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INEQUALITIES CONCERNING THE EXCIRCLE QUADRILATERAL CORRESPONDING TO CYCLIC QUADRILATERALS AND BICENTRIC QUADRILATERALS

OVIDIU T. POP and MARIO DALCÍN

Abstract. The centers of the excircles of a quadrilateral determine the excircle quadrilateral. In this paper we will demostrate some inequalities related to the perimeter and the area of the excircle quadrilateral of a cyclic quadrilateral and of a bicentric quadrilateral.

1. Introduction

In this section, we recall some known results that hold in a quadrilateral.

In a given convex quadrilateral $ABCD$, we denote the lengths of the sides by $a = AB,\ b = BC,\ c = CD,\ d = DA$, the measures of the angles by A, B, C, D, the area and semiperimeter by F and s, respectively. If the quadrilateral $ABCD$ is cyclic, we denote by $\mathcal{C}(O, R)$ the circumscribed circle, where O is its center and R its radius. If the quadrilateral $ABCD$ is tangential, we denote by $\mathcal{C}(I, r)$ the inscribed circle, where I is the center and r is the radius. A quadrilateral $ABCD$ is bicentric if and only if is cyclic and tangential.

Let $ABCD$ be a convex quadrilateral. The circle tangent to side AB and tangent to the extensions of its two adjacent sides, is called the excircle of the quadrilateral corresponding to the side AB. Let $\mathcal{C}(I_a, r_a)$ be this circle, where I_a is the center and r_a is the radius. The excircles $\mathcal{C}(I_b, r_b)$, $\mathcal{C}(I_c, r_c)$, $\mathcal{C}(I_d, r_d)$ of $ABCD$ tangent to the sides b, c and d, respectively, are defined similarly (see Figure 1). The quadrilateral $I_aI_bI_cI_d$ is

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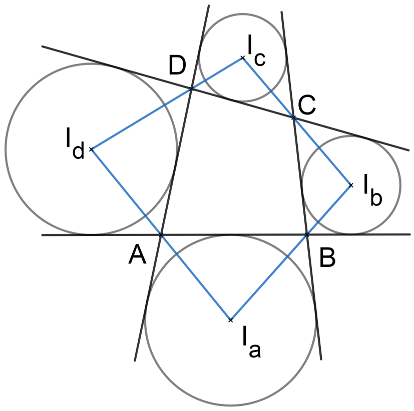

Figure 1

called the excircle quadrilateral corresponding to the quadrilateral $ABCD$.

In what follows, we recall some known results for the cyclic quadrilateral, see for example [1], pages 96-98, 163 and 168-171.

Theorem 1. *In the cyclic quadrilateral $ABCD$, we have*

$$F=\sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{1}$$

$$R=\frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4F} \tag{2}$$

and

$$\cos\frac{A}{2}=\sin\frac{C}{2}=\sqrt{\frac{(s-b)(s-c)}{ad+bc}}. \tag{3}$$

If, in addition, the quadrilateral $ABCD$ is also tangential, then

$$ac+bd=2r(\sqrt{4R^2+r^2}+r). \tag{4}$$

Theorem 2. *Let $ABCD$ be a bicentric quadrilateral. The inequality*

$$2\sqrt{2r(\sqrt{4R^2+r^2}-r)}\leq s \tag{5}$$

holds.
If $R=r\sqrt{2}$, then $ABCD$ is square, both circles are concentric and the equality in (5) holds. If $R\neq r\sqrt{2}$, then the equality holds if and only if $ABCD$ is an isosceles trapezoid.
Moreover, we have the inequality

$$s\leq\sqrt{4R^2+r^2}+r, \tag{6}$$

which becomes equality if $ABCD$ is orthodiagonal.
The following inequalities also hold:

$$2\sqrt{2r(\sqrt{4R^2+r^2}-r)}\leq s\leq\sqrt{4R^2+r^2}+r. \tag{7}$$

If $R = r\sqrt{2}$, then both inequalities become equalities and in this case $ABCD$ is a square. If $R \neq r\sqrt{2}$, then at least one of the inequalities is strict.
The inequality of Fejes-Tóth holds:

$$R \geq r\sqrt{2}. \tag{8}$$

In [2] we proved the following results that we will use in this paper.

Theorem 3. *In the cyclic quadrilateral $ABCD$, we have the following relation*

$$I_cI_d = \frac{s(ad+bc)\sqrt{ab+cd}}{(a+c)(b+d)}\frac{1}{\sqrt{(s-c)(s-d)}} \tag{9}$$

and its analogues.

Theorem 4. *If the quadrilateral $ABCD$ is cyclic, then the quadrilateral $I_aI_bI_cI_d$ is orthodiagonal.*

2. Main results

In this section, we denote by s' the semiperimeter of the quadrilateral $I_aI_bI_cI_d$.

Lemma 1. *If $ABCD$ is a cyclic quadrilateral, then*

$$2s' = \frac{4sR}{(a+c)(b+d)\sqrt{ac+bd}}\Big(\sqrt{ad+bc}\Big(\sqrt{(s-a)(s-b)}+\sqrt{(s-c)(s-d)}\Big)+ +\sqrt{ab+cd}\Big(\sqrt{(s-a)(s-d)}+\sqrt{(s-b)(s-c)}\Big)\Big). \tag{10}$$

If, in addition, the quadrilateral $ABCD$ is also tangential, then

$$2s' = \frac{s}{(a+c)(b+d)}\Big((ad+bc)\sqrt{ab+cd}\Big(\frac{1}{\sqrt{cd}}+\frac{1}{\sqrt{ab}}\Big)+ +(ab+cd)\sqrt{ad+bc}\Big(\frac{1}{\sqrt{ad}}+\frac{1}{\sqrt{bc}}\Big)\Big). \tag{11}$$

Proof. Taking (9) into account, we have

$$2s' = I_aI_b + I_bI_c + I_cI_d + I_dI_a =$$

$$= \frac{s(ad+bc)\sqrt{ab+cd}}{(a+c)(b+d)}\frac{1}{\sqrt{(s-a)(s-b)}} + \frac{s(ba+cd)\sqrt{bc+da}}{(a+c)(b+d)}\frac{1}{\sqrt{(s-b)(s-c)}} +$$

$$+\frac{s(cb+da)\sqrt{cd+ab}}{(a+c)(b+d)}\frac{1}{\sqrt{(s-c)(s-d)}} + \frac{s(dc+ab)\sqrt{da+bc}}{(a+c)(b+d)}\frac{1}{\sqrt{(s-d)(s-a)}} =$$

$$= \frac{s\sqrt{(ab+cd)(ad+bc)}}{(a+c)(b+d)}\Big(\sqrt{ad+bc}\Big(\frac{1}{\sqrt{(s-a)(s-b)}}+\frac{1}{\sqrt{(s-c)(s-d)}}\Big)+ +\sqrt{ab+cd}\Big(\frac{1}{\sqrt{(s-b)(s-c)}}+\frac{1}{\sqrt{(s-d)(s-a)}}\Big)\Big).$$

Using the relation (2) in the identity above, the identity (10) follows.
If the quadrilateral $ABCD$ is tangential, we have $a+c = b+d$, then $s-a = c$ and its analogues. Taking this observation into account, relation (11) is immediately obtained.

Theorem 5. *If $ABCD$ is a cyclic quadrilateral, the following inequality*

$$s' \leq \frac{2s^2R\sqrt{2}}{\sqrt{(a+c)(b+d)(ac+bd)}} \tag{12}$$

holds, with equality if and only if $ABCD$ is a square. If $ABCD$ is bicentric quadrilateral, then

$$s' \leq s\sqrt{\frac{\sqrt{4R^2+r^2}-r}{r}}, \tag{13}$$

with equality if and only if $ABCD$ is a square.

Proof. Using the means inequality we have

$$\sqrt{(s-a)(s-b)}+\sqrt{(s-c)(s-d)} \leq \frac{s-a+s-b}{2}+\frac{s-c+s-d}{2}=s,$$

with equality if and only if $a=b$ and $c=d$. Similarly we have that $\sqrt{(s-a)(s-d)}+\sqrt{(s-b)(s-c)} \leq s$, with equality if and only if $a=d$ and $b=c$. Taking into account the above, from (10) we have $2s' \leq \frac{4s^2R}{(a+c)(b+d)\sqrt{ac+bd}}\Big(\sqrt{ad+bc}+\sqrt{ab+cd}\Big)$, with equality if and only if $a=b=c=d$, equivalent to $ABCD$ is a square.
Using the well-known inequality $x+y \leq \sqrt{2(x^2+y^2)}$, $x,y>0$, with equality if and only if $x=y$, we have

$$\sqrt{ad+bc}+\sqrt{ab+cd} \leq \sqrt{2\Big((ad+bc)+(ab+cd)\Big)}=\sqrt{2(a+c)(b+d)}.$$

From the above, we obtain that $2s' \leq \frac{4s^2R}{(a+c)(b+d)\sqrt{ac+bd}}\sqrt{2(a+c)(b+d)}$, from where (12) follows.
If $ABCD$ is bicentric quadrilateral, then $a+c=b+d=s$ and taking (4) into account, from (12) we obtain (13).

Theorem 6. *If $ABCD$ is a bicentric quadrilateral, then*

$$s\sqrt{2} \leq s', \tag{14}$$

with equality if and only if $ABCD$ is a square.

Proof. Using the AM-GM inequality we have $\sqrt{ab+cd} \geq \sqrt{2\sqrt{abcd}}$ and $\frac{1}{\sqrt{cd}}+\frac{1}{\sqrt{ab}} \geq \frac{2}{\sqrt{\sqrt{abcd}}}$, with equality if and only if $ab=cd$ and $cd=ab$, respectively, from where $\sqrt{ab+cd}\Big(\frac{1}{\sqrt{cd}}+\frac{1}{\sqrt{ab}}\Big) \geq 2\sqrt{2}$, with equality if and only if $ab=cd$. Similarly $\sqrt{ad+bc}\Big(\frac{1}{\sqrt{ad}}+\frac{1}{\sqrt{bc}}\Big) \geq 2\sqrt{2}$, with equality if and only if $ad=bc$. From the above it results that the equality holds if and only if $a=c$ and $b=d$. But $a+c=b+d$, so the equality holds if and only if $a=b=c=d$, equivalent to $ABCD$ being a square.
Taking into account the inequalities above, relation (11) becomes

$$2s' \leq \frac{2s\sqrt{2}}{(a+c)(b+d)}\Big((ad+bc)+(ab+cd)\Big) = \frac{2s\sqrt{2}}{(a+c)(b+d)}(a+c)(b+d),$$
from where (14) follows.

Corollary 1. *If $ABCD$ is a bicentric quadrilateral, then*

$$\sqrt{2} \leq \frac{s'}{s} \leq \sqrt{\frac{\sqrt{4R^2+r^2}-r}{r}}. \tag{15}$$

Proof. It results from (13) and (14).

Lemma 2. *If $ABCD$ is a cyclic quadrilateral, then*

$$I_bI_d = \frac{s\sqrt{(ad+bc)(ab+cd)}}{(b+d)\sqrt{(s-b)(s-d)}} \tag{16}$$

and its analogues.

Proof. From the Law of Cosines in triangle $I_bI_aI_d$ (see Figure 2), we have $I_bI_d^2 = I_aI_b^2 + I_aI_d^2 - 2I_aI_b \cdot I_aI_d \cos\widehat{AI_aB}$. But $\widehat{AI_aB} = \dfrac{A+B}{2}$ and then $\cos\widehat{AI_aB} = \cos\dfrac{A+B}{2} = \cos\dfrac{A}{2}\cos\dfrac{B}{2} - \sin\dfrac{A}{2}\sin\dfrac{B}{2}$. Taking (3) into account,

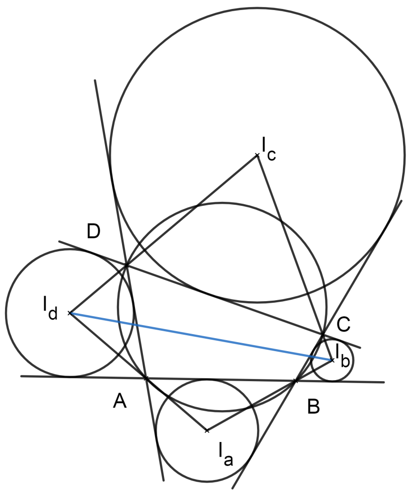

Figure 2

we have

$$\cos\widehat{AI_aB} = \cos\frac{A}{2}\cos\frac{B}{2} - \cos\frac{C}{2}\cos\frac{D}{2} =$$
$$= \sqrt{\frac{(s-b)(s-c)}{ad+bc}} \cdot \sqrt{\frac{(s-c)(s-d)}{ba+cd}} - \sqrt{\frac{(s-d)(s-a)}{cb+ad}} \cdot \sqrt{\frac{(s-a)(s-b)}{dc+ab}} =$$
$$= \frac{\sqrt{(s-b)(s-d)}\Big((s-c)-(s-a)\Big)}{\sqrt{(ab+cd)(ad+bc)}} = \frac{(a-c)\sqrt{(s-b)(s-d)}}{\sqrt{(ab+cd)(ad+bc)}}.$$

Using (9) and substituting above, we have

$$I_bI_d^2 = \frac{s^2(ad+bc)^2(ab+cd)}{(a+c)^2(b+d)^2}\cdot\frac{1}{(s-a)(s-b)} + \frac{s^2(ab+cd)^2(ad+bc)}{(a+c)^2(b+d)^2}\cdot$$

$$\cdot\frac{1}{(s-a)(s-d)} - 2\cdot\frac{s^2(ad+bc)(ab+cd)\sqrt{(ab+cd)(ad+bc)}}{(a+c)^2(b+d)^2}\cdot$$

$$\cdot\frac{1}{(s-a)\sqrt{(s-b)(s-d)}}\cdot\frac{(a-c)\sqrt{(s-b)(s-d)}}{\sqrt{(ab+cd)(ad+bc)}} =$$

$$= \frac{s^2(ab+cd)(ad+bc)}{(a+c)^2(b+d)^2(s-a)}\left(\frac{ad+bc}{s-b} + \frac{ab+cd}{s-d} - 2(a-c)\right) =$$

$$= \frac{s^2(ab+cd)(ad+bc)}{(a+c)^2(b+d)^2(s-a)}\cdot$$

$$\cdot\frac{(ad+bc)(a+b+c-d)+(ab+cd)(a-b+c+d)-(a-c)(a-b+c+d)(a+b+c-d)}{2(s-b)(s-d)}$$

and after calculus we obtain (16).

In the following, we denote by F' the area of the quadrilateral $I_aI_bI_cI_d$.

Lemma 3. *If $ABCD$ is a cyclic quadrilateral, then*

$$F'F = \frac{s^2(ab+cd)(ad+bc)}{2(a+c)(b+d)}. \tag{17}$$

If, in addition, the quadrilateral $ABCD$ is also tangential, then

$$F'F = \frac{1}{2}(ab+cd)(ad+bc) \tag{18}$$

or

$$F' = s(\sqrt{4R^2+r^2} - r). \tag{19}$$

Proof. Taking Theorem 4, (16) and (1) into account, we have $F' = \frac{1}{2}I_aI_c\cdot I_bI_d = \frac{1}{2}\frac{s\sqrt{(ab+cd)(ad+bc)}}{(a+c)\sqrt{(s-a)(s-c)}}\cdot\frac{s\sqrt{(ad+bc)(ab+cd)}}{(b+a)\sqrt{(s-b)(s-d)}}$, from where (18) follows.

If the quadrilateral $ABCD$ is also tangential, then $a+c = b+d = s$ and then from (17), (18) results.

From (2) we have that $(ab+cd)(ad+bc) = \dfrac{16R^2F^2}{ac+bd}$.

Taking (4) and that $F = rs$ into account, from (18) we obtain that $F' = \dfrac{1}{2}\,\dfrac{16R^2rs}{2r(\sqrt{4R^2+r^2}+r)} = \dfrac{4R^2s}{\sqrt{4R^2+r^2}+r}$, whence (19) results.

Theorem 7. *If $ABCD$ ia a bicentric quadrilateral, then*

$$F' \geq 2F, \tag{20}$$

with equality if and only if the quadrilateral $ABCD$ is a square.

Proof. From the AM-GM inequality we have that $ab+cd \geq 2\sqrt{abcd}$, with equality if and only if $ab = cd$, and $ad+bc \geq 2\sqrt{abcd}$, with equality if and only if $ad = bc$. The case of equality is identical to that of Theorem 6. Taking (18) and that $F = \sqrt{abcd}$ into account, relation (20) is obtained.

Theorem 8. *If $ABCD$ is a bicentric quadrilateral, then*

$$2(\sqrt{4R^2+r^2}-r)\sqrt{2r(\sqrt{4R^2+r^2}-r)} \le F' \le 4R^2. \tag{21}$$

Proof. Inequalities (21) results immediately from (19) and (7).

Corollary 2. *If $ABCD$ is a bicentric quadrilateral, the inequalities*

$$8r^2 \le F' \le 4R^2 \tag{22}$$

hold.

Proof. The left-hand side of the inequality (22) is obtained using the inequality from (8).

References

[1] Pop, O.T., Minculete, N. and Bencze, M., *An introduction to quadrilateral geometry*, Ed. Didactică şi Pedagogică, Bucureşti, **2013**.

[2] Pop, O.T. and Dalcín, Mario, *Equalities and Inequalities in the Excircle Quadrilateral of a Convex Quadrilateral.* Journal for Geometry and Graphics, **27**(**2**) (**2023**) (In press) 171–179.

NATIONAL COLLEGE “MIHAI EMINESCU”
5 MIHAI EMINESCU, 440014 SATU MARE, ROMANIA
E-mail address: ovidiutiberiu@yahoo.com

’ARTIGAS’ SECONDARY SCHOOL
TEACHERS INSTITUTE - CFE
MONTEVIDEO, URUGUAY
E-mail address: mdalcin00@gmail.com