^{1/n}$. This ratio is a golden ratio for any choice of integer $n \geq 2$. In western music, the widely used golden ratio of $r_{12} = 2^{1/12} \simeq 1.059463$ has heretofore gone unappreciated because of the unfortunate widespread perception that the only golden ratio is the Euclidean $\varphi = (1 + \sqrt{5})/2 \simeq 1.618034$ that results from the section of a line segment into only two noncongruent parts. Confusion has also been compounded by lack of distinction between golden ratios of lengths versus those of discrete physical values (such as frequencies in music). In this work, we clarify these issues, and we extend the $n = 2$ Euclidean golden section to $n \geq 3$ partitions.

1. Introduction

The dimensionless number $\varphi = (1 + \sqrt{5})/2 \simeq 1.618034$ [24], the positive root of the unitless equation

$$\frac{1}{\varphi} = \varphi - 1\,, \tag{1}$$

was termed the 'golden ratio' [16] in the early 19$^{\text{th}}$ century, when it captivated the imagination of scientists and laymen alike [3, 12, 13, 18, 21]. In the next 200 years, the golden ratio was "found" to appear virtually everywhere in nature, music, the arts, architecture, the sciences, the pseudosciences, and the human body itself [13], although many such accounts were eventually discredited, or altogether disproven, or found to originate from loose rational approximations of φ [9, 10, 13, 18, 19, 21, 26] or its conjugate ('silver') ratio $\varphi^\star \equiv \varphi^{-1} \simeq 0.618034$ [25].

In recent times, the controversy over the appearance of φ continues unabated (e.g., [1, 4, 7, 19, 29]). With the notable exception of closed elliptical orbits in Newtonian and Hookean potentials [4], an academic exercise in which each set of orbits includes precisely one exact golden ellipse, other realizations of φ are rough approximations, thus manifestly open to debate. We highlight two examples:

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(a) The axisymmetric dumbbell shapes of red blood cells [29]:
The spontaneous curvature introduced to a shape model of a biological vesicle, properly normalized, was predicted to have a value of $-\varphi$ [17]. A fit to the data of a new analytic model in pressure equilibrium with its surroundings and with zero tensile stress [15] produced a value of -1.67 [29]. So, it seems that favoring $-\varphi$ rather than $-5/3$ may be debatable.

(b) A conspicuous key change in Lady Gaga's song *Perfect Illusion* [7, 22]:
It has been claimed that the key changes 111 seconds into the song whose duration is 179 seconds. Then, $179/111 = 1.6\overline{126}$ with a deviation of only -0.335% from φ—one of the best rational approximations ever reported. The writers of the song have not declared that the timing was influenced by the golden ratio φ, so this match may be a coincidence.

In musical compositions, the golden ratio has been professed to appear (either "naturally" or by deliberate design) when counting and timing bars, notes within bars, and systemic tones and semitones [1, 10, 12, 19, 26]. Such investigations have unavoidably considered fundamentally different sets or groups of physical intervals (i.e., continuous lengths) or acoustic frequencies (i.e., discrete values) without ever making a distinction between the groups. This state of numerological confusion occurred for two reasons: (1) an underlying assumption (logarithmic arrangement of values) common to all studied groups; and (2) the naive belief that the resulting geometric sequencies are beautiful only if they exhibit the golden ratio φ (or $\varphi^\star$).

The root cause of this problem can be traced to the Euclidean mean-extreme theorems [8, 11] that produced φ in the first place; but to be fair, our ancestors bare no responsibility for the contemporary fascination regarding φ that has transformed its searches to purely numerological endeavors. It is beyond belief that mathematicians did not realize that the theorems of Euclid dealt with the simplest asymmetric section of a line segment, the one described by a geometric sequence of only two terms, viz.$\{1, \varphi\}$ or, equivalently, $\{1-\varphi^\star, \varphi^\star\}$; and more so, that they did not extend the Euclidean problem to more than two sections, not even after they witnessed the emergence and widespread adoption of the 12 tone equal temperament (12 TET) tuning system in western music (e.g., [3, 6, 18, 20, 21, 26, 28]).

The popular 12 TET system divides the octave into $n = 12$ frequencies, but there have been other scales with up to $n = 96$ logarithmic intervals and down to just two intervals (the *ditonic scale*, generally confined to *prehistoric music*) [14, 20, 28]; whereas the analogous Euclidean program of subdividing a given length scale was never advanced past its original inception of $n = 2$ logarithmic intervals. For the reasons that have just become apparent, we proceed as follows:

- In Section 2, we use geometric sequences with $r_n > 1$ to consolidate the mathematical properties of n TET sound tuning frequencies spread over a harmonic interval of one or more octaves, where $n \geq 2$ is a positive integer.

- In Section 3, we solve the analogous Euclidean problem with $r_n < 1$, where $n \geq 2$ is the number of logarithmically-spaced lengths of the partitions of a given line segment of unit length.

In our view, the common ratios r_n of the geometric progressions are golden in both cases. In particular, the controversial Euclidean ratio $r_2 = \varphi^\star$ assumes its proper place in the world as the upper limit of golden ratios r_n of lengths in flat space, where $r_n \in (1/2, \varphi^\star]$ for integers $n \in (+\infty, 2]$.

2. Musical Frequencies in an Octave

One octave is the frequency interval between a note and its nearest harmonic (e.g., [6, 19, 20, 26]). In musical TET theory, the octave is generally divided into n frequencies whose consecutive ratios are equal. The equal ratios across such a sequence p_n dictate that it is geometric in nature [23], thus it can be represented by

$$p_n = f \cdot \left\{(r_n)^i\right\}_{i=0}^{i=n} , \tag{2}$$

where f is the fundamental frequency and $r_n > 1$ is the common ratio of a sequence with $n + 1$ members. The octave terminates at the first harmonic of frequency $2f$, thus $f(r_n)^n = 2f$, and the common ratio then is

$$r_n = 2^{1/n} , \tag{3}$$

a result that is very well-known in musical theory as well as in mathematical theory. In western musical practice, the *chromatic scale* of $n = 12$ semitones has been widely adopted, so that $r_{12} = 2^{1/12} \simeq 1.059463$ [20]. This is the golden ratio of the chosen frequencies in the immensely popular 12 TET chromatic scale of our times, although the system dates back to Johann Sebastian Bach (1685–1750) and his contemporaries [2, 5].

Taking a step further, a number of probing musicians have experimented with musical intervals across two octaves, in which case the terminal frequency becomes $3f$ and the golden ratios then are $r_n = 3^{1/n}$. Thus, we establish the well-known property of n logarithmically-spaced frequencies spread out over V octaves that determine a generalized golden ratio of $r_n = V^{1/n}$.

For $n = 2$ frequencies, equation (3) gives $r_2 = \sqrt{2}$ and the geometric progression is $p_2 = f \cdot \left\{1, \sqrt{2}, 2\right\}$. This modern ditonic scale [14, 20, 28] may be taken as an indication that primitive musicians may have been empirically aware of the effect of $\sqrt{2}$ on sounds, but, at the same time, they were certainly unaware of φ as a number [9, 13]. Over the past 200 years, geometers were also unaware of the simple $n = 2$ musical scale and its specious disagreement with mean-extreme ratios and φ. This is most unfortunate; realizing this superficial discrepancy would have revealed earlier the obvious fundamental difference between mean-extreme ratios of Euclidean lengths versus the ratios of physical quantities such as frequencies or times (i.e., continuous spatial intervals versus discrete temporal stamps).

3. Partitions of a Euclidean Line Segment

In trying to catch up with musical n TET theory, we solve now the long overdue Euclidean problem of partitioning a given segment to $n \geq 2$ asymmetric sections, such that the ratios of adjacent lengths are the same. Sidestepping tradition [8, 11], we set to 1 the length of the given segment, which implies that the Euclidean $n = 2$ case produces a partition of the golden ratio conjugate length $\varphi^\star \equiv \varphi^{-1} \simeq 0.618034$ [25] and the smaller length $1 - \varphi^\star \simeq 0.381966$. The golden ratio conjugate $\varphi^\star$ can be obtained from equation (1) above by the transformation $\varphi \to 1/\varphi^\star$, and then $\varphi^\star$ turns out to be the positive root of the unitless equation

$$\frac{1}{\varphi^\star} = \varphi^\star + 1 . \tag{4}$$

In the general case of n logarithmic sections of a unit length, the lengths of the partitions establish a geometric progression of the form $p_n = \left\{x, x^2, x^3, \cdots, x^n\right\}$ or

$$p_n = \left\{x^i\right\}_{i=1}^{i=n} \quad (\text{where } \ x < 1)\,, \tag{5}$$

with sum

$$\mathcal{S}_n = \sum_{i=1}^{n} x^i \ = \ x + x^2 + x^3 + \cdots + x^n \ \equiv \ 1\,. \tag{6}$$

This is a geometric series of n terms with leading term x and common ratio also x. The resulting polynomial equation can be solved analytically by *Mathematica* [27] for $n = 2, 3$, and 4. The positive real roots $x \simeq 0.6180, 0.5437, 0.5188, 0.5087, \cdots$ show a rapid decline with increasing n toward the lower limiting value of $x = 1/2$.

Alternatively, applying the summation property of geometric sequences ([23], § 10.2) to the last equality of equation (6), we obtain the proportion

$$\frac{1}{x} = \frac{1 - x^n}{1 - x}\,. \tag{7}$$

In the Euclidean case of $n = 2$ partitions, we recover equation (4) for $x = \varphi^\star$.

Since the common ratio $x \neq 1$, then equation (7) also represents a polynomial equation of degree $n + 1$, viz.

$$x^{n+1} - 2x + 1 = 0\,, \tag{8}$$

in which the additional real root $x = 1$ is rejected. We recognize now that, for $n = 2$, the golden ratio conjugate $\varphi^\star$ is also a root of the *depressed cubic equation* $x^3 - 2x + 1 = 0$ that can be solved analytically (roots $x = \varphi^\star, -\varphi, 1$).

Another interesting property of all sections with $n \geq 2$ partitions is derived as follows: It is a well-known property of equation (4) that the golden ratio $\varphi = 1/\varphi^\star$ can be obtained by adding 1 to the root $\varphi^\star$, i.e., $\varphi = \varphi^\star + 1$ [13]. Much less known (if at all) is the property implied by equations (8) for the reciprocal ratios $1/x$ that

$$\frac{1}{x} = 2 - x^n\,. \tag{9}$$

In the particular case of $n = 2$, the Euclidean golden ratio φ can then be obtained from the identity

$$\varphi = 2 - \left(\varphi^\star\right)^2\,. \tag{10}$$

For comparison purposes, the converse relation reads $\varphi^\star = 1/(\varphi^2 - 1)$.

Finally, we note that equation (8) with $n = 2$ produces slightly different cubic polynomial equations for φ and $\varphi^\star$: as written above, it shows that

$$\left(\varphi^\star\right)^3 - 2\varphi^\star + 1 = 0\,, \tag{11}$$

whereas its transformation shows that

$$\varphi^3 - 2\varphi^2 + 1 = 0\,. \tag{12}$$

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LOWELL CENTER FOR SPACE SCIENCE AND TECHNOLOGY
UNIVERSITY OF MASSACHUSETTS LOWELL
LOWELL, MA 01854, USA
E-mail address: dimitris_christodoulou@uml.edu

DEPARTMENT OF MUSIC SCIENCE AND ART
UNIVERSITY OF MACEDONIA
THESSALONIKI 54636, GREECE
E-mail address: a-zervas@uom.edu.gr