INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 13 (2024), No. 3, 109 - 117

LOCALLY CONFORMALLY SYMMETRIC CONDITION ON FOUR DIMENSIONAL STRICT WALKER METRICS

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Abstract. In this paper, conditions for four dimensional strict Walker manifolds to be locally conformally flat and locally conformally symmetric are given. Finally, we obtain a classification of essentially conformally symmetric four dimensional strict Walker manifolds.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold of dimension m with ∇ its Levi-Civita connection. Let $\mathcal{R}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ be the curvature operator and $R(X,Y,Z,T) = g(\mathcal{R}(X,Y)Z,T)$ its Riemann curvature tensor. Let $\{e_i\}$ be a local frame for the tangent bundle. We set $g_{ij} := g(e_i, e_j)$ and let g^{ij} be the inverse matrix. The associated Ricci tensor $\rho(\cdot,\cdot)$, the scalar curvature τ and the Weyl conformal curvature operator $\mathcal{W}$ are given by

$$\begin{aligned} \rho(X,Y) :=& \quad \sum_{i,j} g^{ij} R(X, e_i, e_j, Y),\ \tau : \sum_{i,j} g^{ij}\rho(e_i, e_j), \\ \mathcal{W}(X,Y)Z \;=& \;\; \mathcal{R}(X,Y)Z + \frac{\tau}{(m-1)(m-2)}\mathcal{R}^0(X,Y)Z \\ & + \frac{1}{(m-2)}\mathcal{L}(X,Y)Z, \end{aligned}$$

where X, Y, Z are vector fields on M and

$$\begin{aligned} \mathcal{R}^0(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ \mathcal{L}(X,Y)Z &= g(\rho Y, Z)X - g(\rho X, Z)Y + g(Y,Z)\rho X - g(X,Z)\rho Y. \end{aligned}$$

The Weyl conformal curvature $\mathcal{W}$ as a conformal invariant is important in the understanding of conformal pseudo-Riemannian geometry [8]. It is well known that an m-dimensional pseudo-Riemannian manifold, $m \geq 4$, is conformally flat if and only if its Weyl conformal curvature vanishes.

Keywords and phrases: Conformally flat,Locally conformally symmetric, Walker metrics
(2010)Mathematics Subject Classification: 53B30, 53B5
Received: 27.01.2024. In revised form: 11.05.2024. Accepted: 19.06.2024.

In [1], the authors construct examples of complete locally conformally flat Riemannian manifolds with nonpositive sectional and Ricci curvatures. Also, the authors [2], gives necessary and sufficient conditions for a static space-time to be locally conformally flat, showing some significant restrictions on the possible warping functions of the space-times. In the paper [15], the author studies hypersurfaces in a conformally flat manifold.

A pseudo-Riemannian manifold is said to be conformally symmetric if its Weyl conformal curvature is parallel [7]. It is known that any conformally symmetric Riemannian manifold is either locally symmetric or locally conformally flat. Recall that, the Weyl tensor vanishes in dimension three, conformally symmetric manifolds have been investigated only in dimension greater than four. In [4], the authors extend the study of conformal symmetric manifolds to the three-dimensional setting, where all the conformal information is codified by the Cotton tensor. It is well-known that any locally conformally flat manifold has vanishing Cotton tensor and the converse is also true in dimension $m = 3$. Moreover, the Cotton tensor plays an important role in Riemannian and pseudo-Riemannian geometry. The study of gradient Ricci solitons in locally conformally flat manifolds [5] or the Goldberg-Sachs theorem in pseudo-Riemannian geometry [16] are examples where the Cotton tensor appears naturally.

The paper is organized in the following way. In Section 2, we describe the curvature of four dimensional strict Walker metrics. In Section 3, we give a theorm for a four dimensional strict Walker metric to be locally conformally flat. In Section 4, the condition for a four dimensional strict Walker metric to be locally conformally symmetric is given.

2. Description of the metric

A Walker manifold [17] is a triple $(M, g, \mathcal{D})$, where M is an m-dimensional manifold, g an indefinite metric, and $\mathcal{D}$ an r-dimensional parallel null distribution. Of special interest are manifolds of even dimensions admitting a field of null planes of maximum dimensional ($r = \frac{m}{2}$). An example of a Walker Osserman metric of signature $(3, 3)$ which admits a field of parallel null 3-planes is given in [10]. Conditions for a restricted four dimensional Walker manifold to be Einstein, locally symmetric, Einstein and locally conformally flat are given in [12]. A lot of examples of Walker structures have appeared, which proved to be important in differential geometry and general relativity as well (see [9, 11, 13] and references terein).

A field of r-plane $\mathcal{D}$ is said to be strictly parallel if each vector in the plane at a point $p \in M$ is carried by a parallel transport by a vector in the plane at another point $q \in M$, the latter vector being the same for all paths from p to q [17]. A four-dimensional Walker manifold is a strict Walker manifold if and only if $\mathcal{D}$ admits two null parallel spanning vector fields or, equivalently, if we can choose a coordinate system so $g_{ij}(x_1, x_2, x_3, x_4) = g_{ij}(x_3, x_4), i, j = 3, 4$. In [3], the authors shown that four dimensional strict Walker manifolds are geodesically complete. Recently, conditions for four dimensional strict Walker manifolds to be locally symmetric are given [14].

In this paper, we consider the family of metrics $g_{a,b,c}$ on $O \subset \mathbb{R}^4$ given by

$$\begin{aligned} g_{a,b,c} &= 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + a(x_3, x_4)dx_3 \circ dx_3 \\ &\quad + b(x_3, x_4)dx_4 \circ dx_4 + 2c(x_3, x_4)dx_3 \circ dx_4, \end{aligned} \tag{1}$$

where a, b and c depend only on (x_3, x_4). We denote by $\partial_i := \frac{\partial}{\partial x_i}$ and $f_i := \frac{\partial f(x_3,x_3)}{\partial x_i}$. A straightforward calculation, shows that the non-zero components of the Levi-Civita connection of the metric (1) are given by:

$$\begin{aligned} \nabla_{\partial_3}\partial_3 &= \frac{1}{2}a_3\partial_1 + \frac{1}{2}(2c_3 - a_4)\partial_2; \\ \nabla_{\partial_3}\partial_4 &= \frac{1}{2}a_4\partial_1 + \frac{1}{2}b_3\partial_2; \\ \nabla_{\partial_4}\partial_4 &= \frac{1}{2}(2c_4 - b_3)\partial_1 + \frac{1}{2}b_4\partial_2. \end{aligned}$$

The non-zero components of the curvature tensor of $(O, g_{a,b,c})$ are given by:

$$\begin{aligned} R(\partial_3, \partial_4)\partial_3 &= \frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_2; \\ R(\partial_3, \partial_4)\partial_4 &= -\frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_1. \end{aligned}$$

The nonzero component of the $(0, 4)$-curvature tensor of the metric (1) is given by:

$$R_{3434} = \frac{1}{2}(a_{44} + b_{33} - 2c_{34}).$$

We find that the Ricci tensor and the scalar curvature of $(O, g_{a,b,c})$ vanish. More precisely, we have: $\rho_{ij} = 0$ and $\tau = 0, \forall i, j = 1, 2, 3, 4$. The nonzero components of the Einstein tensor $G_{ij} = \rho_{ij} - \frac{\tau}{4}g_{ij}$ are given by $G_{ij} = 0$, $\forall i, j = 1, 2, 3, 4$. Hence a strict four dimensional Walker manifold $(M, g_{a,b,c})$ is Ricci flat and so Einstein [14]. So, the components of the Weyl conformal curvature for $(M, g_{a,b,c})$ are given by

$$W_{3434} = \frac{1}{2}(b_{33} + a_{44} - 2c_{34}). \tag{2}$$

3. Locally conformally flat strict Walker metrics

Recall that a Riemannian manifold (M, g) is locally conformally flat if every point in M admits a coordinate neighborhood U which is conformal to Euclidean space $\mathbb{R}^n$; equivalently, if there is a diffeomorphism $\Phi : V \subset \mathbb{R}^n \to U$ such that $\Phi^* g = \Psi^2 g_{\mathbb{R}^s}$ for some positive function Ψ. Any surface is locally conformally flat, but not every higher-dimensional Riemannian manifold admits a locally conformally flat structure. Necessary and sufficient conditions for the existence of such a structure are the nullity of the Weyl tensor W when $\dim M \geq 4$, and, in dimension three, the condition that the Schouten tensor be a Codazzi tensor [1].

Next, we give the form of a locally conformally flat strict Walker metric.

Theorem 3.1. *A strict four dimensional Walker of the form (1) is locally conformally flat if and only the functions a, b and c satisfying the following forms:*

$$\begin{aligned} a(x_3,x_4) &= x_4\overline{a}(x_3)+\widehat{a}(x_3)+G_1(x_3,x_4); \\ b(x_3,x_4) &= x_3\overline{b}(x_4)+\widehat{b}(x_4)+G_2(x_3,x_4)); \\ c(x_3,x_4) &= \int \overline{c}(x_4)dx_4+\widehat{c}(x_3) \\ &\quad +\frac{1}{2}\int\Big(\int\Big[\frac{\partial^2 G_1(x_3,x_4)}{\partial x_4^2}+\frac{\partial^2 G_2(x_3,x_4)}{\partial x_3^2}\Big]dx_3\Big)dx_4 \end{aligned}$$

where $\overline{a}(x_3), \widehat{a}(x_3), G_1(x_3,x_4)$, $\overline{b}(x_4), \widehat{b}(x_4), G_2(x_3,x_4)$, $\overline{c}(x_4)$ and $\widehat{c}(x_4)$ are smooth functions.

Proof. From (2), we have :

$$W_{3434}=\frac{1}{2}\Big(\frac{\partial^2 a}{\partial x_4^2}+\frac{\partial^2 b}{\partial x_3^2}-2\frac{\partial^2 c}{\partial x_4\partial x_3}\Big).$$

A pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes, that is $W=0$. Thus we have:

$$\frac{\partial^2 a}{\partial x_4^2}=\frac{2\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 b}{\partial x_3^2}. \tag{3}$$

By a integration with respect to x_4 of both side of (3), we get:

$$\frac{\partial a}{\partial x_4}=\overline{a}(x_3)+\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 b}{\partial x_3^2}\Big)dx_4.$$

A integration with respect to x_4 give us the followin:

$$a(x_3,x_4) = x_4\overline{a}(x_3)+\int\Big[\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big]dx_4+\widehat{a}(x_3).$$

We set: $G_1(x_3,x_4)=\int\Big[\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big]dx_4$. Hence we obtain

$$a(x_3,x_4)=x_4\overline{a}(x_3)+\widehat{a}(x_3)+G_1(x_3,x_4). \tag{4}$$

By analogously, with the same routine, we get:

$$b(x_3,x_4)=x_3\overline{b}(x_4)+\widehat{b}(x_4)+G_2(x_3,x_4), \tag{5}$$

where $G_2(x_3,x_4)=\int\Big[\int\Big(2\frac{\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 a}{\partial x_4^2}\Big)dx_3\Big]dx_3$. Using (3), (4) and (5), we obtain:

$$\frac{\partial^2 c}{\partial x_4\partial x_3}=\frac{1}{2}\frac{\partial^2 G_1(x_3,x_4)}{\partial x_4^2}+\frac{1}{2}\frac{\partial^2 G_2(x_3,x_4)}{\partial x_3^2}.$$

By integration with respect to x_3, we get:

$$\frac{\partial c}{\partial x_4} = \overline{c}(x_4)+\frac{1}{2}\int\Big[\frac{\partial^2 G_1(x_3,x_4)}{\partial x_4^2}+\frac{\partial^2 G_2(x_3,x_4)}{\partial x_3^2}\Big]dx_3$$

By integration with respect to x_4, we get:

$$\begin{aligned} c(x_3,x_4) &= \int \overline{c}(x_4)dx_4 + \widehat{c}(x_3) \\ &+ \frac{1}{2}\int\Big(\int\Big[\frac{\partial^2 G_1(x_3,x_4)}{\partial x_4^2}+\frac{\partial^2 G_2(x_3,x_4)}{\partial x_3^2}\Big]dx_3\Big)dx_4. \end{aligned} \tag{6}$$

The proof is complete.

Remark 3.1. *From (2), a strict four dimensional Walker manifold is locally conformally flat if only if it is locally flat.*

Example 3.1. *Let $M_{a,b,c}$ be as in (1). With the following choices of a, b and c: $a(x_3,x_4) = x_3x_4 + x_3^2x_4 + x_3$, $b(x_3,x_4) = x_3x_4^2 + x_3x_4^2 + x_4$, and $c(x_3,x_4) = \frac{x_4^2}{2} + x_3^2$. Then $M_{a,b,c}$ is locally conformally flat.*

Example 3.2. *Let $M_{a,b,c}$ be as in (1). With the following choices of a, b and c: $a(x_3,x_4) = x_3x_4 + x_3x_4 + x_3$, $b(x_3,x_4) = x_3x_4 + x_3x_4^2 + x_4$, and $c(x_3,x_4) = \frac{x_4^3}{3} + x_3^3$. Then $M_{a,b,c}$ is loccally conformally flat.*

Corrolary 3.1. *A strict four dimensional Walker of the form $a(x_3,x_4) \neq 0$, $b(x_3,x_4) \neq 0$ and $c(x_3,x_4) = 0$ is locally conformally flat if and only the following equation is satisfied*

$$\frac{\partial^2 a}{\partial x_4^2} + \frac{\partial^2 b}{\partial x_3^2} = 0. \tag{7}$$

Corrolary 3.2. *A strict four dimensional Walker of the form $a(x_3,x_4) \equiv b(x_3,x_4) \equiv c(x_3,x_4)$ is locally conformally flat if and only the function a is solution of the following system of partial differential equation:*

$$\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2} + \frac{1}{2}\frac{\partial^2 a}{\partial x_3^2} - \frac{\partial^2 a}{\partial x_3 \partial x_4} = 0. \tag{8}$$

Corrolary 3.3. *A strict four dimensional Walker of the form $a(x_3,x_4) \equiv b(x_3,x_4) \equiv 0$ and $c(x_3,x_4) \neq 0$ is locally conformally flat if and only the function c is a linear function.*

4. Locally conformally symmetric strict Walker metrics

A pseudo-Riemannian manifold (M, g) of dimension $m \geq 4$ is said to be conformally symmetric if the Weyl tensor W of M is parallel. Obvious examples arise when M is Riemannian conformally flat or locally symmetric. Conformally symmetric manifolds which are neither conformally flat nor locally symmetric have usually been referred to as essentially conformally symmetric (ECS). Such manifolds must be non-Riemannian. The local and global geometry of essentially conformally symmetric pseudo-Riemannian manifolds has been extensively investigated by Derdzinski and Roter in a series of papers (see [6, 7] and the references therein for further information).

Theorem 4.1. *A strict four dimensional Walker of the form (1) is locally conformally symmetric if and only the functions a, b and c satisfying the following forms:*

$$\begin{aligned} a(x_3,x_4) &= \frac{K}{2}x_4^2 + G_1(x_3,x_4) + x_4H_1(x_3) + F_1(x_3), \\ b(x_3,x_4) &= \frac{K}{2}x_3^2 + G_2(x_3,x_4) + x_3H_2(x_4) + F_2(x_4), \\ c(x_3,x_4) &= G_3(x_3,x_4) - \frac{K}{2}x_3x_4 + x_4H_3(x_4) + F_3(x_3). \end{aligned}$$

where $F_i, G_i, H_i, i = 1,2,3,$ are smooth functions.

Proof. From (2), the non vanishing components of the covariant derivative of the Weyl conformal operator are:

$$\begin{aligned} (\nabla_{\partial_3}W)(\partial_3,\partial_4,\partial_3,\partial_4) &= \frac{1}{2}\frac{\partial^3 a}{\partial x_4^2\partial x_3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^3} - \frac{\partial^3 c}{\partial x_3^2\partial x_4}, \\ (\nabla_{\partial_4}W)(\partial_3,\partial_4,\partial_3,\partial_4) &= \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^2\partial x_4} - \frac{\partial^3 c}{\partial x_3\partial x_4^2}. \end{aligned}$$

The locally conformally symmetric property means $\nabla W = 0$, that is:

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2\partial x_3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^3} - \frac{\partial^3 c}{\partial x_3^2\partial x_4} = 0, \tag{9}$$

and

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} + \frac{1}{2}\frac{\partial^3 b}{\partial x_3^2\partial x_4} - \frac{\partial^3 c}{\partial x_3\partial x_4^2} = 0. \tag{10}$$

From (9), we have:

$$\frac{\partial^2 a}{\partial x_4^2} + \frac{\partial^2 b}{\partial x_3^2} - 2\frac{\partial^2 c}{\partial x_4\partial x_3} = A(x_4), \tag{11}$$

and, from (10), we have:

$$\frac{\partial^2 b}{\partial x_3^2} + \frac{\partial^2 a}{\partial x_4^2} - 2\frac{\partial^2 c}{\partial x_3\partial x_4} = B(x_3). \tag{12}$$

By the use of (11) and (12), we remark that: $A(x_4) = B(x_3) = K \in \mathbb{R}$. Hence, from (11), we have:

$$\frac{\partial^2 a}{\partial x_4^2} = K + \Big(\frac{2\partial^2 c}{\partial x_3\partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big),$$

this implies

$$\frac{\partial a}{\partial x_4} = Kx_4 + \int\Big(\frac{2\partial^2 c}{\partial x_3\partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big)dx_4 + H_1(x_3).$$

and so

$$\begin{aligned} a(x_3,x_4) &= \frac{K}{2}x_4^2 + \int\Big[\int\Big(\frac{2\partial^2 c}{\partial x_3\partial x_4} - \frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big]dx_4 \\ &\quad + x_4H_1(x_3) + F_1(x_3). \end{aligned}$$

We set $G_1(x_3,x_4)=\int\Big[\int\Big(\frac{2\partial^2 c}{\partial x_3\partial x_4}-\frac{\partial^2 b}{\partial x_3^2}\Big)dx_4\Big]dx_4$. We obtain:

$$a(x_3,x_4)=\frac{K}{2}x_4^2+G_1(x_3,x_4)+x_4H_1(x_3)+F_1(x_3). \tag{13}$$

This is our first equation on the Theorem 4.1.
By analogy, with the same routine, we get:

$$b(x_3,x_4)=\frac{K}{2}x_3^2+G_2(x_3,x_4)+x_3H_2(x_4)+F_2(x_4), \tag{14}$$

where $G_2(x_3,x_4)=\int\Big[\int\Big(\frac{2\partial^2 c}{\partial x_4\partial x_3}-\frac{\partial^2 a}{\partial x_4^2}\Big)dx_3\Big]dx_3$. The expression (14) is our second equation on the Theorem 4.1.
Using (12), we have:

$$\frac{\partial^2 b}{\partial x_3^2}+\frac{\partial^2 a}{\partial x_4^2}-\frac{2\partial^2 c}{\partial x_4\partial x_3}=B(x_3)=K,$$

that implie

$$\frac{\partial}{\partial x_3}\Big(\frac{\partial c}{\partial x_4}\Big)=\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2}+\frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}-\frac{K}{2}.$$

By a first integration with respect to x_3, we get:

$$\frac{\partial c}{\partial x_4}=\int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2}+\frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3-\frac{K}{2}x_3+H_3(x_4).$$

By a second integration with respect to x_4, we get:

$$\begin{aligned} c(x_3,x_4) \;&=\; \int\Big[\int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2}+\frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3\Big]dx_4 \\ &\quad -\frac{K}{2}x_3x_4+.x_4H_3(x_4)+F_3(x_3)\end{aligned}$$

We set $G_3(x_3,x_4)=\int\Big[\int\Big(\frac{1}{2}\frac{\partial^2 a}{\partial x_4^2}+\frac{1}{2}\frac{\partial^2 b}{\partial x_3^2}\Big)dx_3\Big]dx_4$, on obtain,

$$c(x_3,x_4)=G_3(x_3,x_4)-\frac{K}{2}x_3x_4+x_4H_3(x_4)+F_3(x_3). \tag{15}$$

This is the last equation on the Theorem 4.1, and the is complete
We have te followin remark

Remark 4.1. *A strict four dimensional Walker manifold is locally conformally symmetric if only if it is locally symmetric* [14].

Example 4.1. *Let M be as in (1). With the following choices of $a(x_3,x_4)=x_4^2+x_3^2x_4+x_3$, $b(x_3,x_4)=x_3^2+x_3x_4^2+x_4$, and $c(x_3,x_4)=0$. Then $M_{a,b,c}$ is loccally conformally symmetric.*

Example 4.2. *Let M be as in (1. With the following choices of $a(x_3,x_4)=x_4^2+x_3^2x_4+x_4x_3+x_3$, $b(x_3,x_4)=x_3^2+x_3^2x_4^2+x_3x_4^2+x_4^2$, and $c(x_3,x_4)=x_3^2x_4-x_3x_4+x_4^2+x_3^2$. Then $M_{a,b,c}$ is loccally conformally symmetric.*

Corrolary 4.1. *A strict four dimensional Walker of the form: $a=a(x_3,x_4)$, $b=b(x_3,x_4)$ and $c\equiv 0$ is locally conformally symmetric if and only if the* *functons a and b are solutions of the following system of partial differential equations:*

$$\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{\partial^3 b}{\partial x_3^3} = 0 \text{ and } \frac{\partial^3 b}{\partial x_3^2 \partial x_4} + \frac{\partial^3 a}{\partial x_4^3} = 0.$$

Corrolary 4.2. *A strict four-dimensional Walker of the form: $a = b = c = f(x_3, x_4)$ is locally conformally symmetric if and only a is solutions of the following system of partial differential equations:*

$$\frac{1}{2}\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{1}{2}\frac{\partial^3 a}{\partial x_3^3} - \frac{\partial^3 a}{\partial x_3^2 \partial x_4} = 0 \text{ and } \frac{1}{2}\frac{\partial^3 a}{\partial x_3^2 \partial x_4} + \frac{1}{2}\frac{\partial^3 a}{\partial x_4^3} - \frac{\partial^3 a}{\partial x_4^2 \partial x_3} = 0.$$

Corrolary 4.3. *A strict four-dimensional Walker of the form: $a = b = f(x_3, x_4)$ and $c \equiv 0$ is locally conformally symmetric if and only a is solutions of the following system of partial differential equations:*

$$\frac{\partial^3 a}{\partial x_4^2 \partial x_3} + \frac{\partial^3 a}{\partial x_3^3} = 0 \text{ and } \frac{\partial^3 a}{\partial x_3^2 \partial x_4} + \frac{\partial^3 a}{\partial x_4^3} = 0.$$

Corrolary 4.4. *A strict four-dimensional Walker of the form: $a = b \equiv 0$ and $c = f(x_3, x_4)$ is locally conformally symmetric if and only c is solutions of the following system of partial differential equations:*

$$\frac{\partial^3 c}{\partial x_3^2 \partial x_4} = 0 \text{ and } \frac{\partial^3 c}{\partial x_4^2 \partial x_3} = 0.$$

Recall tat, a pseudo-Riemannian manifold of dimension $m \geq 4$ is called essentially conformally symmetric if it is conformally symmetric (in the sense that its Weyl conformal tensor is parallel) without being conformally flat or locally symmetric.

Remark 4.2. *There does not existe a four strict Walker manifold which is essentially locally symmetric.*

Acknowledgments

The authors would like to thank the referee for his/her valuable suggestions and comments that helped them improve the paper.

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