INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 13 (2024), No. 4, 12 - 25

INVARIANTS OF POLYNOMIALS AND PENCILS OF LINES

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Abstract. Classical invariant theory is mainly concerned with form invariance. This involves finding some function of the coefficients of a homogeneous polynomial which, under a linear transformation of the variables in the polynomial, has its value changed by a factor which only depends upon the transformation. In this article a wider class of invariants is considered. The polynomials now involves only a single variable. If a function of the coefficients can be found which is unchanged by a simple change of origin then it is said to be polynomial invariant. It is shown that there is an intimate connection between such an invariant and pencils of lines associated with an algebraic curve and this is extended to cover a collection of polynomials.

1. Polynomial and Form Invariants

The subject of invariance for polynomials has a long history and is still very much alive. The references [2] and [4] are particularly recommended as providing context and history as well as exposition. [1] also gives credit to the founders of the subject such as Boole, Cayley, Gordan, Salmon and Sylvester while [2] is particularly impressed by the perseverance of these originators: Gordan 'reams of calculations', Cayley 'page after page of extensive explicit tables'. Few would wish to emulate Salmon, who in his own book [7] states that in the *Philosophical Transactions* of 1858 he published a result involving nearly 900 terms. [There is surely scope for a thesis on why these 19th century mathematicians were willing to expend so much time and effort on such calculations. Perhaps it was the mathematical equivalent of finding the North-West passage or the source of the Nile. It certainly was not to further their careers by publishing.]

This article is concerned with the links between algebra and geometry. On the one hand there are invariants of polynomials (both single polynomials and also of several unrelated ones) and on the other hand there are pencils of lines associated with one or more algebraic curves.

Keywords and phrases: Invariants, Polynomials, Pencils of lines
(2020)Mathematics Subject Classification: 51M04, 51M15
Received: 27.03.2024. In revised form: 29.05.2024. Accepted: 04.07.2024.

1.1. **Invariants for a Single Polynomial.** Consider the polynomial function f defined by

$$f(t) = \sum_{i=0}^{n} a_i \binom{n}{i} t^i \tag{1}$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ is a set of $(n+1)$ complex constants and t an indeterminate. If t is replaced by $(t+T)$ then

$$\begin{aligned} f(t+T) &= \sum_{i=0}^{n} a_i \binom{n}{i} (t+T)^i \\ &= \sum_{i=0}^{n} \tilde{a}_i(T) \binom{n}{i} t^i. \end{aligned}$$

Any function $I = I(\mathbf{a})$ which has the property that

$$I(\mathbf{a}) = I(\tilde{\mathbf{a}}) \ \forall \ T \tag{2}$$

is called an *invariant* of the polynomial f. When $n = 2$

$$f(t) = a_0 + 2a_1 t + a_2 t^2 \tag{3}$$

so that

$$\tilde{a}_0 = a_0 + 2a_1 T + a_2 T^2, \ \tilde{a}_1 = a_1 + a_2 T, \ \tilde{a}_2 = a_2$$

and it is easily seen that

$$\tilde{a}_0 \tilde{a}_2 - \tilde{a}_1^2 = a_0 a_2 - a_1^2. \tag{4}$$

Thus, as expected, $I = (a_0 a_2 - a_1^2)$ is an invariant of (3). An even simpler example is furnished by a_2; indeed, for any polynomial of degree n, a_n is an invariant.

It has been found (historically, algebraically and geometrically) extremely useful to homogenize (1) to give

$$F(x, y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i y^{n-i}. \tag{5}$$

Nowadays such an expression is referred to as a *form* although this writer regrets the demise of the term *quantic* used by Cayley and Salmon. A wider class of change of variable is now available while still preserving linearity. Specifically, x and y may be replaced by X and Y where

$$x = \alpha X + \beta Y \quad \text{and} \quad y = \gamma X + \delta Y$$

(the Greek letters are constants) to give the new homogeneous polynomial

$$\tilde{F}(X, Y) = \sum_{i=0}^{n} \tilde{a}_i(\alpha, \beta, \gamma, \delta) \binom{n}{i} X^i Y^{n-i}.$$

If a function J of $\mathbf{a}$ is found such that

$$J(\mathbf{a}) = (\alpha\delta - \beta\gamma)^k J(\tilde{\mathbf{a}}) \ \forall \ \alpha, \beta, \gamma, \delta \tag{6}$$

then J is called an invariant of the form F with *weight* k. If it should happen that k is zero then I is called an *absolute invariant.* A simple example of an invariant is again furnished by $n = 2$. Now

$$F(x, y) = a_0 y^2 + 2a_1 yx + a_2 x^2 \tag{7}$$

and

$$\tilde{a}_0\tilde{a}_2 - \tilde{a}_1^2 = (\alpha\delta - \beta\gamma)^2(a_0a_2 - a_1^2) \tag{8}$$

and so $J = (a_0a_2 - a_1^2)$ is an invariant with weight two for the form (7). However, a_2 is not an invariant of the form (7). It is desirable to distinguish between *polynomial invariants*, e.g. a_2 for (3) and *form invariants* e.g. J for (7). Form invariance implies polynomial invariance but not conversely. More significantly, with $n = 3$

$$f(t) = a_0 + 3a_1t + 3a_2t^2 + a_3t^3 \tag{9}$$

and it may be verified that

$$I'(\mathbf{a}) = a_0a_3^2 - 3a_1a_2a_3 + 2a_2^3 \tag{10}$$

is an invariant for this polynomial but is not invariant for the associated form

$$F(x, y) = a_0y^3 + 3a_1y^2x + 3a_2yx^2 + a_3x^3.$$

However this form does have an invariant J of weight six given by

$$J = a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2. \tag{11}$$

An invariant can often be related to some property of the roots $\theta_1, \theta_2, \ldots$ of the original polynomial. For example, referring once again to the quadratic case (3),

$$4I = a_2^2(\theta_1 - \theta_2)^2$$

and for the cubic (9)

$$\begin{aligned} 27I' &= a_3^3(\theta_2 + \theta_3 - 2\theta_1)(\theta_3 + \theta_1 - 2\theta_2)(\theta_1 + \theta_2 - 2\theta_3) \\ \text{and}\ \ 27J &= -a_3^4(\theta_2 - \theta_3)^2(\theta_3 - \theta_1)^2(\theta_1 - \theta_2)^2. \end{aligned}$$

It is not difficult to show that a form invariant of (5) with weight n is

$$I = \sum_{i=0}^{n} a_ia_{n-i}\binom{n}{i}(-1)^i. \tag{12}$$

For n odd this produces nothing and with $n = 2$ it only gives the very familiar result for I. However, when $n = 4$ it gives the form invariant of weight four

$$S = (a_0a_4 - 4a_1a_3 + 3a_2^2)/12 \tag{13}$$

(the factor of 12 is customary, see e.g. [8] or [5]). In terms of the roots $\theta_1, \theta_2, \theta_3, \theta_4$ of the quartic

$$144S = -a_4^2(BC + CA + AB)$$

where

$$A = (\theta_1 - \theta_2)(\theta_3 - \theta_4),\ B = (\theta_2 - \theta_4)(\theta_3 - \theta_1),\ C = (\theta_1 - \theta_4)(\theta_2 - \theta_3) = -A - B.$$

However, the quartic form has another invariant (this one with weight six) given by

$$\begin{aligned} T &= a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3 \\ &= \begin{vmatrix} a_4 & a_3 & a_2 \\ a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 \end{vmatrix} \end{aligned} \tag{14}$$

and now

$$432T = a_4^3(B - C)(C - A)(A - B).$$

Other invariants may be constructed from S and T, specifically:

- S^3/T^2 has weight zero and so is an absolute invariant. There exists a sextic equation whose coefficients depend only upon this absolute invariant which has as its six roots all the possible values of
$$\frac{(\theta_1 - \theta_2)(\theta_3 - \theta_4)}{(\theta_1 - \theta_4)(\theta_2 - \theta_3)}.$$
- $(4S)^3 - T^2 = a_4^6 A^2 B^2 C^2/6912$ and so the condition for a quartic equation to have a repeated root is $(4S)^3 = T^2$.

These results will be very familiar to some readers and mysterious to others. It is hoped that when the geometrical significance of them is explained that all will have gained something. Before doing that, it is pointed out that the notion of invariance is not confined to a single polynomial.

1.2. **Invariants for Two or More Forms.**

1.2.1. *One of the Forms is Linear.* The simplest case involving two forms is when one is linear. Let

$$F(x, y) = \sum_{i=0}^{n} a_i \binom{n}{i} y^{n-i} x^i \quad \text{and} \quad G(x, y) = b_0 y + b_1 x. \tag{15}$$

If it is required that these two expressions shall have a common root then clearly $F(-b_0, b_1)$ is zero or

$$I(\mathbf{a}, \mathbf{b}) \equiv \sum_{i=0}^{n} a_i \binom{n}{i} b_0^i b_1^{n-i} (-1)^i = 0. \tag{16}$$

The common-root property will be preserved under a linear transformation and so I is a form invariant with weight n for the pair (15). Also, if $\theta_1, \theta_2, \ldots$ are the roots of F (that is, the values of x/y which make F zero) and ϕ is the root of G then

$$I = b_1^n a_n \prod_{i=1}^{n} (\phi - \theta_i).$$

1.2.2. *Two Quadratic Forms.* The condition that the two forms

$$F(x, y) = a_0 y^2 + 2a_1 yx + a_2 x^2 \quad \text{and} \quad G(x, y) = b_0 y^2 + 2b_1 yx + b_2 x^2 \tag{17}$$

shall have a common root is

$$(a_0 b_2 - 2a_1 b_1 + a_2 b_0)^2 - 4(a_0 a_2 - a_1^2)(b_0 b_2 - b_1^2) = 0 \tag{18}$$

and setting I_1 equal to this expression gives an invariant of weight four. Furthermore, letting θ_1, θ_2 be the roots of F and ϕ_1, ϕ_2 be the roots of G gives

$$I_1 = a_2^2 b_2^2 (\theta_1 - \phi_1)(\theta_1 - \phi_2)(\theta_2 - \phi_1)(\theta_2 - \phi_2).$$

Also each of

$$I_2 = a_2 b_0 - 2a_1 b_1 + a_0 b_2, \ a_0 a_2 - a_1^2 \quad \text{and} \quad b_0 b_2 - b_1^2 \tag{19}$$

is a form invariant of weight two.

1.2.3. *Three Quadratic Forms.* The three forms

$$F(x,y) = a_0y^2 + 2a_1yx + a_2x^2,\\ G(x,y) = b_0y^2 + 2b_1yx + b_2x^2,\\ H(x,y) = c_0y^2 + 2c_1yx + c_2x^2 \tag{20}$$

have the form invariant with weight three given by

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}. \tag{21}$$

This will recur in section 7.1.

1.2.4. *A Quadratic and a Cubic.* In [7] it is stated that the condition for the quadratic and cubic polynomials

$$Ax^2 + Bxy + Cy^2 \text{ and } ax^3 + bx^2y + cxy^2 + dy^3 \tag{22}$$

shall have a common root is

$$a^2C^3 - abBC^2 + acC(B^2 - 2AC) - ad(B^3 - 3ABC) + b^2AC^2\\ - bcABC + bdA(B^2 - 2AC) + c^2A^2C - cdBA^2 + d^2A^3. \tag{23}$$

Although not stated by Salmon, this expression is form invariant with weight six. The relevance of this result will be apparent later (section 7.6).

2. Pencils of Lines

Only figures lying in a plane will be considered and homogeneous coordinates $\mathbf{r} = (r_1, r_2, r_3)$ will be employed. Any homogeneous function of degree n, say $P(\mathbf{r})$, will correspond to a curve in the plane. Clearly when $n = 1$ the curve is a straight line, $n = 2$ gives a conic, $n = 3$ a cubic curve, etc. In general, if a curve P is associated with a homogeneous polynomial of degree n then it will be referred to as an n-curve. If $X(\boldsymbol{\alpha})$ is any point in the plane, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_, \alpha_3)$, then the *polar curve* of X in the n-curve P is the $(n-1)$-curve P_X defined by

$$nP_X \equiv \alpha_1 \frac{\partial P}{\partial r_1} + \alpha_2 \frac{\partial P}{\partial r_2} + \alpha_3 \frac{\partial P}{\partial r_3} = 0.$$

Two crucial observations need to be made (see [9]):

- If P is an n-curve and Q an m-curve then

$$(m+n)(PQ)_X \equiv nP_XQ + mPQ_X. \tag{24}$$

- If P is an n-curve and $P_X \equiv 0 \;\forall\, \mathbf{r}$ then P consists of n straight lines with common point X.

In what follows it will be necessary to refer not only to P_X and P_{XX} but to P_{XXX} etc. which is ungainly. Instead, with P any n-curve the following notation will be used:

$$p_0 \equiv P(\mathbf{r}),\ p_1 \equiv P_X,\ p_2 \equiv P_{XX}, \ldots,\ p_n \equiv P(\boldsymbol{\alpha}). \tag{25}$$

Thus p_k is homogeneous in $\mathbf{r}$ with degree $(n-k)$ and homogeneous in $\boldsymbol{\alpha}$ with degree k. Also, considering p_j as a function of $\boldsymbol{\alpha}$ and $\mathbf{r}$, it is seen that

$$p_j(\boldsymbol{\alpha}, \mathbf{r}) \equiv p_{n-j}(\mathbf{r}, \boldsymbol{\alpha}).$$

3. Abbreviated Notation

Some new notation is now introduced which will allow the results to be given in a compact form. The idea behind it is best described by an example. Suppose that P and Q are given conics and define $\theta(\boldsymbol{\alpha}, \mathbf{r})$ by

$$\theta(\boldsymbol{\alpha}, \mathbf{r}) \equiv PQ_{XX} - 2P_X Q_X + P_{XX}Q.$$

Using (25), this is written as

$$\theta = p_0 q_2 - 2p_1 q_1 + p_2 q_0.$$

Now make the formal identification

$$p_k \equiv \not{p}^k \quad \text{and} \quad q_k \equiv \not{q}^k$$

so as to give

$$\tilde{\theta} = (\not{p} - \not{q})^2.$$

More generally, let each of $P, Q, \ldots$ be homogeneous in $\mathbf{r}$ with degrees $n_P, n_Q, \ldots$ and let θ be a summation of terms $A_{k\,l\ldots} p_k q_l \ldots$ where the A's are constants. Thus $\tilde{\theta}$ will become a summation of terms involving $\not{p}^k \not{q}^l, \ldots$

Clearly some care is required in order to retrieve the original form of θ from such an expression, especially if $\tilde{\theta}$ has been factorised (which will usually be the case). The retrieval is achieved by the following steps.

- Expand $\tilde{\theta}$
- Replace any $\not{p}^k$ by p_k.
- If there is a term without any p's then multiply that term by p_0.
- Continue in this manner for each of the powers of $\not{q}, \ldots$ in turn.

The restriction that θ is linear in $P, Q, \ldots$ may appear to be very severe. But in principle any number of curves may be included and when a suitable θ has been found (i.e. one in which $\theta_X \equiv 0$) and the retrieval process just described has been performed, then one may take any subset of $P, Q, \ldots$ to be equal (provided that the members have the same degree). In particular, with $P = Q = \ldots$ a result involving only P will be obtained. Although it is not necessary for all of $P, Q, \ldots$ to have the same degree this will commonly be used subsequently to restrict the possibilities. The most notable exception is for a collection of conics and cubics (section 7.6).

4. The Relationship between Invariants and Pencils

Let $X(\boldsymbol{\alpha})$ be any point, P be any curve of degree n_P and, with $\mathbf{p} = (p_0, p_1, \ldots, p_{n_P})$ as in (25), define the polynomial f by

$$f(t) = \sum_{i=0}^{n_P} p_i \binom{n_P}{i} t^i.$$

Then

$$f(t+T) = \sum_{i=0}^{n_P} \tilde{p}_i \binom{n_P}{i} t^i$$

where (suppressing the dependence of p_i and $\tilde{p}_i$ on $\boldsymbol{\alpha}$)

$$\tilde{p}_i = \tilde{p}_i(\mathbf{r}, T) = \sum_{j=i}^{n_P} p_j \binom{n_P - i}{j - i} T^{j-i}.$$

But

$$P(\mathbf{r}+t\boldsymbol{\alpha}) = \sum_{i=0}^{n_P} p_i \binom{n_P}{i} t^i \text{ and } p_i(\mathbf{r}+t\boldsymbol{\alpha}) = \sum_{j=i}^{n_P} p_j \binom{n_P - i}{j-i} t^{j-i}.$$

Hence

$$\tilde{p}_i(\mathbf{r}, T) = p_i(\mathbf{r} + T\boldsymbol{\alpha}).$$

With $P, Q, \ldots$ being a collection of algebraic curves and $I(\mathbf{p}, \mathbf{q}, \dots)$ being some polynomial expression in $(p_0, \ldots, p_{n_P}, q_0, \ldots, q_{n_Q}, \ldots)$, it follows that

$$I(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \ldots) = I(\mathbf{p}(\mathbf{r}+T\boldsymbol{\alpha}), \mathbf{q}(\mathbf{r}+T\boldsymbol{\alpha}), \ldots).$$

Now write $I(\mathbf{p}, \mathbf{q}, \dots)$ as $\theta(\boldsymbol{\alpha}, \mathbf{r})$. If θ is a pencil of lines with vertex $X(\boldsymbol{\alpha})$ then

$$\theta(\boldsymbol{\alpha} + T\boldsymbol{\alpha}, \mathbf{r}) = \theta(\boldsymbol{\alpha}, \mathbf{r}) \tag{26}$$

and so I is an invariant for the collection of polynomials

$$\sum_{i=0}^{n_P} p_i \binom{n_P}{i} t^i, \; \sum_{i=0}^{n_Q} q_i \binom{n_Q}{i} t^i \ldots \tag{27}$$

Also if I is an invariant for these polynomials, then (26) will hold and θ will be a pencil of lines.

It is to be noticed that form invariance is not required to establish this result.

5. Pencils from Two Curves

5.1. One of the Curves is a Straight Line. Let P be an n-curve and Q the straight line $\boldsymbol{\beta} \cdot \mathbf{r} = 0$ so that

$$q_0 = Q = \boldsymbol{\beta} \cdot \mathbf{r} \text{ and } q_1 = Q_X = \boldsymbol{\beta} \cdot \boldsymbol{\alpha}.$$

With $X(\boldsymbol{\alpha})$ as any point, define the n-curve θ by

$$\theta(\boldsymbol{\alpha}, \mathbf{r}) \equiv \sum_{i=0}^{n} p_i \binom{n}{i} q_0^i q_1^{n-i} (-1)^i = 0. \tag{28}$$

It is readily confirmed that θ_X is identically zero and so θ consists of a pencil of n lines with common point X. Furthermore, at a point at which P and Q are both zero so is θ and so θ gives the set of lines joining X to the n common points of P and Q. The similarity of (28) and (16) will be evident.

5.2. Two n-Curves. From the two n-curves P and Q construct the new n-curve $\theta(\boldsymbol{\alpha}, \mathbf{r}) = 0$ where

$$\theta(\boldsymbol{\alpha}, \mathbf{r}) \equiv \sum_{i=0}^{n} p_i q_{n-i} \binom{n}{i} (-1)^i \;\Rightarrow\; \tilde{\theta} = (\not{p} - \not{q})^n$$

which, since $\theta_X \equiv 0$, gives n lines with common poin $X(\boldsymbol{\alpha})$. Any line of this pencil will meet P in n points, say $A_1, A_2, \ldots, A_n$, and also meet Q at $B_1, B_2, \ldots, B_n$. It is shown in [9] that

$$\sum_{\pi} \prod_{i=1}^{n} A_i B_{\pi(i)} = 0$$

where the summation is over all permutations of $\{1, 2, \ldots, n\}$.

The simplest cases are:

- $\mathbf{n = 1}$. With the lines P and Q given by

$$P \equiv \mathbf{a} \cdot \mathbf{r} = 0 \quad \text{and} \quad Q \equiv \mathbf{b} \cdot \mathbf{r} = 0$$

  θ is given by

$$\theta(\boldsymbol{\alpha}, \mathbf{r}) = (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \boldsymbol{\alpha}) - (\mathbf{a} \cdot \boldsymbol{\alpha})(\mathbf{b} \cdot \mathbf{r}) = \mathbf{r} \cdot [\boldsymbol{\alpha} \times (\mathbf{a} \times \mathbf{b})].$$

  [The use of the notation for scalar and vector products is very convenient but of course only coordinates of points are involved.] If the common point of P and Q is Y then θ is the line XY.
- $\mathbf{n = 2}$. Here the lines forming θ are the two tangents from X to the harmonic envelope of the conics P and Q, see [8] or [6] for more details.
- $\mathbf{P = Q}$ This gives, in essence, the result (12).

6. Multiple Curves

The method to be used is best explained by an example. Let P, Q, R be cubics and suppose what is sought is a function θ which is linear in the coefficients of each of P, Q, R and has degree three in $\mathbf{r}$. Crucially, it is also required that θ_X shall be identically zero so that θ will consist of three lines. Now a typical term in θ involves $p_k q_l r_m$ where $k + l + m = 6$ (the degree of $\mathbf{r}$ in p_k is $3 - k$). The number of such terms is ten and so the expression for θ involves ten unknowns.

Now

$$(p_k q_l r_m)_X = (3 - k) p_{k+1} q_l r_m + (3 - l) p_k q_{l+1} r_m + (3 - m) p_k q_l r_{m+1}$$

or, in the abbreviated notation,

$$\mathpzc{p}^k \mathpzc{q}^l \mathpzc{r}^m \to [(3 - k)\mathpzc{p} + (3 - l)\mathpzc{q} + (3 - m)\mathpzc{r}]\mathpzc{p}^k \mathpzc{q}^l \mathpzc{r}^m.$$

The condition $\theta_X \equiv 0$ produces six constraints and so leaves four linearly independent solutions of which only two are permutationally distinct; they are

$$\mathpzc{r}^3(\mathpzc{p} - \mathpzc{q})^3 \text{ and } \mathpzc{q}\mathpzc{r}^2(\mathpzc{p} - \mathpzc{q})^2(\mathpzc{p} - \mathpzc{r}).$$

When the procedure described in Section 3 is carried out the results are

$$R_{XXX}(PQ_{XXX} - 3P_X Q_{XX} + 3P_{XX} Q_X - P_{XXX} Q)$$

and

$$\begin{aligned} &PQ_{XXX}R_{XXX} - 2P_X Q_{XX} R_{XXX} - P_X Q_{XXX} R_{XX} \\ &\qquad + P_{XX} Q_X R_{XXX} + 2P_{XX} Q_{XX} R_{XX} - P_{XXX} Q_X R_{XX}. \end{aligned}$$

When P, Q, R are set equal, the first solution disappears but the second gives

$$P(P_{XXX})^2 - 3P_X P_{XX} P_{XXX} + 2(P_{XX})^3$$

which is equivalent to I' given in equation (10).

In general, one chooses a collection of N curves $P, Q, R \ldots$ and K, the number of lines in the hoped-for pencil θ. Then a typical term in θ will be $p_k q_l r_m \ldots$ where there are N factors and the sum of the indices is the sum of the degrees in the collection less K. Unsurprisingly, the number of solutions varies enormously. For the case just considered there were four solutions and they could be easily split into two permutationally distinct sets. But with five quintics and $K = 7$ the number of terms in θ is initially 305, the condition $\theta_X \equiv 0$ imposes 205 constraints which leads to exactly 100 solutions.

7. A Selection of the Results

Even when a solution θ (i.e. $\theta_X \equiv 0$) is found it may not be of interest because:

- $\tilde{\theta}$ is found to be separable, e.g. $\tilde{\theta}(\mathscr{p}, \mathscr{q}, \mathscr{r}, \mathscr{s}) = \phi(\mathscr{p}, \mathscr{q})\psi(\mathscr{r}, \mathscr{s})$, so that θ splits into two separate pencils,
- θ does not contain p_0. In this case θ is a solution for a curve of lower degree.

A solution for $\tilde{\theta}$ will be referred to as a $\mathscr{p}$-solution. Usually only the total number of solutions is quoted so that this includes permutationally equivalent solutions (when a distinction is made then *distinct* will mean permutationally distinct). Two permutationally distinct solutions may give the same solution when the curves involved are set equal.

7.1. **Conics.**

- **Two Conics - Two Lines.** There is one $\mathscr{p}$-solution

$$\tilde{\theta} = (\mathscr{p} - \mathscr{q})^2 \;\Rightarrow\; \theta = p_2 q_0 - 2p_1 q_1 + p_0 q_2 = P_{XX}Q - 2P_X Q_X + PQ_{XX}.$$

This is the invariant I_2 of (19). The envelope of these lines is the classic harmonic envelope of the two conics and setting $P = Q$ gives the pair of tangents from X to the conic.

- **Three Conics - Three Lines.** The only $\mathscr{p}$-solution is

$$\tilde{\theta} = (\mathscr{q} - \mathscr{r})(\mathscr{r} - \mathscr{p})(\mathscr{p} - \mathscr{q})$$

or

$$\tilde{\theta} = \begin{vmatrix} 1 & \mathscr{p} & \mathscr{p}^2 \\ 1 & \mathscr{q} & \mathscr{q}^2 \\ 1 & \mathscr{r} & \mathscr{r}^2 \end{vmatrix} \Rightarrow \theta = \begin{vmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \\ r_0 & r_1 & r_2 \end{vmatrix} \tag{29}$$

which is essentially (21). Because the process of forming θ_X from θ is so akin to differentiation, it is apparent that, for the θ just displayed, θ_X is identically zero and so (21) gives an invariant for three quadratics (involving 9 coefficients) whereas (29) gives an invariant for three conics (involving 18 coefficients).

This result can be immediately generalized to give an invariant for a collection of $(n + 1)$ polynomials of degree n. Alternatively it may be viewed as giving a pencil of $n(n + 1)/2$ lines from $(n + 1)$ n-curves.

- **Four Conics.** There are three $\mathscr{p}$-solutions but only two of these are distinct. They are:

$$(\mathscr{p} - \mathscr{q})^2(\mathscr{r} - \mathscr{s})^2 \quad \text{and} \quad (\mathscr{p} - \mathscr{q})(\mathscr{q} - \mathscr{r})(\mathscr{r} - \mathscr{s})(\mathscr{s} - \mathscr{p}).$$

The first of these is separable and so not, by itself, interesting but a combination of these solutions is

$$\tilde{\theta} = (\mathscr{p} - \mathscr{q})(\mathscr{r} - \mathscr{s})[2(\mathscr{p}\mathscr{q} + \mathscr{r}\mathscr{s}) - (\mathscr{p} + \mathscr{q})(\mathscr{r} + \mathscr{s})]$$

which gives

$$\theta = (p_0q_2 - p_2q_0)(r_0s_2 - r_2s_0) - 2(p_0q_1 - p_1q_0)(r_1s_2 - r_2s_1) \\ + 2(p_1q_2 - p_2q_1)(r_1s_0 - r_0s_1)$$

When $S = P$ and $R = Q$ this becomes

$$\theta = (p_0q_2 - 2p_1q_1 + p_2q_0)^2 - 4(p_0p_2 - p_1^2)(q_0q_2 - q_1^2)$$

which is the pencil of four lines joining the point X to the common points of the conics P and Q. This is essentially the invariant I_1 in (18).

7.2. **Cubic Curves.**

- **Two Cubics - Three Lines.** There is a unique $\mathscr{p}$-solution

$$\tilde{\theta} = (\mathscr{p} - \mathscr{q})^3 \ \Rightarrow\ \theta = p_0q_3 - 3p_1q_2 + 3p_2q_1 - p_3q_0.$$

- **Three Cubics - Three Lines.** There are four $\mathscr{p}$-solutions of which two are distinct, viz.

$$(\mathscr{p} - \mathscr{q})^3\mathscr{r}^3 \quad \text{and} \quad (\mathscr{p} - \mathscr{q})^2\mathscr{q}(p - \mathscr{r})\mathscr{r}^2.$$

When the curves are all the same there is a unique solution

$$\theta = p_op_3^2 - 3p_1p_2p_3 + 2p_2^3$$

which corresponds to I' in (10).

- **Four Cubics**
  - **Three Lines.** There are ten $\mathscr{p}$-solutions of which two are distinct; they are

$$(\mathscr{p} - \mathscr{q})^3\mathscr{r}^3\mathscr{s}^3 \quad \text{and} \quad (\mathscr{p} - \mathscr{q})(\mathscr{p} - \mathscr{r})(\mathscr{p} - \mathscr{s})\mathscr{q}^2\mathscr{r}^2\mathscr{s}^2.$$

  The first is separable (includes the three-line solution for two cubics) but the second gives

$$\theta = p_0q_3r_3s_3 - p_1(q_2r_3s_3 + q_3r_2s_3 + q_3r_3s_2) \\ + p_2(q_3r_2s_2 + q_2r_3s_2 + q_2r_2s_3) - p_3q_2r_2s_2.$$

  If the curves coincide this reduces again to the three-line solution.
  - **Six Lines.** Here there are four solutions which may be written as

$$(\mathscr{p} - \mathscr{q})^2(\mathscr{r} - \mathscr{s})^2(\mathscr{p} - \mathscr{r})(\mathscr{q} - \mathscr{s}), \\ (\mathscr{p} - \mathscr{q})^2(\mathscr{r} - \mathscr{s})^2(\mathscr{p} - \mathscr{s})(\mathscr{q} - \mathscr{r}) \\ (\mathscr{p} - \mathscr{q})^3(\mathscr{r} - \mathscr{s})^3 \quad \text{and} \quad (\mathscr{p} - \mathscr{q})(\mathscr{p} - \mathscr{r})(\mathscr{p} - \mathscr{s})(\mathscr{q} - \mathscr{r})(\mathscr{q} - \mathscr{s})(\mathscr{r} - \mathscr{s}).$$

  The last solution may also be expressed in terms of determinants as in (29). When the curves are all the same, the only solution is

$$\theta = p_0^2p_3^2 - 6p_0p_1p_2p_3 + 4p_0p_2^3 + 4p_1^3p_3 - 3p_1^2p_2^2$$

  which is J in (11). This pencil of six lines gives all of the tangents from X to the cubic curve P.

7.3. **Quartic Curves.**

- **Two Quartics - Two Lines.** There is a unique $\mathscr{p}$-solution, $\tilde{\theta} = (\mathscr{p} - \mathscr{q})^4$, which when $P = Q$ gives

$$\theta = p_0p_4 - 4p_1p_3 + 3p_2^2$$

which corresponds to S in (13).
- **Three Quartics.**
  - **Four Lines.** There are five $\mathscr{p}$-solutions of which two are distinct; they are

$$(\mathscr{p} - \mathscr{q})^4\mathscr{r}^4 \text{ and } \mathscr{q}^2\mathscr{r}^2(\mathscr{p} - \mathscr{r})^2(\mathscr{p} - \mathscr{q})^2.$$

  When $P = Q = R$ both give

$$\theta = p_4(p_0p_4 - 4p_1p_3 + 3p_2^2)$$

  and so is the S solution again.
  - **Five Lines.** There are three $\mathscr{p}$-solutions of which only one is distinct, viz.

$$\tilde{\theta} = \mathscr{r}^2(\mathscr{p} - \mathscr{q})^3(\mathscr{p} - \mathscr{r})(\mathscr{q} - \mathscr{r})$$

  and this evaporates when the quartics coincide.
  - **Six Lines.** There is a unique $\mathscr{p}$-solution

$$\tilde{\theta} = (\mathscr{q} - \mathscr{r})^2(\mathscr{r} - \mathscr{p})^2(\mathscr{p} - \mathscr{q})^2$$

  which, when the quartics coincide, gives

$$\theta = \begin{vmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{vmatrix} = p_0p_2p_4 - p_0p_3^2 - p_1^2p_4 + 2p_1p_2p_3 - p_2^3$$

  which corresponds to T in (14).
- **Six Quartics - Twelve Lines.** There are 65 $\mathscr{p}$-solutions, which when all the curves are the same, give just two solutions for θ which may be written as S^3 and T^2. The combination $(4S)^3 - T^2$ gives the discriminant given by [7]. It was noted in section 1.1 that $(4S)^3 = T^2$ is the condition that a quartic polynomial should have a repeated root.

  The twelve lines represented by $(4S)^3 = T^2$ are the twelve tangents from X to the quartic.

7.4. **Quintic Curves.**

- **Three Quintics - Five Lines.** There are six $\mathscr{p}$-solutions, including

$$\tilde{\theta} = (\mathscr{p} - \mathscr{r})^2\mathscr{r}^3(\mathscr{p} - \mathscr{q})^3\mathscr{q}^2$$

  which gives, when all the curves are the same,

(30) $$\theta = p_0p_5^2 - 5p_1p_4p_5 + 2p_2p_3p_5 + 8p_2p_4^2 - 6p_3^2p_4.$$

  This is only polynomial invariant for a quintic polynomial.
- **Four Quintics.**

- **Seven Lines.** There are eight $\mathscr{p}$-solutions, including

$$\tilde{\theta} = (\mathscr{p} - \mathscr{q})^3(\mathscr{p} - \mathscr{r})^2(\mathscr{q} - \mathscr{r})(\mathscr{q} - \mathscr{s})\mathscr{r}^2\mathscr{s}^4$$

which gives, when all the curves are the same,

$$\theta = p_0p_3p_5^2 - p_0p_4^2p_5 - p_1p_2p_5^2 - 2p_1p_3p_4p_5 + 3p_1p_4^3 + 4p_2^2p_4p_5 - p_2p_3^2p_5 - 6p_2p_3p_4^2 + 3p_3^3p_4.$$

This is only polynomial invariant for a quintic polynomial.

- **Eight Lines.** There are 21 $\mathscr{p}$-solutions of two are

$$(\mathscr{p} - \mathscr{q})^4(\mathscr{p} - \mathscr{r})(\mathscr{q} - \mathscr{s})\mathscr{r}^2\mathscr{s}^2(\mathscr{r} - \mathscr{s})^2$$

and

$$(\mathscr{p} - \mathscr{q})^3(\mathscr{p} - \mathscr{r})^2(\mathscr{q} - \mathscr{s})\mathscr{r}\mathscr{s}^3(\mathscr{r} - \mathscr{s})(\mathscr{q} - \mathscr{r}).$$

When curves are the same, there are two solutions which may be written as

$$(p_1p_5 - 4p_2p_4 + 3p_3^2)^2$$

and

$$p_0(-p_2p_5^2 + 3p_3p_4p_5 - 2p_4^3) + 5p_1(p_2p_4p_5 - 2p_3^2p_5 + p_3p_4^2) + 4p_2^2p_3p_5 - 14p_2^2p_4^2 + 16p_2p_3^2p_4 - 6p_3^4. \tag{31}$$

The first of these is a solution for quartics (because p_0 is absent) and the second is not unique (because any multiple of the first one may be added to it). Neither is form invariant.

- **Ten Lines.** There are three $\mathscr{p}$-solutions including

$$\tilde{\theta} = (\mathscr{p} - \mathscr{q})^4(\mathscr{r} - \mathscr{s})^4(\mathscr{q} - \mathscr{s})(\mathscr{p} - \mathscr{r})$$

which gives, when all the curves are the same,

$$\theta = p_0^2p_5^2 - 10p_0p_1p_4p_5 + 4p_0p_2p_3p_5 - 76p_1p_2p_3p_4 + 16p_0p_2p_4^2 + 16p_1^2p_3p_5 - 12p_0p_3^2p_4 - 12p_1p_2^2p_5 + 48p_1p_3^3 + 48p_2^3p_4 + 9p_1^2p_4^2 - 32p_2^2p_3^2.$$

This gives one of the four form invariants of a quintic polynomial. They are all given explicitly in [3] where θ is given in a more compact form (but the product of the first two brackets lacks a factor of 4).

• **Five Quintics.** When the five curves are the same, all the solutions are combinations of ones previously found. However when they are distinct there is a plethora of new solutions which are too numerous to investigate properly.

- **Eight Lines.** There are 115 $\mathscr{p}$-solutions one of which is

$$\mathscr{p}\mathscr{q}\mathscr{r}^4\mathscr{s}^3\mathscr{t}^4(\mathscr{p} - \mathscr{q})^3(\mathscr{p} - \mathscr{r})(\mathscr{q} - \mathscr{s})(\mathscr{t} - \mathscr{s}).$$

When the curves are the same there are three solutions;

$$(p_3p_5 - p_4^2)(p_1p_3p_5 - p_1p_4^2 + 2p_2p_3p_4 - p_2^2p_5 - p_3^3),$$
$$p_5(p_1p_5 - 4p_2p_4 + 3p_3^2)^2$$

and a third which is the eight-line solution of (31) multiplied by p_5.

7.5. **Sextic Curves.**

- **Two Sextics - Six Lines.** There is a unique $\mathscr{p}$-solution given by $\tilde{\theta} = (\mathscr{p} - \mathscr{q})^6$ which, when all the curves are the same, becomes

$$\theta = p_0p_6 - 6p_1p_5 + 15p_2p_4 - 10p_3^2.$$

- **Four Sextics - Twelve Lines.** There are 7 $\mathscr{p}$-solutions which when all the curves are the same give two solutions for θ, viz.

$$(p_0p_6 - 6p_1p_5 + 15p_2p_4 - 10p_3^2)^2 \quad \text{and} \quad \begin{vmatrix} p_0 & p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_5 \\ p_3 & p_4 & p_5 & p_6 \end{vmatrix}.$$

7.6. **Conics and Cubic Curves.**

- **One Conic and Two Cubic Curves - Four Lines.**
  When it is required to associate four lines with a conic (P) and two cubics (S and T), the only $\mathscr{p}$-solution is

$$\tilde{\theta} = (\mathscr{s} - \mathscr{t})^2(\mathscr{p} - \mathscr{s})(\mathscr{p} - \mathscr{t})$$

which gives

$$\theta = p_0(s_1t_3-2s_2t_2+s_3t_1)+p_1(s_1t_2+s_2t_1-s_0t_3-s_3t_0)+p_2(s_0t_2-2s_1t_1+s_2t_0).$$

If the two cubics coincide ($T = S$) this becomes

(32) $$p_0(s_1s_3 - s_2^2) + p_1(s_1s_2 - s_0s_3) + p_2(s_0s_2 - s_1^2)$$

which gives a form invariant of weight four for the polynomials

$$p_0y^2 + 2p_1yx + p_2x^2 \quad \text{and} \quad s_0y^3 + 3s_1y^2x + 3s_2yx^2 + s_3x^3.$$

The pencil of lines defined by (32) has an envelope which is a 12-curve.

- **Three Conics and Two Cubic Curves - Six Lines.** Here P, Q, R are conics and S, T are cubic curves. There are 7 $\mathscr{p}$-solutions of which two are distinct;

$$(\mathscr{r} - \mathscr{s})(\mathscr{q} - \mathscr{s})(\mathscr{p} - \mathscr{t})(\mathscr{r} - \mathscr{t})(\mathscr{q} - \mathscr{t})(\mathscr{p} - \mathscr{s})$$

and

$$(\mathscr{r} - \mathscr{s})(\mathscr{q} - \mathscr{s})(\mathscr{p} - \mathscr{t})(\mathscr{r} - \mathscr{t})(\mathscr{s} - \mathscr{t})(\mathscr{p} - \mathscr{q}).$$

When the three conics are the same and the two cubic curves are the same the result is two independent solutions. One is equivalent to (23) (but the coefficients are different because the usual binomial coefficients are absent in (22)) and the other separates into $(p_0p_2-p_1^2)$ and the solution (32).

8. Comment on Dual Curves

If $P(\mathbf{r})$ is an n-curve and $X(\boldsymbol{\alpha})$ a point then $P(\boldsymbol{\alpha}\times\mathbf{r})$ will be a pencil of n lines with vertex X. Furthermore, each of these lines (for any position of X) will be a tangent to an algebraic curve, the dual curve to P. For example if P is a conic with matrix A (so that $P(\mathbf{r}) = \mathbf{r}^T A\mathbf{r}$) then the dual curve is the conic with matrix adj(A). However, given that $\theta(\boldsymbol{\alpha}, \mathbf{r})$ is a pencil of lines with vertex $X(\boldsymbol{\alpha})$ for any point X it does not follow that θ can be written as a function of $\boldsymbol{\alpha} \times \mathbf{r}$ nor that these lines necessarily have an envelope. For example if P is now a cubic curve and θ is given by

$$\theta = PP_{XXX}^2 - 3P_X P_{XX} P_{XXX} + 2P_{XX}^3,$$

c.f. equation(10), then θ is a pencil of three lines but there is no envelope. (If X is confined to lie on a line then an envelope will exist but it will depend upon the line chosen.)

As a final example, with P an n-curve, Q the line $\boldsymbol{\beta}\cdot\mathbf{r} = 0$ and θ defined by (28), the geometrical interpretation implies that

$$\theta = P((\boldsymbol{\alpha} \times \mathbf{r}) \times \boldsymbol{\beta}).$$

This is a degenerate case since the envelope of the lines of θ consists of the n common points of P and Q.

The few cases investigated are consistent with the suggestion that if $P, Q, \ldots$ are algebraic curves and there exists a form invariant $I(\mathbf{p}, \mathbf{q}, \ldots)$ (where $\mathbf{p}, \mathbf{q}, \ldots$ are as in (25)), then I can be written as a function of $\boldsymbol{\alpha} \times \mathbf{r}$ and the associated pencil has an envelope.

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