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A NOTE ON CARNOT'S THEOREM

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Abstract. In this article we establish a connection of Carnot's theorem to Ceva's theorem.

1. Introduction

Carnot's theorem expresses a necessary and sufficient condition [2, p.289] for couples of points $\{(X, X'), (Y, Y'), (Z, Z')\}$ lying respectively on the sides $\{BC, CA, AB\}$ of the triangle ABC, guaranteeing that they are on a conic (see Figure 1). Since the geometric configuration illustrated by figure 1 is

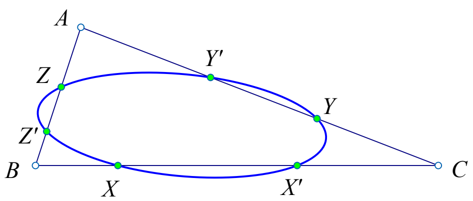

Figure 1. Carnot's condition : $\frac{BX}{XC} \cdot \frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} \cdot \frac{AZ'}{Z'B} = 1$

invariant under projective transformations, the corresponding Carnot condition should be connected with their invariant which is the cross ratio. In this note we show that this is indeed the case, the corresponding relation involving Ceva's theorem.

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2. Equivalent condition

Theorem 2.1. *Carnot's condition is equivalent with the following one. There is a point* P *of the plane not lying on the sides of the triangle with corresponding Cevians* $\{AA', BB', CC'\}$ *and the couples of points* (X, X'), (Y, Y'), *and* (Z, Z') *are harmonic conjugate correspondingly to well defined other couples* $\{(A', A''), (B', B''), (C', C'')\}$ *(see Figure 2)*

$$(XX'; A'A'') = (YY'; B'B'') = (ZZ'; C'C'') = -1 \quad \textit{with}$$
$$(A'A''; BC) = (B'B''; CA) = (C'C''; AB) = -1 \ .$$

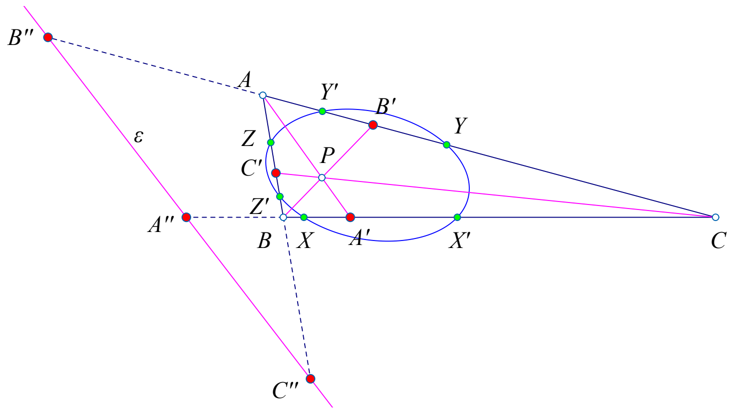

Figure 2. (A', A'') : common harmonics of $\{(X, X'), (B, C)\}$

Proof. The determination of the crucial point P relates to the notion of *"common harmonics"* of a quadruple of points $\{B, C, X, X'\}$ on a line ε. These are two points $\{A', A''\}$ simultaneous harmonic conjugate to (B, C) and (X, X'). The existence of these points results from existence of a *"homographic invonlution"* of the line ε interchanging the members of the couples $\{(B, C), (X, X')\}$ ([1]). In line coordinates and denoting with small letters the coordinates of the corresponding points, the involution is of the form

$$y = f(x) = \frac{ux + v}{wx - u} \quad \text{with} \quad u^2 + vw \neq 0 \ \text{ the constants being determined}$$

$$\text{through: } c = f(b) = \frac{ub + v}{wb - u} \quad \text{and} \quad x' = f(x) = \frac{ux + v}{wx - u} \ .$$

This suffices to completely define f by determining the constants $\{u, v, w\}$ up to a factor:

$$\text{If} \quad u = 0 \ , \ y = \frac{v}{wx} \quad \text{and if} \quad u \neq 0$$

$$\frac{v}{u} = \frac{(b + c)xx' - bc(x + x')}{bc - xx'} \ ,$$

$$\frac{w}{u} = \frac{(b + c) - (x + x')}{bc - xx'} \ .$$

The common harmonics are determined through the fixed points of f which are the two roots $\{a', a''\}$ of equation $x = f(x) \quad \Leftrightarrow \quad wx^2 - 2ux - v = 0.$

Using this calculation we determine the *common harmonics* $\{A', A''\}$ of the couples $\{(B, C), (X, X')\}$ and analogously also the common harmonics

$\{(B', B''), (C', C'')\}$ of the corresponding couples on the other sides. It turns out, that, eventually relabeling the members of the couples, the corresponding points $\{A', B', C'\}$ define Cevians intersecting at a point P, precisely when the Carnot condition is valid.

To see this we notice, that using projective coordinates for the points on the side BC :

$$A'(b', c') \ , \ A''(b'', c'') \ , \ X(x, y) \ , \ X'(x', y') \ ,$$

the fact that (A', A'') are harmonic conjugate to $\{B, C\}$ implies $\frac{b''}{c''} = -\frac{b'}{c'}$. Also points $\{X, X'\}$ being harmonic conjugate to $\{A', A''\}$ translates to

$$-1 \ = \ (XX'; A'A'') = \frac{(x/y) - (b'/c')}{(x'/y') - (b'/c')} : \frac{(x/y) - (b''/c'')}{(x'/y') - (b''/c'')} =$$

$$\frac{(x/y) - (b'/c')}{(x'/y') - (b'/c')} : \frac{(x/y) + (b'/c')}{(x'/y') + (b'/c')} \quad \Leftrightarrow \quad \frac{2(b'^2yy' - c'^2xx')}{(b'y + c'x)(b'y' - c'x')} \ = \ 0$$

$$\Leftrightarrow \quad \frac{c'^2}{b'^2} \ = \ \frac{y}{x} \cdot \frac{y'}{x'} \ = \ \frac{BX}{XC} \cdot \frac{BX'}{X'C} \ .$$

Doing the analogous work for the other sides, we see that Carnot's condition is equivalent to Ceva's condition for the points $\{A', B', C'\}$ to be Cevians for a point P, as claimed, the points $\{A'', B'', C''\}$ being then on the *"trilinear polar"* ε of P (see Figure 2).

Remark 2.1. *We notice that the involutive homography f used in the proof, through which, for given $\{X, X'\}$ we determine the common harmonics $\{A', A''\}$, coincides with the involutive homography of the theorem of Desargues, according to which (*[3, I,p.128]*), "a pencil of conics through four points $\{Y, Y', Z, Z'\}$ in general position defines through the intersections $\{X, X'\}$ of its members with a fixed line an involutive homography on that line. " In particular, the two conics through $\{Y, Y', Z, Z'\}$ which are also tangent to the line BC are precisely tangent at the points $\{A', A''\}$ (see Figure 3). An analogous remark holds true also for the couples of points*

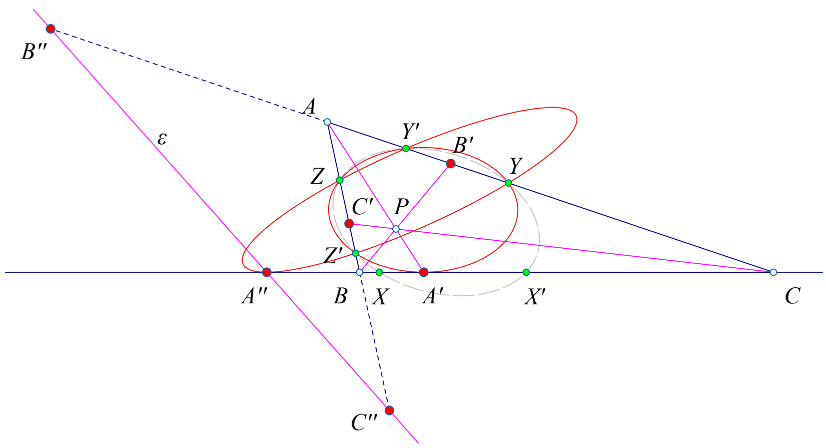

FIGURE 3. The two conics through $\{Y, Y', Z, Z'\}$ which are also tangent to BC

(B', B'') and (C', C'') on the other sides.

Remark 2.2. *The theorem implies that all conics passing through six points, lying by two on the sides of the triangle ABC are obtained in the following way:*

(1) *Take a point P not lying on the sides of the triangle ABC.*
(2) *Define the corresponding Cevians $\{AA', BB', CC'\}$ and the harmonic conjugates $\{A'' = A'(BC), B'' = B'(CA), C'' = C'(AB)\}$ lying on the trilinear polar of P.*
(3) *Take arbitrary $\{(X, X'), (Y, Y'), (Z, Z')\}$ harmonic conjugate couples on the sides $\{(XX'; A'A'') = (YY'; B'B'') = (ZZ'; C'C'') = -1\}$. The points $\{X, X', Y, Y', Z, Z'\}$ are then on a conic.*

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