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EXTENSIVE COMPILATION OF CHARACTERIZATIONS OF RECTANGLES

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Abstract. In this paper we study 84 characterizations of rectangles. A quarter of these are new sufficient conditions, as far as we know.

1. Introduction

There is no doubt the rectangle is the type of quadrilateral we experience most in our daily life. Just look around you in your home. You see rectangles everywhere. They are the shape of most floors, walls, doors, tables, windows, book shelves, picture frames, books, television screens, and so on. With its four right angles, it's one of the simplest quadrilaterals and one of the most studied in early math classes.

Most basic math books that include a section on rectangles list a few (usually no more than five) characterizations of rectangles, but that is it. As we shall see in this paper, there are many more, and about half of them are very basic, but they have been found scattered in many different sources. This is the first time, as far as we know, anyone has attempted to collect these in one place. At the time of writing this paper, even the English *Wikipedia* page lists only eight characterizations of rectangles, and the extensive encyclopedia *MathWorld*, to our surprise, lists none.

There are several possible definitions of a rectangle in use in different textbooks, including that it is a:

- Parallelogram with at least one right angle
- Parallelogram with four right angles
- Quadrilateral with four right angles
- Quadrilateral with four equal angles

All of these are equally valid. Having to choose one as the definition we will use, we have chosen the fourth of these as it is the most general and the one that is best suited when studying duality between different types of quadrilaterals (see [34]).

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Hence, the basis for our study of characterizations of rectangles is the following:

Definition 1.1. *A quadrilateral is a rectangle if and only if it has four equal angles.*

Another way to state this is that a rectangle is the only equiangular quadrilateral. That this definition is equivalent to the third in the list (the one with four *right* angles instead) is a direct consequence of the fact that the angle sum of a quadrilateral is 360°.

When proving characterizations, we have to prove they are both necessary and sufficient conditions. Necessary conditions are the properties that a certain object have, and these are in most math books studied in greater detail than sufficient conditions. Just think about the rectangle. How many properties can you list, and how many of these do you know to also be sufficient conditions? Chances are big the first category include many more items than the second.

Since properties of rectangles are so well known, we list the most basic here that are relevant for this study and expect the reader to be familiar with these and how they are proved. For that reason, *we will mainly focus on proving the sufficient conditions of rectangles.* If any of these are not known necessary conditions for the reader, we set it as an exercise to prove them.

Here are some of the basic properties of a rectangle that are well-known or easy to confirm:

- All vertex angles are right angles
- Opposite sides are parallel
- Opposite sides have equal length
- The diagonals have equal length
- The diagonals divide each other in four equal parts
- It has area $K = ab$ where a and b are two adjacent sides
- The bimedians are axes of symmetry
- It is a parallelogram
- It is an isosceles trapezoid
- It is cyclic
- The diagonals are diameters of the circumcircle
- The diagonals intersect at the circumcenter
- The bimedians intersect at the circumcenter

A *bimedian* is a line segment connecting the midpoints of two opposite sides. A circumcircle is a circle that goes through all four vertices; its center is the circumcenter.

We have recently studied characterizations of two other basic types of quadrilaterals: the square [37] and the parallelogram [38], [39]. The former is a special case of the rectangle while the latter is a generalization. For that reason, it will not be a surprise that many characterizations of rectangles are formulated as restrictions on parallelograms, which is the subject of the next section.

2. Parallelograms

Here we study no less than 19 conditions for when a parallelogram is a rectangle. The first three of these were stated in the old books [47], [54], and [63] respectively. *(d)* is also very fundamental, but we have been unable to find an old reference for it, while *(e)* and *(f)* are from the old books [56] and [62] respectively. To prove *(g)*, *(j)* and *(k)* were exercises in [11, p. 53], [58, p. 42], and [58, p. 38] respectively. The sufficient condition in *(l)* was proved in the old book [19], and *(r)* was discussed at [45]. Condition *(s)* and its proof are cited from a 2004 Regional Mathematics Competition in Romania for Grade 11 [7, pp. 103–104]; it was proposed by M. Becheanu.

Theorem 2.1. *A parallelogram $ABCD$ with adjacent sides a and b, bimedians m and n, diagonals p and q, and diagonal intersection P satisfies any one of:*

- *(a) it has four right angles*
- *(b) it has four equal angles*
- *(c) it has at least one right angle*
- *(d) it has two adjacent equal angles*
- *(e) it is cyclic*
- *(f) it has diagonals of equal length*
- *(g) triangles ABD and DCA are congruent*
- *(h) one bimedian is perpendicular to a side*
- *(i) it has perpendicular bimedians*
- *(j) the midpoints of the sides are the vertices of a rhombus*
- *(k) $AM = BM$, where M is the midpoint of CD*
- *(l) it has maximum area for a given base and perimeter*
- *(m) it has area $K = ab$*
- *(n) it has area $K = mn$*
- *(o) $a^2 + b^2 = pq$*
- *(p) $m^2 + n^2 = pq$*
- *(q) $a^2 + b^2 = e^2$, where e is the length of any diagonal*
- *(r) $\left(\frac{AQ}{QS}\right)^2 + \left(\frac{BC}{QR}\right)^2 = 1$, where a line through P intersects AB, CD, AD at Q, R, S respectively*
- *(s) $EF = EG$ and $a \neq b$, where E is the foot of the perpendicular from D to AC and the line through E perpendicular to BD intersects DC in F and AD in G*

if and only if it's a rectangle.

Proof. *(a), (b), (c)* These are trivially equivalent to the definition.

(d) Two adjacent angles in a parallelogram $ABCD$ are supplementary, so we get

$$\begin{cases} \angle A + \angle B = 180^\circ \\ \angle A = \angle B \end{cases} \quad \Leftrightarrow \quad \begin{cases} \angle A = 90^\circ \\ \angle B = 90^\circ \end{cases}$$

and $ABCD$ is a rectangle according to *(c)*.

(e) Two opposite angles in a cyclic quadrilateral $ABCD$ are supplementary, so in a cyclic parallelogram we get

$$\begin{cases} \angle A + \angle C = 180^\circ \\ \angle A = \angle C \end{cases} \Leftrightarrow \begin{cases} \angle A = 90^\circ \\ \angle C = 90^\circ \end{cases}$$

meaning that $ABCD$ is a rectangle according to *(c)*.

(f) We directly get that triangles ABD and DCA are congruent (SSS), which is condition *(g)*, and since vertex angles at A and D are both equal and supplementary, both *(f)* and *(g)* imply that $ABCD$ is a rectangle according to *(d)*.

(h) The bimedians in a parallelogram divide it into four congruent smaller parallelograms (see Figure 1), so if one bimedian is perpendicular to a side, then the parallelogram has a right vertex angle, making it a rectangle according to *(c)*.

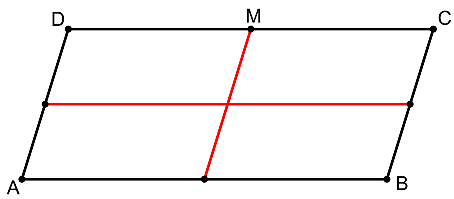

FIGURE 1. The two bimedians

(i) The bimedians in a parallelogram are parallel to the sides (see Figure 1), so this condition is a direct consequence of *(c)*.

(j) It is well-known that the midpoints of the sides in any quadrilateral are the vertices of Varignon's parallelogram, with the bimedians as its diagonals. A parallelogram is a rhombus if and only if its diagonals are perpendicular, so this condition is a consequence of *(i)*.

(k) When $AM = BM$, we directly get that triangles ADM and BCM are congruent (SSS), see Figure 1, so $\angle D = \angle C$, making $ABCD$ a rectangle according to *(d)*.

(l) If the perimeter $L = 2a + 2b$ is a constant, then the area satisfies

$$K = ab \sin A = a\left(\tfrac{L}{2} - a\right) \sin A \le a\left(\tfrac{L}{2} - a\right)$$

where we have equality if and only if $\angle A = 90^\circ$, so $ABCD$ is a rectangle according to *(c)*.

(m) We directly get for the area

$$K = ab \sin A \le ab$$

where we have equality if and only if $\angle A = 90^\circ$, so $ABCD$ is a rectangle according to *(c)*.

(n) In a parallelogram, the bimedians satisfy $m = a$ and $n = b$, so this is a direct consequence of *(m)*.

(o) We have $p^2 + q^2 = 2(a^2 + b^2)$ according to the parallelogram law (see Figure 2), and applying the algebra rule $(p - q)^2 = p^2 + q^2 - 2pq$, we get

$$2(a^2 + b^2) = (p - q)^2 + 2pq \le 2pq$$

where equality holds if and only if $p = q$, that is, only when the parallelogram is a rectangle according to *(f)*.

(p) In a parallelogram, the bimedians satisfy $m = a$ and $n = b$, so this is a direct consequence of *(o)*.

(q) By the law of cosines, we have (see Figure 2)

$$p^2 = a^2 + b^2 - 2ab\cos B \gtreqless a^2 + b^2$$

where equality holds if and only if $\angle B = 90^\circ$, so $ABCD$ is a rectangle according to *(c)*. For the other diagonal we similarly have

$$q^2 = a^2 + b^2 - 2ab\cos A \lesseqgtr a^2 + b^2$$

where equality holds if and only if $ABCD$ is a rectangle.

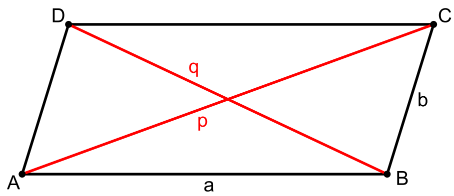

FIGURE 2. The two diagonals

(r) Since AB is parallel to DC, triangles AQS and DRS are similar (see Figure 3), so we get

$$\frac{BC}{QR} = \frac{AD}{QR} = \frac{AS}{QS}.$$

Then

$$\left(\frac{AQ}{QS}\right)^2 + \left(\frac{BC}{QR}\right)^2 = 1 \quad \Leftrightarrow \quad \left(\frac{AQ}{QS}\right)^2 + \left(\frac{AS}{QS}\right)^2 = 1$$

which in turn is equivalent to

$$AQ^2 + AS^2 = QS^2$$

and this holds if and only if $AB \perp AD$ according to the Pythagorean theorem and its converse, which characterizes a rectangle according to *(c)*.

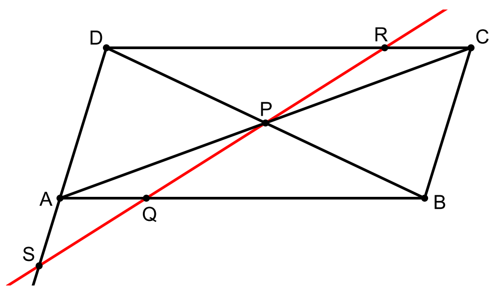

FIGURE 3. Intersection points Q, R, S

(s) This is hardly a well-known property, so we prove both directions of this characterization. When $ABCD$ is a rectangle, DE and DH are heights in the right triangles ADC and FDG respectively, where H is the intersection of FG and BD. From similar triangles, we get $\angle ADE = \angle ACD$

and $\angle DGF = \angle HDC$ (see Figure 4). But in a rectangle, we also have $\angle HDC = \angle ACD$, so it follows that $\angle GDE = \angle DGF$. Then $DE = GE$ according to the isosceles triangle theorem. We also have $\angle EDF = \angle EFD$ since they are both complementary angles to GDE and DGF. It follows that $DE = FE$, and we get $FE = GE$.

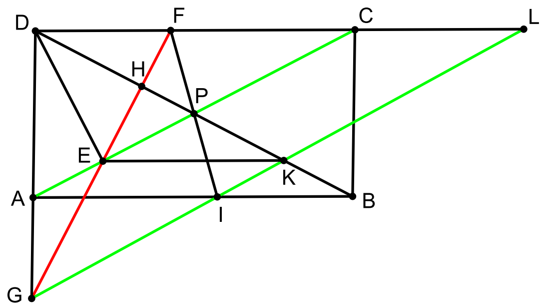

FIGURE 4. Parallel line segments AC and GL

Conversely, when $ABCD$ is a parallelogram with $FE = GE$, let P be the intersection of the diagonals. Also, let the line FP intersect AB in I, and the line GI intersect BD in K and DC extended in L respectively (see Figure 4). A property of parallelograms is that any line segment through P connecting a pair of opposite sides is bisected by P (see Theorem 2.1 *(d)* in [39]), so $FP = PI$. Together with $FE = GE$, this means that EP is parallel to GI according to the intercept theorem applied in triangle FGI. Hence the quadrilateral $GECL$ is a trapezoid. A property of trapezoids is that the midpoints of the bases and the intersection point of the legs are three collinear points. This implies that K is the midpoint of GL.

Next we consider triangle FGL. Here E and K are the midpoints of GF and GL respectively, so it follows that EK and FL are parallel. In triangle GDK we know that GH and DE are heights, so E is its orthocenter and EK is therefor perpendicular to GD and thus to AD. In triangle GFL, E and K are the midpoints of GF and GL, so EK is parallel to FL and thus to DC. We can finally conclude that DC and AD are perpendicular, proving that $ABCD$ is a rectangle according to *(c)*. □

3. Various trapezoids

We use the inclusive definition for all quadrilaterals, so a trapezoid is defined to be a quadrilateral with at least one pair of opposite parallel sides. An *isosceles trapezoid* is defined to be a quadrilateral $ABCD$ with two pairs of distinct equal angles, for instance $\angle A = \angle B$ and $\angle C = \angle D$ (as was done in [12, p. 30]). It's easy to prove that this definition leads to a symmetric trapezoid, which is called isosceles since it has a pair of equal opposite sides.[1] A rectangle is a special case of an isosceles trapezoid, with

[1]We can however not define it in terms of this pair of sides being equal, since then parallelograms would also qualify when using inclusive definitions, and parallelograms are not a special case of isosceles trapezoids but they are a special case of general trapezoids.

the property of all four angles being equal. Any trapezoid has two pairs of adjacent supplementary angles, and if at least one of these pairs is a pair of right angles (for instance $\angle A = \angle D = 90°$), then it's called a *right trapezoid.*

In a quadrilateral $ABCD$ with diagonals intersecting at P, we call the segments AP, BP, CP, DP the *semidiagonals* as was done in [61] and [16].

Next we consider 11 conditions on isosceles trapezoids or right trapezoids that characterize a rectangle. A few of these were discovered while preparing to write this paper. Condition *(b)* is from [23], where it was formulated differently. *(d)* and *(g)* are from [26, p. 123] and [28] respectively, while *(h)*, *(i)*, and *(j)* were stated in [4, p. 162] but not proved there.

Theorem 3.1. *A convex quadrilateral is:*

(a) an isosceles trapezoid with a right angle
(b) an isosceles trapezoid with two opposite equal angles
(c) an isosceles trapezoid with equal bases
(d) an isosceles trapezoid and a parallelogram
(e) an isosceles trapezoid with two opposite equal semidiagonals
(f) an isosceles trapezoid with area $K = ab$, where a and b are the lengths of the longest base and its legs respectively
(g) a trapezoid with two opposite right angles
(h) an isosceles right trapezoid
(i) a cyclic right trapezoid
(j) an equidiagonal right trapezoid
(k) a right trapezoid with perpendicular bimedians

if and only if it's a rectangle.

Proof. *(a)* Any angle in an isosceles trapezoid has one adjacent equal angle and one adjacent supplementary angle, so if one of these three angles is a right angle, then all three of them are right angles. Hence, by the angle sum, all four angles are right angles and thus equal, making it a rectangle according to the definition.

(b) In an isosceles trapezoid, opposite angles are supplementary and since it by definition has two pairs of adjacent equal angles, we directly get that all four angles are right angles. Then it is a rectangle according to *(a).*

(c) Equal bases means that both pairs of opposite sides are equal, so it's a parallelogram. Then the angle argument in the proof of Theorem 2.1 *(d)* shows that it's a rectangle.

(d) Two adjacent angles in a parallelogram $ABCD$ are supplementary and in an isosceles trapezoid they are equal, so we get

$$\begin{cases} \angle A + \angle B = 180° \\ \angle A = \angle B \end{cases} \quad \Leftrightarrow \quad \begin{cases} \angle A = 90° \\ \angle B = 90° \end{cases}$$

and $ABCD$ is a rectangle according to Theorem 2.1 *(c).*

(e) In an isosceles trapezoid, the semidiagonals are pairwise equal: $AP = BP$ and $CP = DP$, so if also $AP = CP$, we get $BP = AP = CP = DP$. Then it's also a parallelogram and the conclusion follows from *(d).*

(f) In an isosceles trapezoid with bases a and $c \le a$, legs b, and height h we have $h \le b$, so its area satisfies

$$K = \tfrac{1}{2}(a+c)h \le \tfrac{1}{2}(a+c)b \le \tfrac{1}{2}(a+a)b = ab$$

where equality holds if and only if $a \perp b$ and $a = c$. Any one of these conditions implies that it's a rectangle according to *(a)* and *(c)*.

(g) A trapezoid has two pairs of supplementary angles. If one angle from each pair is a right angle, then all four angles are right angles and thus equal, making it a rectangle by definition.

(h) In an isosceles right trapezoid, a typical case is

$$\begin{cases} \angle A = \angle B \\ \angle C = \angle D \\ \angle A = \angle D = 90^\circ \end{cases}$$

which directly yields that all four angles are right angles and thus equal, making $ABCD$ a rectangle by definition.

(i) In a cyclic right trapezoid, a typical case is

$$\begin{cases} \angle A + \angle C = 180^\circ \\ \angle B + \angle C = 180^\circ \\ \angle A = \angle D = 90^\circ \end{cases}$$

and it follows that all four angles are equal, so $ABCD$ a rectangle.

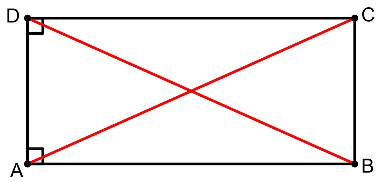

FIGURE 5. An equidiagonal right trapezoid

(j) Suppose $\angle A = \angle D = 90^\circ$. Then $\triangle ABD \cong \triangle DCA$ (RHS), so $AB = DC$ (see Figure 5). Next we get that $\triangle ABC \cong \triangle DCB$ (SSS), implying $\angle B = \angle C$, and since $\angle B + \angle C = 180^\circ$, it follows that $\angle B = \angle C = 90^\circ$. Hence all four angles are equal, making $ABCD$ a rectangle.

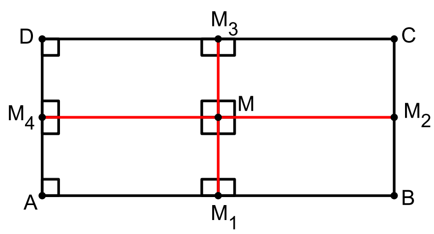

FIGURE 6. A right trapezoid with perpendicular bimedians

(k) Suppose $\angle A = \angle D = 90^\circ$. Together with the condition of perpendicular bimedians, we get the situation in Figure 6 with twelve right angles. Let the bimedians be M_1M_3 and M_2M_4, intersecting at M. Then quadrilaterals AM_1MM_4 and M_1BM_2M are congruent (SASAS), as are M_4MM_3D and

MM_2CM_3 (SASAS). Hence all four angles of $ABCD$ are equal, so it's a rectangle. (For a proof of the SASAS congruence theorem for quadrilaterals, see [24, pp. 98–100]. An online resource for the relevant chapter is [25].) □

4. Angles and semidiagonals

In this section we will prove 18 sufficient conditions for when a quadrilateral $ABCD$ is a rectangle that are expressed as angle relations and/or in terms of the semidiagonals AP, BP, CP, DP where P is the diagonal intersection. *(a)* and *(c)* are from [60] and [13] respectively according to [57, pp. 94–95], while *(b)* was used by Euclid in his famous *Elements* to define a rectangle. Conditions *(d)* and *(f)* are from [55, p. 123] and [44, p. 195] respectively, while *(e)* was a problem on the 2019 Iranian Geometry Olympiad (Intermediate level, Problem 2, proposed by Morteza Saghafian) according to [14]. Condition *(g)* is from [26, p. 123], *(h)* was stated in [27] and to prove that it is a sufficient condition was given as an exercise in [24, p. 114]. The rest were stated in [15], but only *(r)* was proved there.

Theorem 4.1. *A convex quadrilateral $ABCD$ with diagonal intersection P satisfies any one of:*

(a) it has four equal angles
(b) it has four right angles
(c) it has three right angles
(d) it has no obtuse angle
(e) triangles ABC, BCD, CDA, DAB are similar to each other
(f) it has two opposite congruent sides perpendicular to a third side
(g) it has equal and bisecting diagonals
(h) $AP = BP = CP = DP$
(i) $AB = CD$, $AD = BC$, $AP = BP$
(j) $\angle A = \angle C$, $\angle B = \angle D$ *and* $AP = BP$
(k) $AB = CD$, $AP = BP$ *and* $CP = DP$
(l) $\angle A = \angle B$, $\angle C = \angle D$ *and* $AP = CP$
(m) $\angle A = \angle C$, $AP = BP$ *and* $CP = DP$
(n) $\angle B = \angle D$ *and* $AP = BP = CP$
(o) $\angle A = \angle B$ *and* $AP = BP = CP$
(p) $\angle A = \angle D$ *and* $AP = BP = CP$
(q) $\angle A = \angle B = \angle C$ *and* $AP = BP$
(r) $\angle A = \angle B = \angle C$ *and* $AB = CD$

if and only if it's a rectangle.

Proof. *(a)* This is the definition of rectangles that we use.

(b), (c) These are, by applying the angle sum of a quadrilateral, equivalent to *(a)*.

(d) Without any obtuse angle, the angle sum of a quadrilateral can only be achieved with four right angles, since three right angles and an acute angle would not make an angle sum of 360°, and more acute angles would make the angle sum even less. Hence it's a rectangle according to *(b)*.

(e) Suppose without loss of generality that $\angle B$ is the maximum angle in the quadrilateral. This implies that $\angle ABC > \angle DBC$ and $\angle ABC \geq$

$\angle ADC \geq \angle BCD$. Triangles ABC and BCD are similar, so we get $\angle ABC = \angle BCD$. Similarly all angles of the quadrilateral must be equal, so it is a rectangle by definition.

(f) We assume that $\angle A = \angle D = 90°$. Then triangles ABD and DCA are congruent (SAS), so $BD = AC$. It now follows from Theorem 3.1 *(j)* that $ABCD$ is a rectangle.

(g) Since bisecting diagonals is a characterization of parallelograms, this is just another way of formulating Theorem 2.1 *(f)*.

(h) This is a special case of *(g)*.

(i) $AB = CD$ and $AD = BC$ implies that $ABCD$ is a parallelogram, and since they have bisecting diagonals, $AP = BP$ implies that $AC = BD$. Then it's a rectangle according to Theorem 2.1 *(f)*.

(j) $\angle A = \angle C$ and $\angle B = \angle D$ implies that $ABCD$ is a parallelogram, and since they have bisecting diagonals, $AP = BP$ implies that $AC = BD$. Hence it's a rectangle according to Theorem 2.1 *(f)*.

(k) Triangles ABP and CDP are isosceles and congruent (ASA) due to equal angles at P. Triangles ADP and BCP are also congruent (SAS), implying that all vertex angles in $ABCD$ are equal. Then it's a rectangle according to the definition.

(l) $\angle A = \angle B$ and $\angle C = \angle D$ implies that $ABCD$ is an isosceles trapezoid. Due to similar triangles ABP and CDP (AA), we have

$$\frac{AP}{CP} = \frac{BP}{DP}$$

and one diagonal bisected implies that both diagonals are bisected, so it's also a parallelogram. Hence it's a rectangle according to Theorem 3.1 *(d)*.

(m) $AP = BP$ and $CP = DP$ imply that $ABCD$ is an isosceles trapezoid. Those have two distinct pairs of equal angles, and if also a pair of opposite angles are equal, then all four angles are equal. Hence it's a rectangle by definition.

(n) $AP = BP = CP$ implies that triangle ABC is inscribed in a half circle with diameter AC and a right angle at B (see Figure 7). Then $\angle B = \angle D$ confirms that also the fourth vertex lies on that circle, so $AP = BP = CP = DP$ and $ABCD$ is a rectangle according to *(f)*.

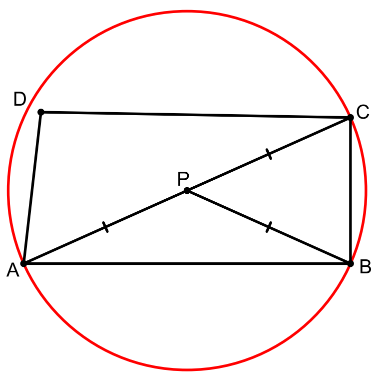

Figure 7. The circumcircle to triangle ABC

(o) $AP = BP = CP$ implies that triangle ABC is inscribed in a half circle with diameter AC and a right angle at B. If D is inside the circle with diameter AC, then $\angle A < 90°$ (see Figure 7), and if D is outside the circle with diameter AC, then $\angle A > 90°$. Both of these cases contradict the assumption $\angle A = \angle B$, so D must lie on that circle. This confirms that $ABCD$ is a rectangle according to *(f)*.

(p) $AP = BP = CP$ implies that triangle ABC is inscribed in a half circle with diameter AC and a right angle at B. If D is inside the circle with diameter AC, then $\angle D > 90°$ but $\angle A < 90°$ (see Figure 7); if D is outside the circle with diameter AC, then $\angle D < 90°$ but $\angle A > 90°$. Both of these cases contradict the assumption $\angle A = \angle D$, so D must lie on that circle. Then $ABCD$ is a rectangle according to *(f)*.

(q) Since triangle ABP is isosceles (see Figure 8), $\angle PAB = \angle PBA$ which implies that $\angle DAP = \angle CBP$. Together with equal vertical angles at P, we have that triangles APD and BPC are congruent (ASA), so $DP = CP$. Then, since also $AP = BP$, $ABCD$ is an isosceles trapezoid with three equal angles, which according to Theorem 3.1 *(b)* means that it's actually a rectangle.

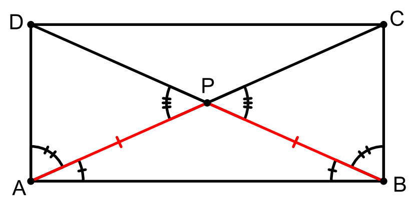

FIGURE 8. Congruent triangles APD and BPC

(r) There are three possibilities for angle A. If $\angle A = 90°$, then $\angle A = \angle B = \angle C = \angle D = 90°$ by the angle sum of a quadrilateral, so $ABCD$ is a rectangle according to *(b)*.

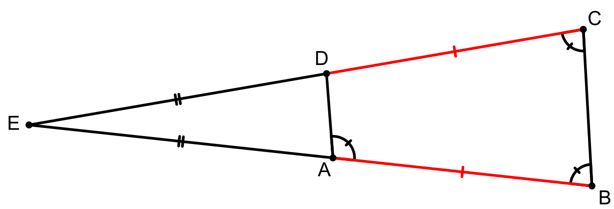

FIGURE 9. Isosceles triangles EBC and EAD

On the other hand, if $\angle A > 90°$ or $\angle A < 90°$, then AB and DC intersect at a point E (see Figure 9). Suppose it is outside of AD, the other case is similar. Then $\angle B = \angle C$ implies that $EB = EC$, and since $AB = DC$, we get $EA = ED$. Hence $\angle EAD = \angle EDA$ by the isosceles triangle theorem, so $\angle A = \angle D$, making all four angles of $ABCD$ equal and thus it's a rectangle by definition. □

5. Symmetry and multitype quadrilaterals

Consider a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ and diagonal intersection P. Let $a' = AP$, $b' = BP$, $c' = CP$, $d' = DP$ be the semidiagonals. In Table 1 we summarize a few of the different types of quadrilaterals that are studied in this section together with one useful characterization.

Quadrilateral	**Definition**	**Characterization**	**Ref.**
Bisect-diagonal	A bisected diagonal	$a' = c' \vee b' = d'$	[35]
Cyclic	Has a circumcircle	$a'c' = b'd'$	[36]
Equidiagonal	Equal diagonals	$a' + c' = b' + d'$	[32]
Extangential	Has an excircle	$\lvert a - c\rvert = \lvert b - d\rvert$	[29]
Semidiagonal	$a' + b' = c' + d'$		[16]
Isosceles trapezoid	$\angle A = \angle B \wedge \angle C = \angle D$	$a' = b' \wedge c' = d'$	

Table 1. Some quadrilaterals

Semidiagonal quadrilaterals is a new type of quadrilaterals that is defined this way in [16] (but there is of course the second possibility $a'+d' = b'+c'$). For the isosceles trapezoid, we only stated one of two possibilities, the other is the definition $\angle A = \angle D \wedge \angle B = \angle C$ and characterization $a' = d' \wedge b' = c'$.

In the next theorem we study another 12 characterizations of rectangles that are about symmetry or different types of quadrilaterals at the same time. The similar first three are taken from [26, p. 123], [57, p. 37], and [17] respectively, while *(d)* is due to Schweizer in 1970 according to [59, p. 183]. Conditions *(f)* and *(g)* are from the quadrilateral classifications in [42] and [34] respectively, while *(h)* to *(k)* were stated in [16] but not proved there. The last characterization was Problem B5019 in the Hungarian mathematical journal KöMaL [43]. We give our proof of it.

Theorem 5.1. *A convex quadrilateral $ABCD$ satisfies any one of:*

(a) each bimedian is a symmetry axis
(b) the perpendicular bisectors of two adjacent sides are symmetry axes
(c) it has symmetry axes through each pairs of opposite sides
(d) it's a parallelogram with a line of symmetry parallel to a side
(e) all four angle bisectors form a square
(f) it's a bisect-diagonal cyclic trapezoid
(g) it's an extangential cyclic trapezoid
(h) it's an extangential equidiagonal trapezoid
(i) it's an extangential equidiagonal semidiagonal quadrilateral
(j) it's an extangential cyclic equidiagonal quadrilateral
(k) it's an extangential cyclic semidiagonal quadrilateral
(l) it's an extangential cyclic quadrilateral where $BA+AC = CD+DB$

if and only if it's a rectangle.

Proof. *(a)* From the definition of a symmetry axis it's evident that the only quadrilateral with a bimedian as a symmetry axis is an isosceles trapezoid, which has a pair of opposite parallel sides. Then the only quadrilateral with two bimedians as symmetry axes must be both an isosceles trapezoid and a parallelogram, that is, a rectangle according to Theorem 3.1 *(d)*.

(b) If the perpendicular bisector of a side is an axis of symmetry, then the quadrilateral must be an isosceles trapezoid. Then the given condition implies that the quadrilateral is both an isosceles trapezoid and a parallelogram, that is, a rectangle according to Theorem 3.1 *(d)*.

(c) If a quadrilateral has a symmetry axis through a pair of opposite sides, then this axis must be the bimedian. Hence this condition is a consequence of *(a)*.

(d) A symmetry line must coincide with a bimedian. Then the two angles between a side and this bimedian are equal, implying that two adjacent vertex angles of the parallelogram are equal. This means it's a rectangle according to Theorem 2.1 *(d)*.

(e) When the four angle bisectors form a square, we get

$$\begin{cases} \frac{\angle A}{2} + \frac{\angle B}{2} = 90^\circ \\ \frac{\angle B}{2} + \frac{\angle C}{2} = 90^\circ \\ \frac{\angle C}{2} + \frac{\angle D}{2} = 90^\circ \\ \frac{\angle D}{2} + \frac{\angle A}{2} = 90^\circ \end{cases} \Rightarrow \begin{cases} \angle A + \angle B = 180^\circ \\ \angle B + \angle C = 180^\circ \\ \angle C + \angle D = 180^\circ \\ \angle D + \angle A = 180^\circ \end{cases} \Rightarrow \begin{cases} \angle A = \angle C \\ \angle B = \angle D \end{cases}$$

so $ABCD$ is a parallelogram. Using notations as in Figure 10, triangles AHD and CFB are congruent (ASA), implying that $w = y$. Triangles AHD and CGD are similar (AA), so

$$\frac{x+y}{w} = \frac{x+z}{z} \quad \Rightarrow \quad \frac{x}{w} = \frac{x}{z},$$

where we used $w = y$. Hence $w = z$, and by the isosceles triangle theorem, we get $\frac{\angle A}{2} = \frac{\angle D}{2}$, so $\angle A = \angle D$. This proves that $ABCD$ is a rectangle according to Theorem 2.1 *(d)*.

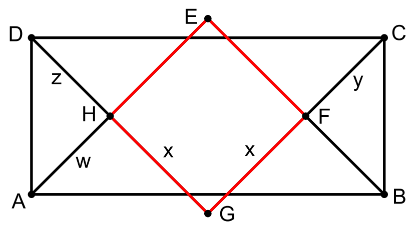

FIGURE 10. The angle bisectors form a square

(f) This is just another way of stating Theorem 4.1 *(l)*.

(g) A trapezoid is cyclic if and only if it's an isosceles trapezoid. It's extangential if and only if $|a - c| = |b - d|$. Since $b = d$ (or $a = c$) in an isosceles trapezoid, we get $a = c$ (or $b = d$), implying that the quadrilateral is a rectangle according to Theorem 3.1 *(c)*.

(h) A trapezoid is equidiagonal if and only if it's an isosceles trapezoid according to Theorem 17 (iii) in [32]. Now the argument is the same as in *(g)* for why it's a rectangle.

(i) In an equidiagonal semidiagonal quadrilateral, we have

$$\begin{cases} a' + c' = b' + d' \\ a' + b' = c' + d' \end{cases} \Leftrightarrow \begin{cases} 2a' = 2d' \\ c' - b' = b' - c' \end{cases} \Leftrightarrow \begin{cases} a' = d' \\ b' = c' \end{cases}$$

which implies an isosceles trapezoid. Since it is also extangential, the quadrilateral is a rectangle according to *(g)*.

(j) A cyclic quadrilateral is equidiagonal if and only if it's an isosceles trapezoid according to Theorem 17 (iv) in [32]. Now the conclusion that it's a rectangle follows from *(g)*.

(k) In a cyclic semidiagonal quadrilateral, we have

$$\begin{cases} a' + b' = c' + d' \\ a'c' = b'd' \end{cases}$$

and by substituting $a' = c' + d' - b'$ into the intersecting chords theorem and factoring, we get $(c' - b')(c' + d') = 0$. Hence $c' = b'$, and thus also $a' = d'$, which implies that the quadrilateral is an isosceles trapezoid. In these $b = d$ (or $a = c$) and then the other of these two equalities also hold according to $|a - c| = |b - d|$. This proves that an extangential cyclic semidiagonal quadrilateral is a cyclic parallelogram, that is, a rectangle according to Theorem 2.1 *(e)*.

(l) Here we have the three metric conditions

$$\begin{cases} |a - c| = |b - d| \\ a + p = c + q \\ ac + bd = pq \end{cases}$$

where the third is the well-known Ptolemy's theorem that characterizes cyclic quadrilaterals. Combining the first two equalities yields

$$|a - c| = |b - d| = |q - p|.$$

Next we need Euler's quadrilateral theorem. It states that the sides of a convex quadrilateral satisfy

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2 \tag{1}$$

where v is the distance between the midpoints of the diagonals p and q (for a proof, see [4, pp. 9–10]). Together with the simple algebra rule

$$(q - p)^2 = q^2 + p^2 - 2pq$$

we get

$$(b - d)^2 = a^2 + b^2 + c^2 + d^2 - 4v^2 - 2(ac + bd)$$

which simplifies into

$$4v^2 = (a - c)^2 \quad \Leftrightarrow \quad v = \tfrac{1}{2}|a - c|.$$

Theorem 12 in [31] states that a convex quadrilateral is a trapezoid with $a \parallel c$ if and only if this formula holds for v. In the same way we derive the similar formula $v = \frac{1}{2}|b - d|$, and by symmetry, this means that the quadrilateral is a trapezoid with $b \parallel d$. Hence both pairs of opposite sides are parallel, so the quadrilateral is a cyclic parallelogram, that is, a rectangle according to Theorem 2.1 *(e)*. □

6. Cyclic quadrilaterals and trigonometric relations

We will need the following trigonometric formula:

Lemma 6.1. *In a cyclic quadrilateral $ABCD$ with consecutive sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, and semiperimeter s,*

$$\tan\frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$$

and there are similar formulas for the other three vertex angles.

Proof. Applying the law of cosines to triangles ABD and BCD, we get

$$a^2 + d^2 - 2ad\cos A = BD^2 = b^2 + c^2 - 2bc\cos C$$

and using $\angle C = \pi - \angle A$, since opposite angles in a cyclic quadrilateral are supplementary angles, yields

$$2(ad + bc)\cos A = a^2 + d^2 - b^2 - c^2.$$

Now we apply the half angle formula for tangent,

$$\begin{aligned}\tan^2\left(\frac{A}{2}\right) &= \frac{1-\cos A}{1+\cos A} = \frac{2(ad+bc) - 2(ad+bc)\cos A}{2(ad+bc) + 2(ad+bc)\cos A} \\ &= \frac{2(ad+bc) - (a^2+d^2-b^2-c^2)}{2(ad+bc) + (a^2+d^2-b^2-c^2)} \\ &= \frac{(b+c)^2 - (a-d)^2}{(a+d)^2 - (b-c)^2} = \frac{(b+c+a-d)(b+c-a+d)}{(a+d+b-c)(a+d-b+c)} \\ &= \frac{2(s-d)\cdot 2(s-a)}{2(s-c)\cdot 2(s-b)} = \frac{(s-a)(s-d)}{(s-b)(s-c)}\end{aligned}$$

completing the derivation.

The formulas for the other three vertices follow by symmetry from cyclic permutations. □

Next we have 12 characterizations of rectangles regarding cyclic quadrilaterals or trigonometric relations. To prove the sufficiency in *(b)* was Problem 2 at the 2008 CentroAmerican Math Olympiad [2], proposed by Aarón Ramírez from El Salvador. Conditions *(c)* and *(d)* were proved in [3]. The corresponding inequality of *(e)* was Problem 2 on the United States Mathematical Olympiad in 1999 [1], but the equality case was not mentioned; that was however stated in [51, pp. 2, 6–7]. The corresponding inequalities of *(f)* and *(g)* were proved in [48, pp. 102–106], but the cases of equality were not stated. Condition *(h)* was proved at [21] (but we have simplified that proof), and *(l)* together with its proof are cited from [22].

An *escribed circle* is a circle tangent to one side of a quadrilateral and the extensions of the two adjacent sides.

Theorem 6.1. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d, corresponding escribed circles with radii r_a, r_b, r_c, r_d, diagonals p, q and diagonal intersection P satisfies any one of:*

(a) it's a cyclic quadrilateral with diagonals being diameters

(b) *it's a cyclic quadrilateral with circumcenter O, diameter AC, and parallelograms $DAOE$ and $BCOF$ such that E and F lie on the circumference*
(c) *it's a cyclic quadrilateral where P coincide with the circumcenter*
(d) *it's a cyclic quadrilateral where the vertex centroid coincide with the circumcenter*
(e) *it's a cyclic quadrilateral where* $|a-c|+|b-d|=2|p-q|$
(f) *it's a cyclic quadrilateral with area* $K=4\sqrt{r_ar_br_cr_d}$
(g) *it's a cyclic quadrilateral with area* $K=(r_a+r_c)(r_b+r_d)$
(h) *it's a cyclic quadrilateral where* $\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}=4$
(i) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}+\sin\frac{D}{2}=2\sqrt{2}$
(j) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}+\cos\frac{D}{2}=2\sqrt{2}$
(k) $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{D}{2}=4$
(l) $\sqrt{2}\sum_{cyc}\cos\frac{A+B}{4}=\sum_{cyc}\cot\frac{A}{2}$

if and only if it's a rectangle.

Proof. *(a)* When the diagonals are diameters, then all vertex angles are right angles according to the inscribed angle theorem, and the cyclic quadrilateral is thus a rectangle according to Theorem 4.1 *(b)*.

(b) When E and F lie on the circumference, $BC=FO=OE=AD$, so $\angle BAC=\angle ACD$ (see Figure 11). This implies that $AB\parallel DC$ so $ABCD$ is a parallelogram. But angles B and D are right angles since AC is a diameter, so $ABCD$ is a rectangle according to Theorem 2.1 *(e)*.

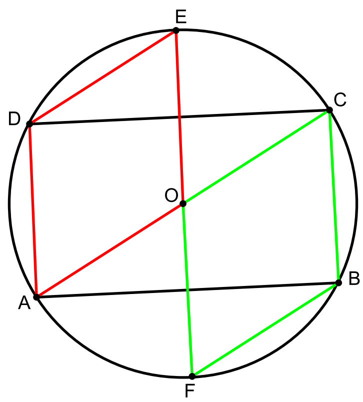

Figure 11. Two parallelograms

(c) If the diagonals intersect at the circumcenter, then they bisect each other and have equal length, so the cyclic quadrilateral is a rectangle according to Theorem 2.1 *(f)*.

(d) It is well known that the vertex centroid in a convex quadrilateral is the point where the bimedians intersect, and that the circumcenter in a cyclic quadrilateral is the intersection of the perpendicular bisectors of the four sides. If these two points coincide, then the two bimedians must be perpendicular to each side of the cyclic quadrilateral (since there is just one line through two points). Then opposite sides of the cyclic quadrilateral are

parallel, making it a cyclic parallelogram, that is, a rectangle according to Theorem 2.1 *(e)*.

(e) In a convex quadrilateral $ABCD$, let M, N, E be the midpoints of AC, BD, BC respectively. Then $ME \parallel AB$, $NE \parallel DC$ and $ME = \frac{1}{2}AB$, $NE = \frac{1}{2}DC$ (see Figure 12). Applying the triangle inequality yields $MN+NE \geq ME$, so $MN \geq ME-NE$. Hence $2v \geq a-c$. There is also the other possibility to have $2v \geq c-a$, so these two cases can be merged into $2v \geq |a-c|$. Equality holds if and only if $AB \parallel ME \parallel MN \parallel NE \parallel DC$, that is, only when $a \parallel c$. By symmetry, there is the similar inequality $2v \geq |b-d|$ where equality holds if and only if $b \parallel d$.

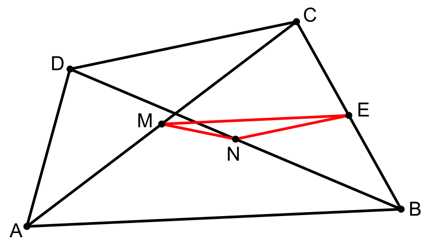

Figure 12. Midpoint triangle MNE

Next we add Ptolemy's theorem $ac + bd = pq$, which holds in a cyclic quadrilateral, to Euler's quadrilateral theorem (1) to get

$$(a-c)^2 + (b-d)^2 = (p-q)^2 + 4v^2.$$

Inserting $2v \geq |a-c|$, we get $|p-q| \leq |b-d|$ with equality only when $a \parallel c$. Similarly $2v \geq |b-d|$ yields $|p-q| \leq |a-c|$ with equality only when $b \parallel d$. By adding, we have proved that in a cyclic quadrilateral,

$$|a-c| + |b-d| \geq 2|p-q|$$

where equality holds if and only if it is a parallelogram, that is, a rectangle according to Theorem 2.1 *(e)*.

(f) In a cyclic quadrilateral,

$$r_a = \frac{aK}{(s-a)(a+c)} \tag{2}$$

where K is the area of the quadrilateral, and similar formulas hold for the other three radii. This formula was partially derived in [37, pp. 28–29]. What was left out was the derivation of the tangent half angle formula, which we derived as a Lemma here. Forming the product of the four escribed radii, we get

$$r_a r_b r_c r_d = \frac{abcdK^4}{(s-a)(s-b)(s-c)(s-d)(a+c)^2(b+d)^2}$$

and using Brahmagupta's formula

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{3}$$

this is simplified into

$$r_a r_b r_c r_d = \frac{abcdK^2}{(a+c)^2(b+d)^2}.$$

Next we apply the following variant of the AM-GM-inequality: $(a+c)^2 \geq 4ac$ and $(b+d)^2 \geq 4bd$ to get

$$r_a r_b r_c r_d \leq \frac{abcdK^2}{4ac \cdot 4bd} = \frac{K^2}{16}$$

and it follows that

$$K^2 \geq 16 r_a r_b r_c r_d$$

in a cyclic quadrilateral, where equality holds if and only if $a = c$ and $b = d$, that is, only when the quadrilateral is a cyclic parallelogram. According to Theorem 2.1 *(e)*, this is true if and only if it's a rectangle.

(g) From (2) and the corresponding formula for r_c, we get

$$\begin{aligned} \frac{r_a + r_c}{K} &= \frac{1}{a+c}\left(\frac{a}{s-a} + \frac{c}{s-c}\right) = \frac{s(a+c) - 2ac}{(a+c)(s-a)(s-c)} \\ &= \frac{s}{(s-a)(s-c)} - \frac{2ac}{a+c}\frac{1}{(s-a)(s-c)} \\ &\geq \frac{s}{(s-a)(s-c)} - \frac{a+c}{2}\frac{1}{(s-a)(s-c)} \\ &= \frac{b+d}{2(s-a)(s-c)} = \frac{(s-a)+(s-c)}{2(s-a)(s-c)} \\ &\geq \frac{\sqrt{(s-a)(s-c)}}{(s-a)(s-c)} = \frac{1}{\sqrt{(s-a)(s-c)}} \end{aligned}$$

where we used $2s = a+b+c+d$ twice. We also applied the AM-GM-inequality twice, so we have equality if and only if $a = c$. Using (3) yields

$$r_a + r_c \geq \frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{\sqrt{(s-a)(s-c)}} = \sqrt{(s-b)(s-d)}$$

where equality holds if and only if $a = c$. By symmetry,

$$r_b + r_d \geq \sqrt{(s-a)(s-c)}$$

where equality holds if and only if $b = d$. Hence

$$(r_a + r_c)(r_b + r_d) \geq \sqrt{(s-b)(s-d)}\sqrt{(s-a)(s-c)} = K$$

where equality holds if and only if $ABCD$ is a cyclic parallelogram, that is, a rectangle according to Theorem 2.1 *(e)*.

(h) From the Lemma, we get

$$\tan\frac{A}{2}\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}\sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}} = \frac{s-a}{s-c}$$

and similar expressions hold for the other three terms in the sum we consider. Thus

$$\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-a}{s-c} + \frac{s-b}{s-d} + \frac{s-c}{s-a} + \frac{s-d}{s-b}. \tag{4}$$

Next we use the inequality $x + \frac{1}{x} \geq 2$, which is equivalent to $(x-1)^2 \geq 0$, so equality holds if and only if $x = 1$. Applying this to the first and third term of (4), and also to the second and fourth term, we get

$$\sum_{cyc} \tan\frac{A}{2}\tan\frac{B}{2} \geq 2+2$$

where equality holds if and only if

$$\frac{s-a}{s-c}=1 \quad \text{and} \quad \frac{s-b}{s-d}=1$$

that is, if and only if $a=c$ and $b=d$. Opposite equal sides is a characterization of a parallelogram, and the only cyclic parallelogram is a rectangle according to Theorem 2.1 *(e)*.

(i) The sine function is concave on the interval $\left(0, \frac{\pi}{2}\right)$. Applying Jensen's inequality yields

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}+\sin\frac{D}{2} \leq 4\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{D}{2}}{4} = 4\sin\frac{\pi}{4} = 2\sqrt{2}$$

where equality holds if and only if $\angle A = \angle B = \angle C = \angle D$, that is, only in a rectangle.

(j) The cosine function is also concave on the interval $\left(0, \frac{\pi}{2}\right)$, so this proof is very similar to that of *(i)*. It is left to the reader.

(k) The tangent function is convex on the interval $\left(0, \frac{\pi}{2}\right)$. Applying Jensen's inequality yields

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{D}{2} \geq 4\tan\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{D}{2}}{4} = 4\tan\frac{\pi}{4} = 4$$

where equality holds if and only if $\angle A = \angle B = \angle C = \angle D$, that is, only in a rectangle.

(l) We note that each cyclic sum has four terms. Applying several times the trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

we get that

$$\begin{aligned}\sqrt{2}\sum_{cyc}\cos\tfrac{A+B}{4} &= \sqrt{2}\cdot 2\cos\tfrac{A+B+C+D}{8}\left(\cos\tfrac{A+B-C-D}{8}\cos\tfrac{A-B-C+D}{8}\right)\\ &= 4\cos\tfrac{A-C}{8}\cos\tfrac{B-D}{8},\end{aligned}$$

where we also used the angle sum of a quadrilateral, so this implies that

$$\sqrt{2}\sum_{cyc}\cos\frac{A+B}{4} \leq 4 \tag{5}$$

where equality holds if and only if both pairs of opposite angles are equal, that is, only in a parallelogram.

The cotangent function is convex on the interval $\left(0, \frac{\pi}{2}\right)$. Applying Jensen's inequality yields

$$\sum_{cyc}\cot\frac{A}{2} \geq 4\cot\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{D}{2}}{4} = 4\cot\frac{\pi}{4} = 4 \tag{6}$$

where equality holds if and only if all angles are equal, that is, only in a rectangle. Combining (5) and (6), we get

$$\sqrt{2}\sum_{cyc}\cos\frac{A+B}{4}\le 4\le\sum_{cyc}\cot\frac{A}{2}$$

where equality between the two cyclic sums holds if and only if $ABCD$ is a rectangle. □

7. Metric relations

We conclude with a theorem about 12 metric characterizations of rectangles. The formula in *(a)* is a famous approximate formula that was used by the ancient Egyptians to calculate the area of a general quadrilateral, which has been studied in many books. It was stated as a characterization in [9] and proving this was Problem 26 at the 1977 Annual High School Mathematics Examination (in the USA) [5, pp. 32, 113]. Condition *(b)* was proved in five different ways in [30] but the proof we give here is from [4, p. 163], *(d)* is from [52], while *(e)* and *(f)* were proved in [33]. The two related *(g)* and *(h)* appeared in [6] and we cite that proof here. The next two related *(i)* and *(j)* are from the final of the 2008 Russian III Southern Tournament Junior Math Fights [50] and [53] respectively. The latter of these has, in the special case when $ABCD$ is a trapezoid, also been a problem on the 1998 Argentina IberoAmerican Training List [46]. The penultimate characterization was a problem proposed by Traian Lălescu at a 1986 Romanian math competition according to [8], but the proof we give is from [20, pp. 298, 301]. The last condition and its proof are cited from [27].

Theorem 7.1. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d, diagonals p, q, and angle θ between the diagonals opposite side a satisfies any one of:*

(a) *it has area* $K=\frac{1}{4}(a+c)(b+d)$
(b) *it has area* $K=\frac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}$
(c) $(a-c)^2+(b-d)^2=(p-q)^2$
(d) $(a+c)^2+(b+d)^2=(p+q)^2$
(e) $a^2+b^2+c^2+d^2=2pq$
(f) $\cos\theta=\frac{|a^2-b^2+c^2-d^2|}{a^2+b^2+c^2+d^2}$
(g) *the sum of distances from each vertex to the other three is constant*
(h) *the distance between any two vertices is equal to the distance between the other two vertices*
(i) *the perimeters of the triangles ABC, BCD, CDA, DAB are equal*
(j) *the inradii of the triangles ABC, BCD, CDA, DAB are equal*
(k) $AX^2+CX^2=BX^2+DX^2$ *for all points X in space*
(l) $\sin\theta=2\sqrt{\frac{KP\cdot LP\cdot MP\cdot NP}{AP\cdot BP\cdot CP\cdot DP}}$, *where $K\in AB$, $L\in BC$, $M\in CD$, $N\in DA$ such that KM and LN are angle bisectors to the angles between the diagonals and P is the diagonal intersection*

if and only if it's a rectangle.

Proof. *(a)* A diagonal can divide a convex quadrilateral into two triangles in two different ways. Adding these four triangle areas yields that the area K of the quadrilateral satisfies

$$\begin{aligned}2K &= \tfrac{1}{2}ab\sin B + \tfrac{1}{2}bc\sin C + \tfrac{1}{2}cd\sin D + \tfrac{1}{2}da\sin A\\ &\le \tfrac{1}{2}ab + \tfrac{1}{2}bc + \tfrac{1}{2}cd + \tfrac{1}{2}da = \tfrac{1}{2}(a+c)(b+d)\end{aligned}$$

where there is equality if and only if $\angle A = \angle B = \angle C = \angle D = \frac{\pi}{2}$, that is, only in a rectangle according to Theorem 4.1 *(b)*.

(b) From the inequality in *(a)*, we directly get

$$K \le \frac{a+c}{2}\cdot\frac{b+d}{2} \le \sqrt{\left(\frac{a^2+c^2}{2}\right)\left(\frac{b^2+d^2}{2}\right)}$$

where the second inequality is obtained by applying the AM-RMS inequality. Equality holds if and only if $\angle A = \angle B = \angle C = \angle D = 90^\circ$, $a = c$ and $b = d$, that is, only when $ABCD$ is a rectangle.

(c) Ptolemy's inequality $ac + bd \ge pq$ can be rewritten as

$$-2ac - 2bd \le -2pq$$

where equality holds if and only if the quadrilateral is cyclic. Adding this to (1) yields

$$(a-c)^2 + (b-d)^2 \le (p-q)^2 + 4v^2.$$

Hence $(a-c)^2 + (b-d)^2 = (p-q)^2$ holds if and only if the quadrilateral is a cyclic parallelogram, which is a rectangle according to Theorem 2.1 *(e)*.

(d) This proof is almost identical to that of *(c)*, so it is left to the reader.

(e) By adding and subtracting $2pq$ to the right hand side of (1), we get

$$a^2 + b^2 + c^2 + d^2 = 2pq + (p-q)^2 + 4v^2 \ge 2pq, \tag{7}$$

where there is equality if and only if $v = 0$ and $p = q$. The first equality is a well-known characterization of parallelograms, and the only parallelograms with equal diagonals are rectangles according to Theorem 2.1 *(f)*.

(f) In all convex quadrilaterals with consecutive sides a, b, c, d, it holds that

$$\left|a^2 - b^2 + c^2 - d^2\right| = 2pq\cos\theta \tag{8}$$

where θ is the acute angle between the diagonals p and q (see [4, pp. 17–18]). Combining (7) and (8), we get

$$\cos\theta \ge \frac{\left|a^2 - b^2 + c^2 - d^2\right|}{a^2 + b^2 + c^2 + d^2}$$

where equality holds if and only if the quadrilateral is a rectangle.

(g) Denoting the constant by L, this condition is

$$\begin{cases} a+p+d = L\\ a+q+b = L\\ b+p+c = L\\ c+q+d = L\end{cases} \quad\Rightarrow\quad \begin{cases} p+d = b+q\\ a+d = b+c\\ a+p = c+q\end{cases} \quad\Rightarrow\quad \begin{cases} d = b\\ p = q\\ a = c\end{cases}$$

where we simplified the system of equations in the following way. First step: by subtracting each of the other three equalities from the first. Second step: by adding two of the equalities at a time and subtracting the third equality.

The final three equalities describe a parallelogram with equal diagonals, that is, a rectangle according to Theorem 2.1 *(f)*.

(h) This condition is just the interpretation of the final three equalities in the proof of *(g)*.

(i) The algebra required to prove this condition is almost the same as in the proof of *(g)*. Here we get

$$a+b+p=b+c+q=c+d+p=d+a+q$$

from which it follows that

$$\begin{cases} a+p=c+q \\ a+b=c+d \\ b+p=d+q \end{cases} \Rightarrow \begin{cases} a=c \\ p=q \\ b=d \end{cases}$$

and again we have a parallelogram with equal diagonals, that is, a rectangle.

(j) Let us denote the inradii of triangles ABC, BCD, CDA, DAB by r_B, r_C, r_D, r_A respectively. In [39] we proved as Theorem 1.1 *(i)* that a quadrilateral is a parallelogram if and only if $r_A = r_C$ and $r_B = r_D$. So when all these four inradii are equal, we can directly conclude that the quadrilateral is a parallelogram.

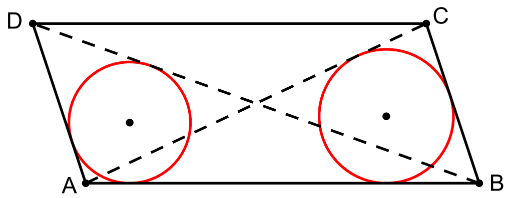

Figure 13. Two of the incircles

Next we consider what $r_A = r_B$ implies (see Figure 13). According to the well-known triangle formula $T = sr$, the area of a triangle is equal to the product of the semiperimeter and the inradius. Triangles DAB and ABC not only have equal inradii, but also equal area since they have equal base and equal height when $ABCD$ is a parallelogram. Hence they also have equal semiperimeter and consequently equal perimeter. This is to say

$$AB+BD+DA=AB+BC+CA$$

which, since $DA = BC$, implies that the diagonals are equal. A parallelogram with equal diagonals is a rectangle according to Theorem 2.1 *(f)*.

(k) For planar points A, B, C, D and an arbitrary point X in space, $AX^2 + CX^2 = BX^2 + DX^2$ is equivalent to

$$(A-X)^2+(C-X)^2=(B-X)^2+(D-X)^2$$

which simplifies to

$$A^2+C^2-B^2-D^2=2X(A+C-B-D).$$

This is valid for *all* points X in space if and only if

$$A+C=B+D, \tag{9}$$

and

$$A^2 + C^2 = B^2 + D^2. \tag{10}$$

We note that (9) is a characterization of a parallelogram (see Theorem 3.1 *(e)* in [39]; it was stated for four complex numbers, so it's also valid when A, B, C, D are the coordinates of planar points). Squaring (9), we have

$$A^2 + C^2 + 2A \cdot C = B^2 + D^2 + 2B \cdot D \tag{11}$$

and subtracting (10) from (11) yields

$$2A \cdot C = 2B \cdot D. \tag{12}$$

Finally subtracting (12) from (10) and factoring, we get

$$(A - C)^2 = (B - D)^2.$$

This means that the parallelogram $ABCD$ has equal diagonals, so it's a rectangle according to Theorem 2.1 *(f)*.

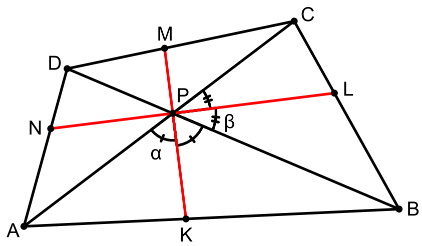

FIGURE 14. Angle bisectors KM and LN

(l) Let $\alpha = \angle APK = \angle BPK = \angle CPM = \angle DPM$ and $\beta = \angle BPL = \angle CPL = \angle DPN = \angle APN$ (see Figure 14). Writing twice the area of triangle ABP in two different ways yields

$$AP \cdot BP \sin 2\alpha = AP \cdot KP \sin\alpha + BP \cdot KP \sin\alpha$$

which by using the double angle formula for sine is equivalent to

$$2AP \cdot BP \cos\alpha = KP(AP + BP) \geq 2KP\sqrt{AP \cdot BP}$$

and this simplifies into

$$\sqrt{AP \cdot BP}\cos\alpha \geq KP$$

where equality holds if and only if $AP = BP$ according the AM-GM-inequality. By symmetry, there are the three similar expressions

$$\sqrt{BP \cdot CP}\cos\beta \geq LP,$$
$$\sqrt{CP \cdot DP}\cos\alpha \geq MP,$$
$$\sqrt{DP \cdot AP}\cos\beta \geq NP$$

where equality holds if and only if $BP = CP$, $CP = DP$, and $DP = AP$ respectively. Multiplying the last four inequalities, we get

$$AP \cdot BP \cdot CP \cdot DP \cos^2\alpha\cos^2\beta \geq KP \cdot LP \cdot MP \cdot NP \tag{13}$$

where equality holds if and only if $AP = BP = CP = DP$. Since $\alpha + \beta = 90°$, we get $\cos\beta = \sin\alpha$, and multiplying (13) with 4 and again using the double angle formula for sine, (13) is equivalent to

$$AP \cdot BP \cdot CP \cdot DP \sin^2(2\alpha) \geq 4KP \cdot LP \cdot MP \cdot NP$$

where equality holds if and only if the quadrilateral is a rectangle according to Theorem 4.1 *(f)*. Since $\theta = 2\alpha$, solving for $\sin 2\alpha$ completes the proof. □

The equation in *(k)* is well known as the British flag theorem. For it to be a characterization, we note that it's important that the equation holds for *all* points X (which could be in the same plane as $ABCD$), otherwise the conclusion will fail. A counterexample is a point on the bimedian that lies on the symmetry line of an isosceles trapezoid.

8. Most useful characterizations

Of the 84 characterizations of rectangles we have studied in this paper, which have been the most useful in the proofs of other characterizations, and how frequently were they used? This is accounted for in Table 2.

Characterization	**Number of proofs**
Quadrilateral with four equal (= right) angles	20
Parallelogram with equal diagonals	12
Parallelogram with one right angle	11
Parallelogram that is cyclic	10
Parallelogram with two adjacent equal angles	6
Parallelogram and isosceles trapezoid	4
Quadrilateral with four equal semidiagonals	4

Table 2. Most frequently used characterizations

We did not distinguish between four equal and four right angles in this table since they are so closely related and the primary difference would arise from the fact that one is stated at the beginning of the paper while the other is not proved until Theorem 4.1 *(b)*, which would give an unfair skewness. In the literature, we get the impression that the former is less often used and when this happens, it is often due to the study of duality in connection with quadrilateral classifications.

9. Chronological compilation

Here we summarize all 84 characterizations of rectangles we have studied in this paper. They are given in chronological order with respect to the oldest source for the *sufficient* condition that we know of, but some of them have surely been published earlier.

The abbreviations Q and Quad stand for quadrilateral, P for parallelogram, IT for isosceles trapezoid, $\mathscr{P}$ for perimeter, w for with, and RSTJMF for Russian Southern Tournament Junior Math Fight. Most notations are not explained here since the purpose of this table is just to get a chronological overview. For the meaning of other labels, please see the corresponding theorems.

#	Year	Source	Short description	Ref.
1	-300	Euclid	Quadrilateral with 4 right angles	[41]
2	1833	Young	Parallelogram with one right angle	[63]
3	1848	Duncan	P w max area for given base & perimeter	[19]
4	1849	Perkins	Parallelogram with 4 right angles	[47]
5	1868	Wright	Parallelogram with equal diagonals	[62]
6	1871	Todhunter	Parallelogram that is cyclic	[56]
7	1879	Henrici	Parallelogram and isosceles trapezoid	[26]
8	1879	Henrici	Quad with equal and bisecting diagonals	[26]
9	1879	Henrici	Q where each bimedian is a symmetry axis	[26]
10	1893	Smith	Parallelogram with 4 equal angles	[54]
11	1950	Welkowitz	Quadrilateral with 4 equal angles	[60]
12	1968	Bottema	Quad with area $K = \frac{1}{4}(a+c)(b+d)$	[9]
13	1970	Schweizer	P with a line of symmetry parallel to a side	[59]
14	1973	Horak	Quad with $AP = BP = CP = DP$	[27]
15	1973	Horak	Quad with $\sin\theta = 2\sqrt{\frac{KP \cdot LP \cdot MP \cdot NP}{AP \cdot BP \cdot CP \cdot DP}}$	[27]
16	1975	Coxford	Quadrilateral with 3 right angles	[13]
17	1981	Garfunkel	Quad with $\sqrt{2}\sum_{cyc}\cos\frac{A+B}{4} = \sum_{cyc}\cot\frac{A}{2}$	[22]
18	1986	Lălescu	$AX^2 + CX^2 = BX^2 + DX^2 \quad \forall X$ in space	[8]
19	1988	Sharygin	Quad with $r_{ABC} = r_{BCD} = r_{CDA} = r_{DAB}$	[53]
20	1992	Seimiya	Quad with $(a+c)^2 + (b+d)^2 = (p+q)^2$	[52]
21	1994	Bech	Sum dist. fr. each vertex to other 3 is const.	[6]
22	1995	Penning	Dist. bw any 2 vert. = dist. bw other 2 vert.	[6]
23	1996	De Villiers	Q w sym. axes thru each pair of opp. sides	[17]
24	2004	Becheanu	Parallelogram with $EF = EG$ and $a \neq b$	[7]
25	?	Jirjahlke	Bisect-diagonal cyclic trapezoid	[42]
26	2008	[RSTJMF]	Quad w $\mathscr{P}_{ABC} = \mathscr{P}_{BCD} = \mathscr{P}_{CDA} = \mathscr{P}_{DAB}$	[50]
27	2008	Ramírez	Cyclic Q w E and F on the circumference	[49]
28	2008	Usiskin	Q w perp. bis. of 2 adj. sides are sym. lines	[57]
29	2009	Al-Sharif	Cyclic quadrilateral with $C = P$	[3]
30	2009	Al-Sharif	Cyclic quadrilateral with $C = G_0$	[3]
31	2010	Byer	Parallelogram with $\triangle ABD \cong \triangle DCA$	[11]
32	2012	Nicula	Parallelogram with $\left(\frac{AQ}{QS}\right)^2 + \left(\frac{BC}{QR}\right)^2 = 1$	[45]
33	2013	Josefsson	Quad w area $K = \frac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}$	[30]
34	2013	Lee	Quad w 2 opp. cong. sides perp. to 3^{rd} side	[44]
35	2014	Solow	Quad with no obtuse angle	[55]
36	2015	Josefsson	Quad with $a^2 + b^2 + c^2 + d^2 = 2pq$	[33]
37	2015	Josefsson	Quad with $\cos\theta = \frac{\lvert a^2-b^2+c^2-d^2\rvert}{a^2+b^2+c^2+d^2}$	[33]
38	2016	Josefsson	Extangential cyclic trapezoid	[34]
39	2017	[Brainly]	Parallelogram w 2 adjacent equal angles	[10]
40	2017	Sedrakyan	Cyclic quad w $\lvert a-c\rvert + \lvert b-d\rvert = 2\lvert p-q\rvert$	[51]
41	2019	Huang	Trapezoid with 2 opposite right angles	[28]
42	2019	[KöMaL]	Extangential cyclic quad w $a + p = c + q$	[43]
43	2019	Saghafian	$\triangle ABC \sim \triangle BCD \sim \triangle CDA \sim \triangle DAB$	[14]
44	2020	Alsina	Isosceles right trapezoid	[4]
45	2020	Alsina	Cyclic right trapezoid	[4]
46	2020	Alsina	Equidiagonal right trapezoid	[4]
47	2020	[Doubtnut]	Cyclic quad w diagonals being diameters	[18]
48	2021	Volchkevich	Parallelogram with $AM = BM$	[58]
49	2021	Volchkevich	Parallelog. w midp. of sides form rhombus	[58]
50	2022	Dalcín	Quad w $AB = CD$, $AD = BC$, $AP = BP$	[15]
51	2022	Dalcín	Quad w $\angle A = \angle C$, $\angle B = \angle D$, $AP = BP$	[15]
52	2022	Dalcín	Quad w $AB = CD$, $AP = BP$, $CP = DP$	[15]
53	2022	Dalcín	Quad w $\angle A = \angle B$, $\angle C = \angle D$, $AP = CP$	[15]
54	2022	Dalcín	Quad w $\angle A = \angle C$, $AP = BP$, $CP = DP$	[15]

55	2022	Dalcín	Quad with $\angle A = \angle D$, $AP = BP = CP$	[15]
56	2022	Dalcín	Quad with $\angle A = \angle B$, $AP = BP = CP$	[15]
57	2022	Dalcín	Quad with $\angle B = \angle D$, $AP = BP = CP$	[15]
58	2022	Dalcín	Quad with $\angle A = \angle B = \angle C$, $AP = BP$	[15]
59	2022	Dalcín	Quad with $\angle A = \angle B = \angle C$, $AB = CD$	[15]
60	2022	Garcia	Cyclic quad with $\sum_{cyc} \tan \frac{A}{2} \tan \frac{B}{2} = 4$	[21]
61	2023	Dalcín	Extangential equidiagonal trapezoid	[16]
62	2023	Dalcín	Extangential equidiagonal semidiagonal Q	[16]
63	2023	Dalcín	Extangential cyclic equidiagonal quad	[16]
64	2023	Dalcín	Extangential cyclic semidiagonal quad	[16]
65	2025	Josefsson	P w 1 bimedian perpendicular to a side	[40]
66	2025	Josefsson	Parallelogram w perpendicular bimedians	[40]
67	2025	Josefsson	Parallelogram with area $K = ab$	[40]
68	2025	Josefsson	Parallelogram with area $K = mn$	[40]
69	2025	Josefsson	Parallelogram with $a^2 + b^2 = pq$	[40]
70	2025	Josefsson	Parallelogram with $m^2 + n^2 = pq$	[40]
71	2025	Josefsson	P with $a^2 + b^2 = e^2$, $e =$ any diagonal	[40]
72	2025	Josefsson	Isosceles trapezoid with a right angle	[40]
73	2025	Josefsson	IT with 2 opposite equal angles	[40]
74	2025	Josefsson	IT with 2 opposite equal semidiagonals	[40]
75	2025	Josefsson	Isosceles trapezoid with equal bases	[40]
76	2025	Josefsson	IT w $K = ab$, $a =$ longest base & $b =$ leg	[40]
77	2025	Josefsson	Right trapezoid with perp. bimedians	[40]
78	2025	Josefsson	Quad where angle bisectors form a square	[40]
79	2025	Josefsson	Cyclic quad with $K = 4\sqrt{r_a r_b r_c r_d}$	[40]
80	2025	Josefsson	Cyclic quad with $K = (r_a + r_c)(r_b + r_d)$	[40]
81	2025	Josefsson	Q w $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} + \sin \frac{D}{2} = 2\sqrt{2}$	[40]
82	2025	Josefsson	Q w $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} + \cos \frac{D}{2} = 2\sqrt{2}$	[40]
83	2025	Josefsson	Q w $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} + \tan \frac{D}{2} = 4$	[40]
84	2025	Josefsson	Quad with $(a - c)^2 + (b - d)^2 = (p - q)^2$	[40]

Table 3. Characterizations of rectangles

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