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CHARACTERIZATIONS OF PARALLELOGRAMS PART 2

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Abstract. This is the concluding part in our extensive study of necessary and sufficient conditions for when a quadrilateral is a parallelogram. In the first part [27] we collected, categorized and proved 62 characterizations, and here we will do the same for an additional 41, making a total of just over a hundred known characterizations of parallelograms.

1. One-dimensional metric relations

In this section we will prove fourteen characterizations of parallelograms that are expressed as one-dimensional metric relations. A different formulation of condition *(b)* is that two medians in triangles ABC and CDA, drawn to opposite sides of the quadrilateral, are parallel and have the same length only when $ABCD$ is a parallelogram. To prove *(d)* was a problem on a Dutch Mathematical Olympiad in 2016 [37] and to prove that the trisection in *(f)* implies a parallelogram was Problem 1 on the Cono Sur Mathematical Olympiad in 2006 [1]. Condition *(g)* is based on Theorem 2.4 in [24], while *(i)* and its proof are related to Problem 225 in [44, pp. 119, 298–299]. Waldemar Pompe suggested *(n)* as a problem in 1997 [39], but our proof differs from the published solution. This also inspired us to consider similar configurations with subtriangle circumradii, medians and angle bisectors, which are *(k)*, *(l)* and *(m)*. With an *escribed circle* we mean a circle tangent to a side of a quadrilateral and the extensions of the two adjacent sides.

Theorem 1.1. *In a quadrilateral $ABCD$ with diagonal intersection P, let M_a, M_b, M_c, M_d be the midpoints of AB, BC, CD, DA respectively, and $m_A = AM_b$, $m_B = BM_c$, $m_C = CM_d$, $m_D = DM_a$ be medians in triangles ABC, BCD, CDA, DAB respectively. Also let r_A, r_B, r_C, r_D be the radii of the incircles in triangles DAB, ABC, BCD, CDA respectively, and R_1, R_2, R_3, R_4 be the radii in the escribed circles tangent to AB, BC, CD, DA*

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respectively. Finally let R_a, t_a, m_a, h_a be the circumradius, the angle bisector, the median, and the height respectively to the side AB in triangle ABP, with similar notations for the other sides of the quadrilateral. Then $ABCD$ satisfies any one of:

(a) equal distances from any two adjacent vertices to the opposite side
(b) $AM_c \parallel CM_a$ and $AM_c = CM_a$
(c) $AH_B = CH_D$ and $\angle ADH_D = \angle CBH_B$ where $\{H_B, H_D\} \in AC$ and $BH_B \perp AC$ and $DH_B \perp AC$
(d) $PX = PY$ where X and Y are the orthocenters of triangles ABP and CDP
(e) $AH = CF$ and $BG = DE$ where $\{E, G\} \in BD$ and $\{F, H\} \in AC$ such that AE, BF, CG, DH are angle bisectors
(f) $DR = RQ = QC$ where $Q = AI \cap DC$ and $R = BI \cap DC$, and where $I = M_dC \cap M_bD$
(g) the internal tangents to the incircles in triangles ABC, CDA and BCD, DAB, other than AC and BD, bisect diagonals BD and AC respectively
(h) $R_1 = R_3$ and $R_2 = R_4$
(i) $r_A = r_C$ and $r_B = r_D$
(j) $m_A = m_C$ and $m_B = m_D$
(k) $R_a = R_c$ and $R_b = R_d$
(l) $t_a = t_c$ and $t_b = t_d$
(m) $m_a = m_c$ and $m_b = m_d$
(n) $h_a = h_c$ and $h_b = h_d$

if and only if it's a parallelogram.

Proof. *(a)* This is a direct consequence of the theorem stating that two lines are parallel if and only if they are everywhere equidistant.

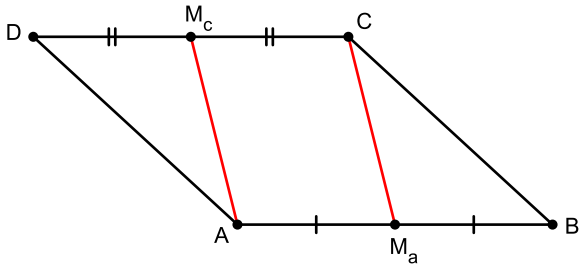

FIGURE 1. Two opposite midpoints

(b) In a parallelogram $ABCD$, $AB = DC$ and $AB \parallel DC$, so $AM_a = M_cC$ and $AM_a \parallel M_cC$ (see Figure 1). This implies that AM_aCM_c is a parallelogram according to Theorem 2.1 *(b)* in part 1; hence $AM_c = CM_a$ and $AM_c \parallel CM_a$.

Conversely, if in a quadrilateral we have $AM_c = CM_a$ and $AM_c \parallel CM_a$, then AM_aCM_c is a parallelogram, so $AM_a = M_aC$ and $AM_a \parallel M_cC$. This implies that $AB = DC$ and $AB \parallel DC$, which according to Theorem 2.1 *(b)* in part 1 confirms that $ABCD$ is a parallelogram.

(c) It holds that $AD = BC$ and $\angle BCA = \angle DAC$ in a parallelogram $ABCD$, so triangles ADH_D and CBH_B are congruent (AAS), see Figure 2. Then $\angle ADH_D = \angle CBH_B$ and $AH_D = CH_B$. Hence $AH_B = CH_D$.

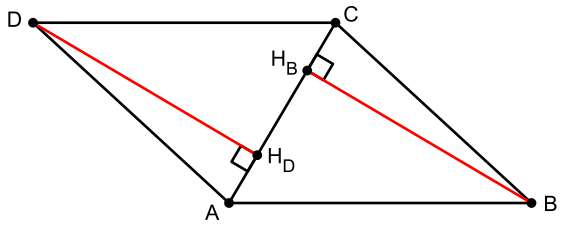

FIGURE 2. Two opposite triangle heights

Conversely, when $AH_B = CH_D$ and $\angle ADH_D = \angle CBH_B$ in a quadrilateral $ABCD$, we have that $AH_D = CH_B$, so triangles ADH_D and CBH_B are congruent (AAS). Then $AD = CB$ and $\angle BCA = \angle DAC$, so $ABCD$ is a parallelogram by Theorem 2.1 *(b)* in part 1.

(d) In a parallelogram $ABCD$, let AK and CL be heights in subtriangles ADP and BCP respectively, with $\{K, L\} \in BD$ (see Figure 3). Since $AP = CP$, right triangles AKP and CLP are congruent (AAS), so $KP = LP$. Then right triangles KPX and LPY are also congruent (AAS); hence $PX = PY$. This proof can be reversed, so that $PX = PY$ implies $AP = CP$, and also $BP = DP$ using two other pairs of congruent triangles. This proves that if $PX = PY$, then $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in part 1.

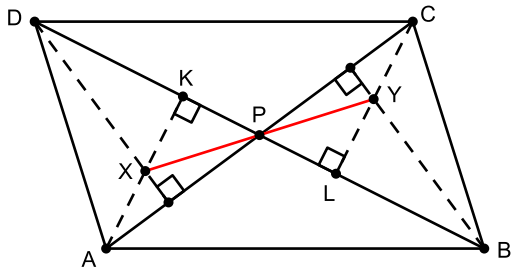

FIGURE 3. Two opposite subtriangle orthocenters

(e) In a parallelogram $ABCD$, we combine Theorem 8.1 *(a)* in part 1 with the angle bisector theorem applied twice (see Figure 4) to get

$$\frac{DG}{BG} = \frac{CD}{BC} = \frac{AB}{AD} = \frac{BE}{DE},$$

from which it follows that

$$\begin{aligned} & DE(DE + EG) = BG(BG + EG) \\ \Leftrightarrow \quad & DE^2 - BG^2 = BG \cdot EG - DE \cdot EG \\ \Leftrightarrow \quad & (DE - BG)(DE + BG + EG) = 0 \end{aligned}$$

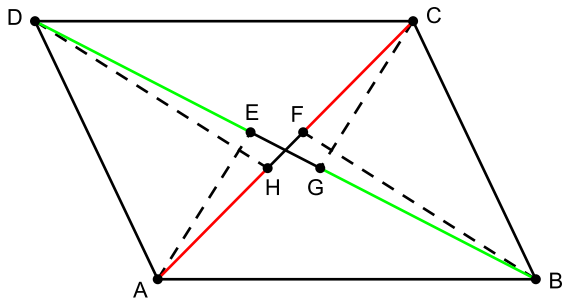

FIGURE 4. Four angle bisectors AE, BF, CG, DH

so $BG = DE$. In the same way we derive $AH = CF$.

Conversely, when $BG = DE$ we can reverse the steps to get

$$\frac{CD}{BC} = \frac{DG}{BG} = \frac{BE}{DE} = \frac{AB}{AD}$$

where we again applied the angle bisector theorem twice, so $AD \cdot CD = AB \cdot BC$. Similarly $AH = CF$ implies $AB \cdot AD = BC \cdot CD$. Hence $ABCD$ is a parallelogram according to Theorem 8.1 *(a)* in part 1.

(f) When $ABCD$ is a parallelogram, M_dM_bCD is also a parallelogram (see Figure 5), so $DI = IM_b$. Then $T_{DIB} = T_{M_bIB} = \frac{1}{2}T_{CIB}$ due to equal heights and the fact that M_b is the midpoint on BC, where T_{XYZ} denotes the area of triangle XYZ. We also get

$$\frac{T_{DRI}}{T_{CRI}} = \frac{DR}{RC} \equiv k \qquad \text{and} \qquad \frac{T_{DRB}}{T_{RCB}} = \frac{DR}{RC} = k$$

where k is a constant. Then

$$\frac{T_{DRI} + T_{DIB}}{T_{CRI} + T_{CIB}} = k.$$

Inserting $T_{DRI} = kT_{CRI}$ and simplifying yields $T_{DIB} = kT_{CIB}$. Together with $T_{DIB} = \frac{1}{2}T_{CIB}$, we get $\frac{DR}{RC} = k = \frac{1}{2}$, so $RC = 2DR$. In the same way, using $M_dI = IC$, we derive $DQ = 2QC$. Hence $DR = QC = RQ = \frac{1}{3}DC$.

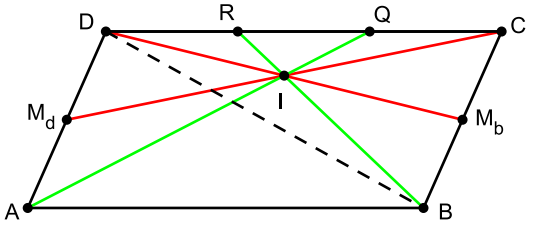

FIGURE 5. Parallelogram with $DR = RQ = QC$

Conversely, when $DR = QC = RQ$ hold in a quadrilateral, we get $T_{DRI} = \frac{1}{2}T_{RCI}$ and $T_{DRB} = \frac{1}{2}T_{RCB}$. These imply that $T_{DIB} = \frac{1}{2}T_{CIB} = T_{M_bIB}$ and we get $DI = IM_b$. In the same way we prove that $M_dI = IC$, so M_dM_bCD is a parallelogram due to bisecting diagonals. Since $AD = 2M_dD = 2M_bC =$

BC and $AD \parallel M_dD \parallel M_bC \parallel BC$, it follows that $ABCD$ is a parallelogram according to Theorem 2.1 *(b)* in part 1.

(g) In [24, pp. 90–92] it was proved that the internal tangent to the incircles in triangles ABC and CDA, other than AC, bisects diagonal BD if and only if $\angle B = \angle D$ by deriving the formula

$$\frac{BU}{DU} = \frac{\tan\frac{D}{2}}{\tan\frac{B}{2}}$$

where U is the intersection between this tangent and BD (see Figure 6). From that formula it directly follows that $BU = DU$ if and only if $\angle B = \angle D$.

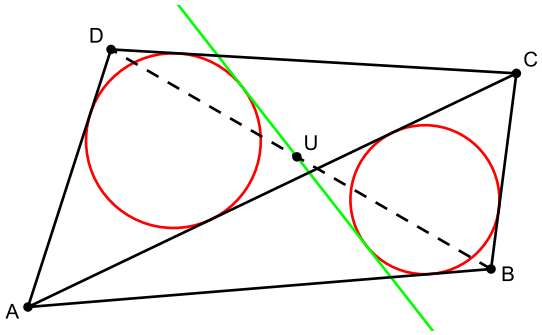

FIGURE 6. $BU = DU \quad \Leftrightarrow \quad \angle B = \angle D$

By symmetry there is the similar formula

$$\frac{AV}{CV} = \frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}$$

regarding the incircles in triangles BCD and DAB, where V is the intersection between this second internal tangent (other than BD) and AC. We have $AV = CV$ if and only if $\angle A = \angle C$. Hence both bisections occur if and only if both pairs of opposite angles are equal, which is equivalent to $ABCD$ being a parallelogram according to Theorem 4.1 *(a)* in part 1.

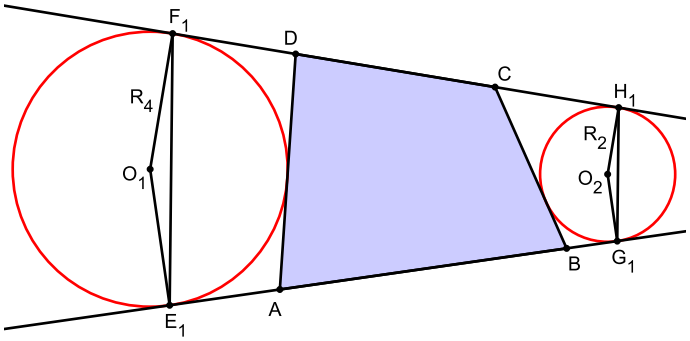

FIGURE 7. $\triangle E_1F_1O_1 \sim \triangle G_1H_1O_2$

(h) In Figure 7, let the extensions of the sides AB and CD be tangent to two of the escribed circles at E_1, F_1, G_1, H_1, and let O_1 and O_2 be the

centers of these circles. Then it's easy to see that isosceles triangles $E_1F_1O_1$ and $G_1H_1O_2$ are similar (AA), so

$$\frac{E_1F_1}{G_1H_1} = \frac{R_4}{R_2}.$$

We have that AB and CD are parallel if and only if $E_1F_1 = G_1H_1$, which is equivalent to $R_2 = R_4$. In the same way, BC and AD are parallel if and only if $R_1 = R_3$. Hence both of these radii equalities characterize parallelograms.

(i) In a parallelogram $ABCD$, triangles ABC and CDA are congruent, as are BCD and DAB. This implies that $r_B = r_D$ and $r_A = r_C$.

We prove the converse by way of contradiction. Suppose $r_B = r_D$ and $r_A = r_C$, but $ABCD$ is not a parallelogram (see Figure 8). First we construct A' and C' such that AA' and CC' are parallel to and have the same lengths as DB. Then $AA'C'C$ is a parallelogram, and triangles ABA', BCC', $A'BC'$ are congruent to triangles ABD, CBD, ADC respectively. Thus the inradii of triangles ABA' and CBC' are equal. Suppose without loss of generality that vertex B is located within triangle COC', where O is the diagonal intersection in $AA'C'C$. Then we have the following relations between different inradii, where three letter subscripts indicate the corresponding triangle:

$$r_A = r_{ABA'} > r_{AA'O} = r_{COC'} > r_{CBC'} = r_C$$

which is a contradiction to $r_A = r_C$. Hence $B = O$. This must also hold from the perspective of the two opposite triangles ABC and ADC. Thus opposite triangles ABC, CDA and BCD, DAB are congruent, implying that $ABCD$ is a parallelogram according to Theorem 3.1 *(c)* in part 1.

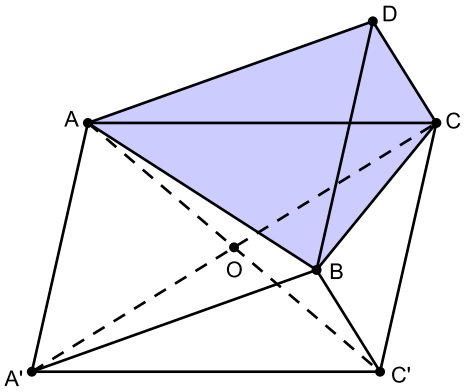

FIGURE 8. The constructed parallelogram $AA'C'C$

(One way of explaining the inradii inequalities used is via the triangle formula

$$r = \frac{AB}{\cot\frac{A}{2} + \cot\frac{B}{2}}.$$

We let the reader derive this formula and use it to fill in the missing steps in the proof.)

(j) In a parallelogram $ABCD$, triangles ABC and CDA are congruent, as are BCD and DAB. This implies that $m_A = m_C$ and $m_B = m_D$.

Conversely, applying the triangle median formula in a quadrilateral $ABCD$ with diagonals $p = AC$ and $q = BD$, we have $4m_A^2 = 2(a^2 + p^2) - b^2$ and $4m_C^2 = 2(c^2 + p^2) - d^2$ (see Figure 9), where $a = AB$, $b = BC$, $c = CD$, $d = DA$. Similar formulas hold for the other two medians, so

$$\begin{cases} 4(m_C^2 - m_A^2) = 2(c^2 - a^2) - (d^2 - b^2) \\ 4(m_D^2 - m_B^2) = 2(d^2 - b^2) - (a^2 - c^2). \end{cases}$$

Since $m_A = m_C$ and $m_B = m_D$, we get

$$\begin{cases} 2(c^2 - a^2) = d^2 - b^2 \\ 2(d^2 - b^2) = a^2 - c^2. \end{cases}$$

Inserting the first equation into the second yields $4(c^2 - a^2) = a^2 - c^2$, that is, $3(c^2 - a^2) = 0$, so $a = c$. This also implies $b = d$, which proves that $ABCD$ is a parallelogram according to Theorem 2.1 *(a)* in part 1.

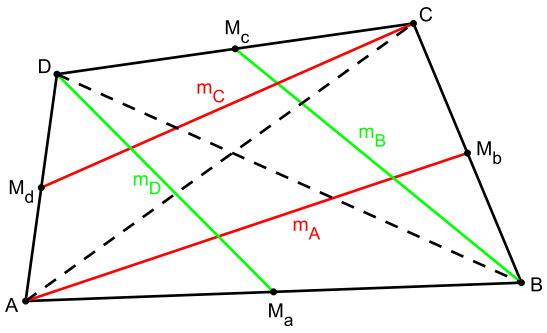

FIGURE 9. The medians m_A, m_B, m_C, m_D

(k) Applying the extended law of sines, we have that a quadrilateral with consecutive sides a, b, c, d is a parallelogram if and only if

$$\begin{cases} a = c \\ b = d \end{cases} \Leftrightarrow \begin{cases} 2R_a \sin\theta = 2R_c \sin\theta \\ 2R_b \sin(\pi - \theta) = 2R_d \sin(\pi - \theta) \end{cases} \Leftrightarrow \begin{cases} R_a = R_c \\ R_b = R_d \end{cases}$$

where θ is the angles between the diagonals opposite side a.

(l) Since opposite subtriangles created by the diagonals are congruent in a parallelogram according to Theorem 3.1 *(b)* in part 1, it follows that $t_a = t_c$ and $t_b = t_d$ hold.

Conversely, let the diagonal parts be $AP = w$, $BP = x$, $CP = y$, $DP = z$ in a quadrilateral $ABCD$. Applying the triangle formula for the length of an angle bisector, we get from $t_a^2 = t_c^2$ the equation

$$wx\left(1 - \frac{a^2}{(w+x)^2}\right) = yz\left(1 - \frac{c^2}{(y+z)^2}\right).$$

Putting both sides on equal denominators and using the law of cosines

$$(1) \qquad a^2 = w^2 + x^2 - 2wx\cos\theta \quad \text{and} \quad c^2 = y^2 + z^2 - 2yz\cos\theta,$$

where θ is one of the angles between the diagonals (see Figure 10), we get after some cancellations

$$\frac{2w^2x^2(1+\cos\theta)}{(w+x)^2} = \frac{2y^2z^2(1+\cos\theta)}{(y+z)^2}$$

and further, by canceling common non-zero factors, inverting and taking the square root, that

$$\frac{w+x}{wx} = \frac{y+z}{yz} \quad \Leftrightarrow \quad \frac{1}{x}+\frac{1}{w} = \frac{1}{z}+\frac{1}{y}. \tag{2}$$

By symmetry, we get a similar expression from $t_b^2 = t_d^2$, which is

$$\frac{1}{y}+\frac{1}{x} = \frac{1}{z}+\frac{1}{w}. \tag{3}$$

Now we subtract (3) from (2) to get

$$\frac{1}{w}-\frac{1}{y} = \frac{1}{y}-\frac{1}{w} \quad \Leftrightarrow \quad \frac{2}{w} = \frac{2}{y}$$

from which it follows that $w = y$, and thus also $x = z$ from (3). Hence the diagonals bisect each other, so $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in part 1.

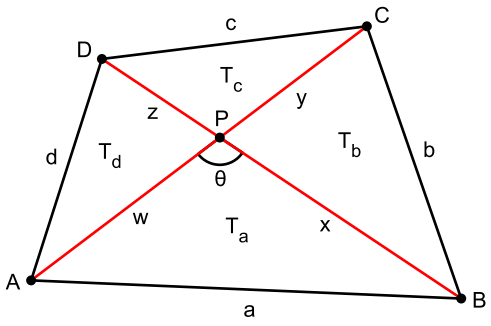

FIGURE 10. The diagonal parts and the four subtriangle areas

(m) Since opposite subtriangles created by the diagonals are congruent in a parallelogram, it's trivial that $m_a = m_c$ and $m_b = m_d$ hold.

Conversely, in a quadrilateral $ABCD$ where $AP = w$, $BP = x$, $CP = y$, $DP = z$, we apply the formula giving the length of a triangle median. Then we get that $4m_a^2 = 4m_c^2$ and $4m_b^2 = 4m_d^2$ implies

$$\begin{cases} 2(w^2+x^2)-a^2 = 2(y^2+z^2)-c^2 \\ 2(x^2+y^2)-b^2 = 2(z^2+w^2)-d^2 \end{cases} \tag{4}$$

where $a = AB$, $b = BC$, $c = CD$, $d = DA$. Applying the law of cosines, we have (1) and $b^2 = x^2+y^2+2xy\cos\theta$ and $d^2 = z^2+w^2+2zw\cos\theta$ since $\cos(\pi-\theta) = -\cos\theta$, so (4) can be rewritten as

$$\begin{cases} w^2+x^2+2wx\cos\theta = y^2+z^2+2yz\cos\theta \\ x^2+y^2-2xy\cos\theta = z^2+w^2-2zw\cos\theta \end{cases}$$

where θ is one of the angles between the diagonals (see Figure 10). Adding and subtracting these two equations yield after simplification

$$\begin{cases} 2(wx - xy - yz + zw)\cos\theta = 2(z^2 - x^2) \\ 2(wx + xy - yz - zw)\cos\theta = 2(y^2 - w^2) \end{cases}$$

which are factorized as

$$\begin{cases} 2(w - y)(x + z)\cos\theta = 2(z - x)(z + x) \\ 2(w + y)(x - z)\cos\theta = 2(y - w)(y + w) \end{cases}$$

and after dividing by common non-zero factors, we get

$$\begin{cases} (w - y)\cos\theta = z - x \\ (x - z)\cos\theta = y - w. \end{cases}$$

Substituting the first equation into the second yields

$$(y - w)(\cos^2\theta - 1) = 0$$

with $y = w$ as the only possible solution (since $\theta \neq 0$ and $\theta \neq \pi$). Then we get $x = z$ from $(w - y)\cos\theta = z - x$, which proves that $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in part 1.

(n) Since opposite subtriangles created by the diagonals are congruent in a parallelogram, it's trivial that $h_a = h_c$ and $h_b = h_d$ hold.

Conversely, consider a quadrilateral $ABCD$ with diagonal parts $AP = w$, $BP = x$, $CP = y$, $DP = z$. Expressing twice the area of triangle ABP in two different ways, we have $AB \cdot h_a = wx\sin\theta$ where θ is one of the angles between the diagonals (see Figure 10). Solving for the inverted height and applying the law of cosines yields

$$\frac{1}{h_a} = \frac{\sqrt{w^2 + x^2 - 2wx\cos\theta}}{wx\sin\theta} = \frac{1}{\sin\theta}\sqrt{\frac{1}{x^2} + \frac{1}{w^2} - \frac{2\cos\theta}{wx}}.$$

In the same way we also have

$$\frac{1}{h_c} = \frac{1}{\sin\theta}\sqrt{\frac{1}{z^2} + \frac{1}{y^2} - \frac{2\cos\theta}{yz}}$$

so $h_a = h_c$ implies

$$\frac{1}{w^2} + \frac{1}{x^2} - \frac{2\cos\theta}{wx} = \frac{1}{y^2} + \frac{1}{z^2} - \frac{2\cos\theta}{yz}.$$

By the same method and applying $\cos(\pi - \theta) = -\cos\theta$, we get from $h_b = h_d$ that

$$\frac{1}{w^2} + \frac{1}{z^2} + \frac{2\cos\theta}{wz} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{2\cos\theta}{xy}.$$

Adding and subtracting the last two equalities and simplifying yields

$$\begin{cases} 2\left(\frac{1}{w^2} - \frac{1}{y^2}\right) = 2\cos\theta\left(\frac{1}{wx} - \frac{1}{yz} + \frac{1}{xy} - \frac{1}{wz}\right) \\ 2\left(\frac{1}{x^2} - \frac{1}{z^2}\right) = 2\cos\theta\left(\frac{1}{wx} - \frac{1}{yz} - \frac{1}{xy} + \frac{1}{wz}\right) \end{cases}$$

which is equivalent to

$$\begin{cases} \frac{2(y-w)(y+w)}{w^2y^2} = \frac{2\cos\theta}{wxyz}(z - x)(y + w) \\ \frac{2(z-x)(z+x)}{x^2z^2} = \frac{2\cos\theta}{wxyz}(y - w)(z + x) \end{cases}$$

and further, after clearing common non-zero factors,

$$\begin{cases} xz(y-w) = wy(z-x)\cos\theta \\ wy(z-x) = xz(y-w)\cos\theta. \end{cases}$$

By substitution, we have

$$xz\cos\theta(y-w)\cos\theta = xz(y-w)$$

which is equivalent to

$$xz(y-w)(\cos^2\theta - 1) = 0$$

and we get $y = w$ as the only possible solution (since $\theta \neq 0$ and $\theta \neq \pi$). Then $z = x$ follows from $wy(z-x) = xz(y-w)\cos\theta$, so the quadrilateral is a parallelogram according to Theorem 3.1 *(a)* in part 1. □

2. Symmetry

In the next theorem we prove eight characterizations about symmetry. The first was used in the old book [17, p. 120] as a definition of a parallelogram and the related *(b)* was discussed in [51, pp. 24–25]. To prove *(e)* was Problem 5 on Round 1 of the British Mathematical Olympiad in 2006 [7, p. 27]. The proof we give is from [6], but another proof can be found in [36]. *(f)* is about a case of Anne's theorem that is often just regarded as an exception (see [25, pp. 146–147]). To prove that the penultimate characterization is a necessary condition in parallelograms was Problem 375 in [54]. We denote by T_{XYZ} the area of triangle XYZ. The last condition was a problem on the Leningrad High School Olympiad in 1981 according to [33], and a vector proof was given in [45, pp. 56–57]. The sufficient part was also Problem 4 (proposed by Xiong Bin) on the first day of the China Western Mathematical Olympiad in 2003 [56, pp. 180–181]. We cite this proof.

Theorem 2.1. *A quadrilateral $ABCD$ with diagonal intersection P satisfies any one of:*

- *(a) it has a center of symmetry*
- *(b) it has rotational symmetry of order 2*
- *(c) P is a bisector of the two segments connecting the intersection points of adjacent perpendicular bisectors of the sides*
- *(d) all line segments through P between two opposite sides is bisected by P*
- *(e) there is a point X inside $ABCD$ such that any line through X bisects the area of $ABCD$*
- *(f) $T_{ABY} + T_{CDY} = T_{BCY} + T_{DAY}$ for any interior point Y*
- *(g) the expression $T_{AYC}\cot\angle AYC - T_{BYD}\cot\angle BYD$ is independent of the location of the interior point Y*
- *(h) the sum of the distances from any interior point to the four sides is constant*

if and only if it's a parallelogram.

Proof. *(a)* That a quadrilateral has a center of symmetry means that each pair of opposite vertices are at equal distance from a common point. Hence

that point must be on each diagonal and the diagonals must bisect each other, so this is just another way of stating Theorem 3.1 *(a)* in part 1.

(b) If we rotate a quadrilateral around the midpoint of one of the diagonals, this characterization states that the quadrilateral overlaps itself after a half-turn if and only if it's a parallelogram. That means this is just another way of stating Theorem 3.1 *(c)* in part 1.

(c) In a parallelogram, triangles ABD and BCD are congruent, so they have equal circumradii. With notations as in Figure 11, this means that $BMDN$ is a rhombus, so its diagonals BD and MN intersect at their midpoint P. In the same way, P bisects UV.

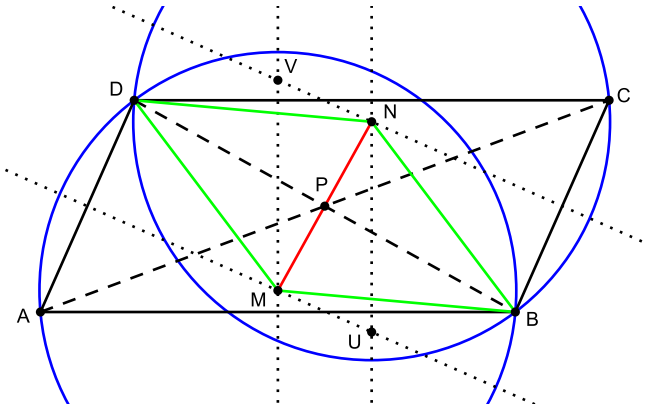

FIGURE 11. Intersections of the perpendicular bisectors

Conversely, suppose that P bisect MN and UV, then $MUNV$ is a parallelogram. This implies that $ABCD$ is also a parallelogram, since its sides are all perpendicular to sides of $MUNV$.

(d) In a parallelogram $ABCD$, draw an arbitrary line segment X_1X_2 through P with $X_1 \in AB$ and $X_2 \in CD$ (see Figure 12). Then triangles X_1BP and X_2DP are congruent (ASA), so $X_1P = X_2P$.

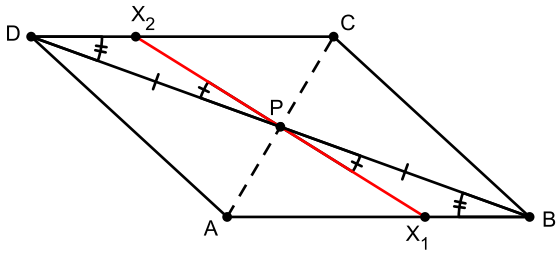

FIGURE 12. One bisected transversal

Conversely, in a quadrilateral $ABCD$ where any line segment through P between two opposite sides is bisected by P, draw two such transversals X_1X_2 and X_3X_4 with $\{X_1, X_3\} \in AB$ and $\{X_2, X_4\} \in CD$ (see Figure 13). Then X_1PX_3 and X_2PX_4 are congruent (SAS), so $\angle X_3X_1P = \angle X_4X_2P$, and thus $X_1X_3 \parallel X_2X_4$, that is, $AB \parallel CD$. In the same way it's proved that $AD \parallel BC$, so $ABCD$ is a parallelogram by definition.

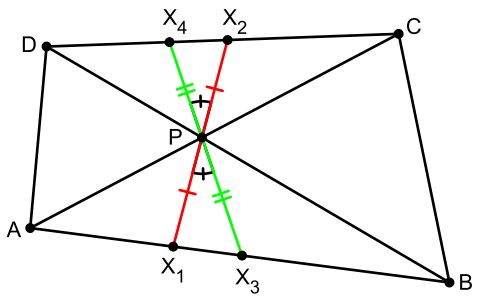

FIGURE 13. Two bisected transversals

(e) In a parallelogram $ABCD$, the point X is the diagonal intersection P. Let X_1X_2 be an arbitrary line segment trough P with $X_1 \in AB$ and $X_2 \in CD$ (see Figure 12). In order to prove that the areas of quadrilaterals AX_1X_2D and BX_1X_2C are the same, we note that triangles ABD and BDC have the same area, so we only need to show that triangles BPX_1 and DPX_2 have the same area. But these triangles are congruent (by ASA), so they have equal areas.

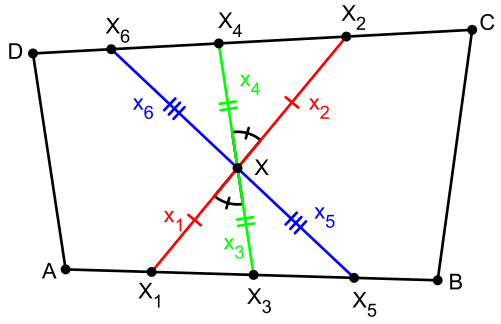

FIGURE 14. Three bisected transversals

Conversely, let $ABCD$ be a quadrilateral having an interior point X such that any line through X bisects its area. We draw three line segments X_1X_2, X_3X_4, X_5X_6 through X such that $\{X_1, X_3, X_5\} \in AB$ and $\{X_2, X_4, X_6\} \in CD$ (see Figure 14). By the assumption we have $[AX_1X_2D] = [X_1BCX_2]$ and $[AX_3X_4D] = [X_3BCX_4]$, where the square brackets denote area. Then we get

$$[AX_1X_2D] - [AX_3X_4D] = [X_1BCX_2] - [X_3BCX_4]$$

which is simplified into

$$[X_2X_4X] - [X_1X_3X] = [X_1X_3X] - [X_2X_4X]$$

and so $[X_1X_3X] = [X_2X_4X]$. If we denote $X_1X = x_1$, $X_3X = x_3$, $X_5X = x_5$, $X_2X = x_2$, $X_4X = x_4$, and $X_6X = x_6$, then

$$\tfrac{1}{2}x_2x_4 \sin \angle X_2XX_4 = \tfrac{1}{2}x_1x_3 \sin \angle X_1XX_3$$

and, since $\angle X_2XX_4 = \angle X_1XX_3$, this implies $x_2x_4 = x_1x_3$. In the same way it's proved that $x_4x_6 = x_3x_5$ and $x_2x_6 = x_1x_5$. Multiplying the first

two of these equalities and dividing by the third yields $x_4^2 = x_3^2$, so $x_3 = x_4$. Then we also get $x_1 = x_2$ and $x_5 = x_6$. It follows that triangles X_1X_3X and X_2X_4X are congruent (SAS), so $\angle X_3X_1X = \angle X_4X_2X$, confirming that $X_1X_3 \parallel X_2X_4$. Hence $AB \parallel DC$. In the same way it's proved that $AD \parallel BC$, so $ABCD$ is a parallelogram by definition.

(f) According to Anne's theorem (for a short proof, see [25]), any point Y in a quadrilateral $ABCD$ satisfying

$$T_{ABY} + T_{CDY} = T_{BCY} + T_{DAY}$$

lies on Newton's line, that is, the line through the midpoints M and N of the diagonals. That line is well-defined in all quadrilaterals but parallelograms, where $MN = 0$ according to Theorem 3.1 *(a)* in part 1. Hence parallelograms are the only quadrilaterals where Y can be any point inside the quadrilateral, since there is always a line trough two points Y and $M = N$.

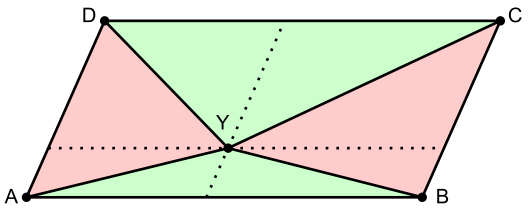

FIGURE 15. A proof without words

For readers who would like another proof, we note that there is the nice PWW of the necessary condition as seen in Figure 15. The sufficient condition can be proved with coordinate methods, applying that the area T of a triangle with vertex coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by $2T = x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3$ if the vertices are labeled in an appropriate way. We let the reader supply the details of this proof.

(g) The expression $4(T_{AYC} \cot \angle AYC - T_{BYD} \cot \angle BYD)$ is in a quadrilateral $ABCD$ equivalent to

$$2AY \cdot YC \sin \angle AYC \cdot \frac{\cos \angle AYC}{\sin \angle AYC} - 2BY \cdot YD \sin \angle BYD \cdot \frac{\cos \angle BYD}{\sin \angle BYD},$$

that is,

$$2AY \cdot YC \cos \angle AYC - 2BY \cdot YD \cos \angle BYD$$

and by applying the law of cosines in triangles AYC and BYD (see Figure 16) this can further be rewritten as

$$AY^2 + YC^2 - AC^2 - BY^2 - DY^2 + BD^2. \tag{5}$$

Now let us put the quadrilateral $ABCD$ in a coordinate system with $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, $D = (d, e)$ and $Y = (x, y)$. By applying the distance formula, (5) is equivalent to

$$x^2+y^2+(b-x)^2+(c-y)^2-b^2-c^2+(d-a)^2+e^2-(a-x)^2-y^2-(d-x)^2-(e-y)^2$$

which is expanded into

$$2x(a + d - b) + 2y(e - c) - 2ad. \tag{6}$$

When $ABCD$ is a parallelogram, then $c = e$ and $b = d+a$, so (6) is simplified to $-2ad$, which is independent of the point (x, y).

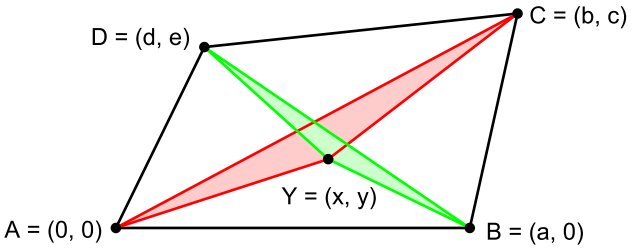

FIGURE 16. Quadrilateral with an arbitrary point Y

Conversely, if the original expression is independent of the point (x, y), then the coefficients of x and y in (6) must both be zero, yielding $a+d-b = 0$ and $e - c = 0$, that is, $a = b - d$ and $c = e$. These imply that the opposite sides AB and DC have equal length and are parallel, making $ABCD$ a parallelogram according to Theorem 2.1 *(b)* in part 1.

(h) In a parallelogram, the sum of the distances from any interior point to the four sides is equal to the sum of the two distances between opposite sides, which is a constant for a parallelogram.

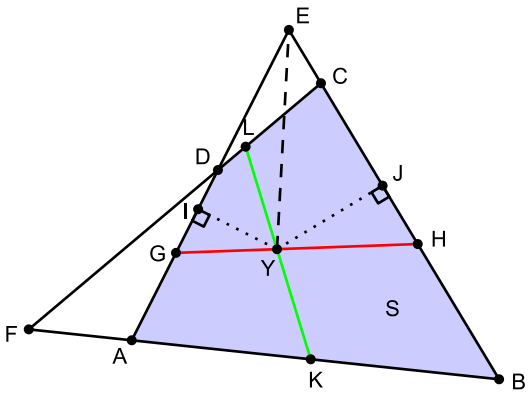

FIGURE 17. $EG = EH$ and $FK = FL$

Before we prove the converse, we first study a locus we need: *In an angle AEB (see Figure 17), the points Y for which the sum of distances to EA and EB is constant, let us call it m, is a line segment GH such that $EG = EH$. For any point Q inside triangle EGH, the sum of distances from Q to EA and EB is less than m, and it's greater than m if Q lies outside of triangle EGH (but still inside angle AEB).* To prove this, we note that all points on GH inside angle AEB satisfy $T_{EGY} + T_{EHY} = T_{EGH}$, so

$$\tfrac{1}{2}EG \cdot IY + \tfrac{1}{2}EH \cdot JY = \tfrac{1}{2}EG \cdot EH \sin\angle E.$$

It follows that $IY + JY = m$ where $EG = EH = \frac{m}{\sin\angle E}$. For any point Q inside triangle EGH, we have $T_{EGQ}+T_{EHQ} < T_{EGH}$, so $IQ+JQ < m$, and points Q outside EGH but still inside angle AEB satisfy $IQ + JQ > m$.

Now we prove the converse, that if the sum of the distances from any interior point to the sides is constant, then the quadrilateral is a parallelogram. There are two possible cases: one pair of opposite sides in $ABCD$ are parallel or neither pair of opposite sides are parallel. We only write the proof in the second (more general) case, but the first case is very similar and is left to the reader. Let opposite sides intersect at E and F as in Figure 17. Through a random point Y, we draw line segments GH and KL such that the sum of distances from any point on GH to AD and BC is constant, and the sum of distances from any point on KL to AB and CD is also constant. Then using the notation $d(Z, XY)$ for the distance between point Z and line segment XY, we get from the assumption and the locus we proved that for any point Q in the region marked S in Figure 17, we have

$$\begin{aligned} &d(Y, AB) + d(Y, BC) + d(Y, CD) + d(Y, DA) \\ =\, &d(Q, AB) + d(Q, BC) + d(Q, CD) + d(Q, DA) \\ =\, &[d(Q, AB) + d(Q, CD)] + [d(Q, BC) + d(Q, DA)] \\ >\, &[d(Y, AB) + d(Y, CD)] + [d(Y, BC) + d(Y, DA)] \end{aligned}$$

which is a contradiction. Hence the assumption that opposite sides are not parallel was wrong, so $ABCD$ is indeed a parallelogram. □

3. Vectors and regular polygons

Here we study thirteen characterizations concerning vectors or regular polygons. We write vectors in **bold** font. Amazingly, we have been unable to find a reference for the very simple condition *(a)* in the following theorem. Conditions *(b)*, *(c)*, *(d)* are from [52, p. 15], [10], [2] respectively. To prove *(e)* was given as an exercise (with a misprint) in [46, p. 65]. Proving the converse in *(f)* was Problem 3 for Grade 9 on the 2006 Romanian District Olympiad [13, p. 38], and we use the vector method from that proof here. The conditions *(h)*, *(j)*, *(l)* were proved in [15] and we cite those proofs here, but *(h)* had been proved earlier with a different method in [11]. The similar cases *(i)*, *(k)*, *(m)* were not covered in [15]. The *external van Aubel quadrilateral* to a quadrilateral is formed by erecting external squares on the sides of a quadrilateral and connecting the centers of adjacent squares, while the *internal van Aubel quadrilateral* is obtained when there are four internal squares on the quadrilateral sides.

Theorem 3.1. *In a quadrilateral $ABCD$ with diagonal intersection P, let $E = AN \cap BD$ and $F = DM \cap AC$ with M and N being the midpoints on AB and BC respectively. Then $ABCD$ satisfies any one of:*

(a) $\boldsymbol{PA} = -\boldsymbol{PC}$ *and* $\boldsymbol{PB} = -\boldsymbol{PD}$
(b) $\boldsymbol{AB} + \boldsymbol{CD} = \boldsymbol{0}$
(c) $\boldsymbol{OA} + \boldsymbol{OC} = \boldsymbol{OB} + \boldsymbol{OD}$ *for an arbitrary point O*
(d) $\boldsymbol{PA} + \boldsymbol{PB} + \boldsymbol{PC} + \boldsymbol{PD} = \boldsymbol{0}$
(e) $A + C = B + D$ *for complex numbers A, B, C, D*
(f) $BE = \frac{1}{3}BD$ *and* $AF = \frac{1}{3}AC$
(g) $BE = \frac{1}{3}BD$ *and* $EN = \frac{1}{3}AN$
(h) its external van Aubel quadrilateral is a square

(i) its internal van Aubel quadrilateral is a square
(j) $UVWX$ is a square, where U, V, W, X are the midpoints of the line segments joining neighboring free vertices of external squares erected on the sides
(k) $U'V'W'X'$ is a square, where U', V', W', X' are the midpoints of the line segments joining neighboring free vertices of internal squares erected on the sides
(l) triangle CIJ is equilateral, where I and J are vertices in external equilateral triangles ADI and BAJ
(m) triangle $CI'J'$ is equilateral, where I' and J' are vertices in internal equilateral triangles ADI' and BAJ'

if and only if it's a parallelogram.

Proof. *(a)* This is just the vector formulation of Theorem 3.1 *(a)* in part 1.

(b) We have that $\mathbf{AB}+\mathbf{CD}=\mathbf{0}$ is equivalent to $\mathbf{AB}=-\mathbf{CD}$, that is, $\mathbf{AB}=\mathbf{DC}$. Since equal vectors have the same length and direction, it's clear that this is the vector formulation of Theorem 2.1 *(b)* in part 1.

(c) The equalities $\mathbf{OA}+\mathbf{OC}=\mathbf{OA}+\mathbf{OB}+\mathbf{BC}$ and $\mathbf{OB}+\mathbf{OD}=\mathbf{OB}+\mathbf{OA}+\mathbf{AD}$ (see Figure 18) yield

$$\mathbf{OA}+\mathbf{OC}-(\mathbf{OB}+\mathbf{OD})=\mathbf{BC}-\mathbf{AD}$$

so

$$\mathbf{OA}+\mathbf{OC}=\mathbf{OB}+\mathbf{OD}\quad\Leftrightarrow\quad\mathbf{BC}=\mathbf{AD}.$$

which is equivalent to $ABCD$ being a parallelogram according to *(b)*, formulated for the other pair of opposite sides.

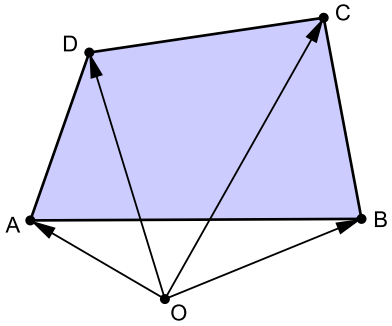

FIGURE 18. Vectors from O

(d) Let $\mathbf{PA}=\mathbf{a}$ and $\mathbf{PC}=k_1\mathbf{a}$ where $k_1<0$ is a real number, and similarly $\mathbf{PB}=\mathbf{b}$ and $\mathbf{PD}=k_2\mathbf{b}$ where $k_2<0$ also is a real number. Then $\mathbf{PA}+\mathbf{PB}+\mathbf{PC}+\mathbf{PD}=\mathbf{0}$ is equivalent to

$$\mathbf{a}+\mathbf{b}+k_1\mathbf{a}+k_2\mathbf{b}=0\quad\Leftrightarrow\quad\mathbf{a}(1+k_1)+\mathbf{b}(1+k_2)=0.$$

Since $\mathbf{a}\nparallel\mathbf{b}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, so we must have $1+k_1=0$ and $1+k_2=0$. We get $k_1=-1$ and $k_2=-1$, so $\mathbf{PA}=-\mathbf{PC}$ and $\mathbf{PB}=-\mathbf{PD}$. Then $ABCD$ is a parallelogram according to *(a)*.

(e) This is a special case of *(c)* when O is the origin of the complex plane, but we give another proof here. The midpoints of AC and BD are $\frac{1}{2}(A+C)$

and $\frac{1}{2}(B+D)$ respectively. Since $ABCD$ is a parallelogram if and only if the diagonals bisect each other, we directly get this characterization from

$$\tfrac{1}{2}(A+C) = \tfrac{1}{2}(B+D).$$

(f) In a parallelogram, DM and AP are medians in triangle ABD (see Figure 19), so

$$AF = \tfrac{2}{3}AP = \tfrac{2}{3}\cdot\tfrac{1}{2}AC = \tfrac{1}{3}AC.$$

In the same way in triangle ABC, we get $BE = \frac{1}{3}BD$.

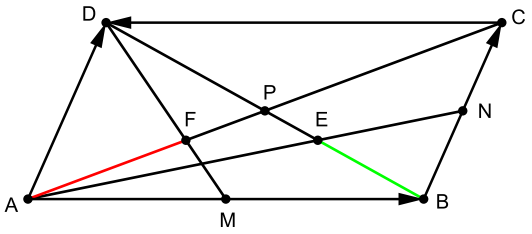

FIGURE 19. $BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$

Conversely, in a quadrilateral $ABCD$ where $BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$, we introduce the vectors $\mathbf{AB} = \mathbf{u}$ and $\mathbf{BC} = \mathbf{v}$. Then we can set $\mathbf{CD} = a\mathbf{u} + b\mathbf{v}$ for some real numbers a and b. It follows that $\mathbf{AN} = \mathbf{u} + \frac{1}{2}\mathbf{v}$ (see Figure 19) and

$$\mathbf{AD} = \mathbf{u} + \mathbf{v} + a\mathbf{u} + b\mathbf{v} = (a+1)\mathbf{u} + (b+1)\mathbf{v}$$

and also

$$\mathbf{BD} = \mathbf{BC} + \mathbf{CD} = a\mathbf{u} + (b+1)\mathbf{v}.$$

Using $\mathbf{BE} = \frac{1}{3}\mathbf{BD}$, we get

$$\mathbf{AE} = \mathbf{AB} + \mathbf{BE} = \left(\tfrac{1}{3}a + 1\right)\mathbf{u} + \tfrac{1}{3}(b+1)\mathbf{v}.$$

Since $E \in AN$, the vectors $\mathbf{AE}$ and $\mathbf{AN}$ are parallel, so

$$\mathbf{AE} \times \mathbf{AN} = \mathbf{0} \quad \Rightarrow \quad \left(\tfrac{1}{3}a + 1\right)\cdot\tfrac{1}{2} - \tfrac{1}{3}(b+1)\cdot 1 = 0$$

which simplifies into

$$a - 2b + 1 = 0. \tag{7}$$

Next we use $\mathbf{AF} = \frac{1}{3}\mathbf{AC}$ to get

$$\mathbf{DF} = -\mathbf{AD} + \mathbf{AF} = -\left(a + \tfrac{2}{3}\right)\mathbf{u} - \left(b + \tfrac{2}{3}\right)\mathbf{v}$$

and

$$\mathbf{DM} = -\mathbf{AD} + \mathbf{AM} = -\left(a + \tfrac{1}{2}\right)\mathbf{u} - (b+1)\mathbf{v}.$$

Since $F \in DM$, the vectors $\mathbf{DF}$ and $\mathbf{DM}$ are parallel, so $\mathbf{DF} \times \mathbf{DM} = \mathbf{0}$, which yields

$$-\left(a + \tfrac{2}{3}\right)(-1)(b+1) - (-1)\left(a + \tfrac{1}{2}\right)(-1)\left(b + \tfrac{2}{3}\right) = 0$$

and simplifies into

$$2a + b + 2 = 0. \tag{8}$$

Now solving the system of equations (7) and (8), we get $a = -1$ and $b = 0$, so we have $\mathbf{AB} = \mathbf{u}$ and $\mathbf{CD} = -\mathbf{u}$. Then $\mathbf{AB} + \mathbf{CD} = \mathbf{0}$, confirming that $ABCD$ is a parallelogram according to *(b)*.

(g) In a parallelogram $ABCD$, we have $EN = \frac{1}{3}AN$ since AN and BP are medians in triangle ABC (see Figure 19), and

$$BE = \tfrac{2}{3}BP = \tfrac{2}{3} \cdot \tfrac{1}{2}BD = \tfrac{1}{3}BD.$$

To prove the converse we use the same notations as in the proof of *(f)*, and by the exact same reasoning as there, we get $a-2b+1$ from $BE = \frac{1}{3}BD$. We find another equality between a and b by using $\mathbf{BD} = a\mathbf{u} + (1+b)\mathbf{v}$ and noting that $EN = \frac{1}{3}AN$ implies $AE = \frac{2}{3}AN$, so

$$\mathbf{BE} = -\mathbf{AB} + \mathbf{AE} = -\mathbf{AB} + \tfrac{2}{3}\mathbf{AN} = -\mathbf{u} + \tfrac{2}{3}\left(\mathbf{u} + \tfrac{1}{2}\mathbf{v}\right) = -\tfrac{1}{3}\mathbf{u} + \tfrac{1}{3}\mathbf{v}.$$

Then $\mathbf{BE} \times \mathbf{BD} = \mathbf{0}$ implies $-\frac{1}{3}(1+b) - \frac{1}{3}a = 0$, that is, $a + b + 1 = 0$. Now solving the system of equations

$$\begin{cases} a - 2b + 1 = 0 \\ a + b + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} a = -1 \\ b = 0 \end{cases}$$

so we get the same solution as in *(f)*, and thus the same conclusion.

(h) We put a quadrilateral $ABCD$ in the complex plane. Let $ADEF$ be an external square on AD with center Q. Then $\mathbf{DA} = A - D$, so $\mathbf{DE} = -i(A - D)$ and thus $E = D - i(A - D) = -iA + (1 + i)D$, see Figure 20, where i is the imaginary unit.

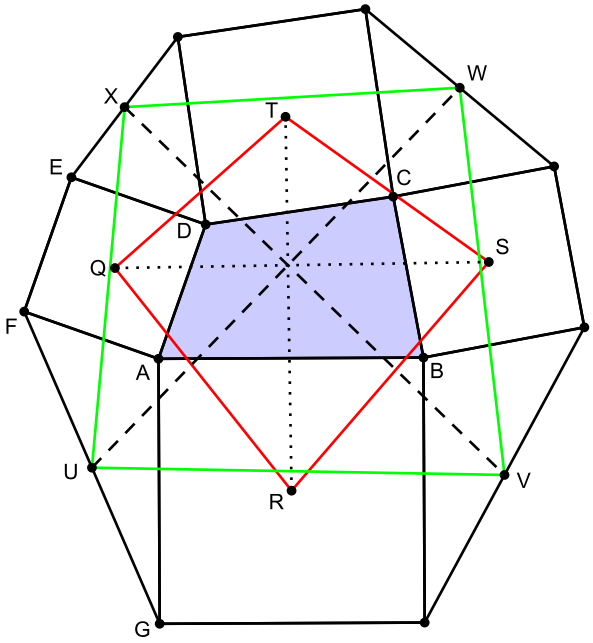

FIGURE 20. The external van Aubel quadrilateral $QRST$

We have that

$$Q = \frac{A + E}{2} = \frac{(1 - i)A + (1 + i)D}{2}$$

and similar for the other vertices of $ABCD$. It's well-known that the centers of the four external squares on the sides of a quadrilateral are vertices of another quadrilateral with diagonals that are equal and perpendicular (called van Aubel's theorem, see [5, pp. 67–68]). It's also well-known that a parallelogram is a square if and only if it has equal and perpendicular diagonals (see Theorem 4.1 (d) in [26, p. 18]), so $QRST$ is a square if and only if it's a parallelogram, which by *(e)* is equivalent to $Q + S = R + T$. This is equivalent to

$$\begin{aligned}&\frac{(1-i)A+(1+i)D}{2}+\frac{(1-i)C+(1+i)B}{2}\\ =\ &\frac{(1-i)B+(1+i)A}{2}+\frac{(1-i)D+(1+i)C}{2}\end{aligned}$$

which is simplified into $-i(A + C) = -i(B + D)$, and by *(e)*, the latter is equivalent to $ABCD$ being a parallelogram.

(i) The only difference between this case and the one in *(h)* is the direction of the four squares. Closely examining the proof of *(h)*, we see that we get this case by rotating a few vectors clock-wise instead of counter clock-wise, which in the end corresponds to multiplying by i instead of $-i$. Hence we can simply change the sign of all i in that proof, so this case yields $i(A + C) = i(B + D)$, which is true if and only if $ABCD$ is a parallelogram according to *(e)*.

(j) We use the same notations as in *(h)*, and by similar arguments we get $F = (1 - i)A + iD$ and $G = -iB + (1 + i)A$ (see Figure 20). Thus

$$U = \frac{G+F}{2} = \frac{2A-iB+iD}{2}$$

and similar for the other midpoints between adjacent vertices on adjacent external squares. In the same way as in *(h)* we have that $UVWX$ is a square if and only if it's a parallelogram, which by *(e)* is equivalent to $U + W = V + X$. This in turn is equivalent to

$$\frac{2A-iB+iD}{2}+\frac{2C-iD+iB}{2}=\frac{2B-iC+iA}{2}+\frac{2D-iA+iC}{2}$$

which is simplified to $A + C = B + D$. This last equality holds if and only if $ABCD$ is a parallelogram according to *(e)*.

(k) As in *(i)*, we get this case by simply changing the sign of all i in the proof of *(j)*, still resulting in $A + C = B + D$.

(l) We will use the fact that a triangle XYZ in the complex plane is equilateral and *positively* oriented if and only if $X + \omega Y + \omega^2 Z = 0$, where $\omega = e^{2i\pi/3}$ (see [5, p. 77]). Since IAD and JBA are positively oriented equilateral triangles (see Figure 21), we have $I = -\omega A - \omega^2 D$ and $J = -\omega B - \omega^2 A$. Hence triangle CIJ is equilateral if and only if $C + \omega I + \omega^2 J = 0$, which is equivalent to

$$C + \omega(-\omega A - \omega^2 D) + \omega^2(-\omega B - \omega^2 A) = 0. \tag{9}$$

We have that $\omega^3 = 1$ and $\omega + \omega^2 = -1$ via Euler's formula. Thus (9) can be simplified into $A + C = B + D$, which is equivalent to $ABCD$ being a parallelogram according to *(e)*.

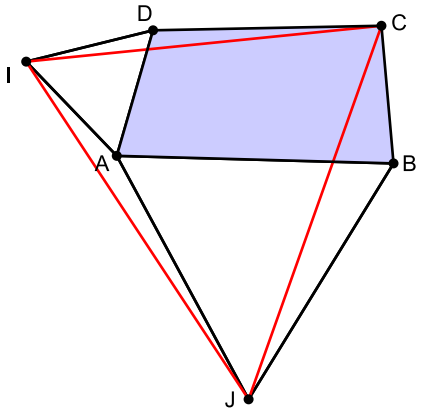

FIGURE 21. Triangle CIJ

(m) According to [5, p. 78], a triangle XYZ in the complex plane is equilateral and *negatively* oriented if and only if $X + \omega^2 Y + \omega Z = 0$, where $\omega = e^{2i\pi/3}$. Making this change in the proof of *(l)*, we get that triangle $CI'J'$ is equilateral if and only if $A + C = B + D$. The details are left to the reader. □

4. Coincidences

In a quadrilateral, the *vertex centroid* G_0 is defined to be the center of mass of four equal masses placed at the vertices (it's also the intersection of the bimedians), the *side centroid* G_1 is the center of mass of four rods of equal uniform density placed over the sides, and the *area centroid* G_2 is the center of mass of a lamina of uniform density placed over the whole quadrilateral. Note that G_0, G_1, G_2 are in general not the same point. Another, not so well-known concept, is an *equipartitioning point*, which means a point Q such that a quadrilateral $ABCD$ is partitioned into four triangles ABQ, BCQ, CDQ, DAQ with *equal* area. Finally, a bisect-diagonal quadrilateral is a quadrilateral where at least one diagonal bisects the other diagonal, and we have previously studied them in section 7 of part 1.

The last theorem in this paper is about six coincidences that characterize parallelograms. The first five and their proofs are from [2, pp. 234–236] while the sixth is cited from [38, p. 20]. We have made some minor modifications to the original proofs to better fit our narrative, both in presentation and notations. Another proof of *(c)* was given in [19].

Theorem 4.1. *A convex quadrilateral $ABCD$ with vertex centroid G_0, side centroid G_1, area centroid G_2, and diagonal intersection P satisfies any one of:*

(a) $G_0 = P$
(b) $G_2 = P$
(c) $G_0 = G_2$
(d) $G_0 = G_1$
(e) $G_1 = P$

(f) it's a bisect-diagonal quadrilateral with equipartitioning point G_2 if and only if it's a parallelogram.

Proof. *(a)* In the proofs of this and the following four coincidences, we shall consider a quadrilateral $ABCD$ in a coordinate system having the diagonal intersection P as its origin O. Then it's well-known that the vertex centroid G_0 has the position vector

$$\mathbf{OG}_0 = \frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}}{4} \tag{10}$$

so we have

$$G_0 = P \quad \Leftrightarrow \quad \mathbf{OG}_0 = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD} = \mathbf{0}$$

which by Theorem 3.1 *(d)* is equivalent to $ABCD$ being a parallelogram.

(b) The position vector of G_2 is

$$\mathbf{OG}_2 = \frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}}{3} \tag{11}$$

according to [2, p. 234], so

$$G_2 = P \quad \Leftrightarrow \quad \mathbf{OG}_2 = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD} = \mathbf{0}.$$

This again is equivalent to $ABCD$ being a parallelogram.

(c) We have that $G_0 = G_2$ if and only if $\mathbf{OG}_0 = \mathbf{OG}_2$, which by (10) and (11) is equivalent to

$$\frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}}{4} = \frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}}{3}.$$

The last equality holds if and only if $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD} = 0$, which characterizes parallelograms according to Theorem 3.1 *(d)*.

(d) The position vector of G_1 in a quadrilateral $ABCD$ with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ is

$$\mathbf{OG}_1 = \frac{(d+a)\mathbf{OA} + (a+b)\mathbf{OB} + (b+c)\mathbf{OC} + (c+d)\mathbf{OD}}{2(a+b+c+d)} \tag{12}$$

according to [2, p. 234] (they have used other notations on the sides). In a parallelogram, where $a = c$ and $b = d$, we have

$$\begin{aligned} \mathbf{OG}_1 &= \frac{(b+a)\mathbf{OA} + (a+b)\mathbf{OB} + (b+a)\mathbf{OC} + (a+b)\mathbf{OD}}{2(a+b+a+b)} \\ &= \frac{(a+b)(\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD})}{4(a+b)} = \mathbf{OG}_0 \end{aligned} \tag{13}$$

so $G_0 = G_1$.

Conversely, in a quadrilateral where $G_0 = G_1$, it holds that

$$\begin{aligned} &\frac{(d+a)\mathbf{OA} + (a+b)\mathbf{OB} + (b+c)\mathbf{OC} + (c+d)\mathbf{OD}}{2(a+b+c+d)} \\ &= \frac{\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}}{4} \end{aligned}$$

and from this equality, we get

$$\begin{aligned} &2(d+a)\mathbf{OA} + 2(a+b)\mathbf{OB} + 2(b+c)\mathbf{OC} + 2(c+d)\mathbf{OD} \\ &= (a+b+c+d)(\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}) \end{aligned}$$

which is equivalent to

$$(a-b-c+d)(\mathbf{OA}-\mathbf{OC})+(a+b-c-d)(\mathbf{OB}-\mathbf{OD})=0.$$

Since $\mathbf{OA}\neq\mathbf{OC}$ and $\mathbf{OB}\neq\mathbf{OD}$ and the fact that the diagonals cannot be parallel in a convex quadrilateral, we get

$$\begin{cases} a-b-c+d=0 \\ a+b-c-d=0. \end{cases}$$

This shows that $ABCD$ is a parallelogram according to Theorem 3.1 *(d)* in part 1.

(e) We have that $G_1=P$ in a parallelogram according to (13) and Theorem 3.1 *(d)*.

Conversely, if $G_1=P=O=0$ in a quadrilateral (the diagonal intersection is still chosen as origin), we get

$$\mathbf{OG}_1=(d+a)\mathbf{OA}+(a+b)\mathbf{OB}+(b+c)\mathbf{OC}+(c+d)\mathbf{OD}=0$$

from (12). This implies that

$$(d+a)\mathbf{OA}+(b+c)\mathbf{OC}=(a+b)\mathbf{OB}+(c+d)\mathbf{OD}=0 \tag{14}$$

since the diagonals in a convex quadrilateral cannot be parallel, so the vectors $\mathbf{AC}$ and $\mathbf{BD}$ are linearly independent. By denoting the diagonal parts $AP=w$, $BP=x$, $CP=y$, $DP=z$, (14) can be rewritten as

$$(d+a)w=(b+c)y \qquad \text{and} \qquad (a+b)x=(c+d)z \tag{15}$$

since the vectors from O to opposite vertices have different directions.

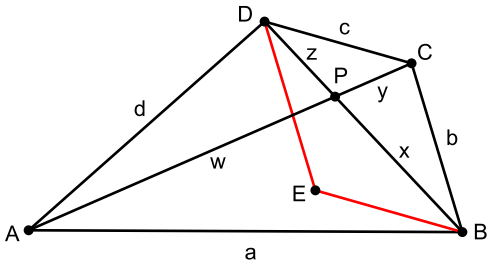

FIGURE 22. $BCDE$ is a parallelogram

Next, we may assume without loss of generality that we have the area inequalities

$$T_{BPA}\geq T_{CPD} \qquad \text{and} \qquad T_{APD}\geq T_{CPB}. \tag{16}$$

This implies $T_{ABC}\geq T_{BCD}$, and further that the height from A to BC is greater than or equal to the height from D to BC (see Figure 22). We conclude that a line from D, parallel to BC, intersect the side AB and that a line from B, parallel to CD, intersect the side DA. Their point of intersection E must then lie inside triangle ABD. By construction, $BCDE$ is a parallelogram. Then we have

$$DA+AB\geq DE+EB$$

according to Proposition I.21 in Euclid's *Elements* [29], with equality if and only if $E = A$, which we can restate as $d + a \geq b + c$ using properties of the parallelogram $BCDE$, where there is equality if and only if $ABCD$ is a parallelogram.

By (16) in combination with the law of sines, we get $wx \geq yz$ and $wz \geq xy$. Multiplying these and canceling common factors yields $w \geq y$ with equality if and only if $wx = yz$ and $wz = xy$, that is, $w = y$ and $x = z$. Hence $w \geq y$ with equality if and only if $ABCD$ is a parallelogram by Theorem 3.1 *(a)* in part 1. Multiplying $w \geq y$ and $d + a \geq b + c$, we have

$$(d + a)w \geq (b + c)y$$

with equality if and only if $ABCD$ is a parallelogram. According to (15), $(d + a)w = (b + c)y$, so we conclude that $ABCD$ is a parallelogram.

(f) In [12, pp. 68–70] and [38, pp. 17–18] it was proved that a quadrilateral has an equipartitioning point Q if and only if one diagonal bisects the other diagonal, and then Q is the midpoint of the bisecting diagonal. First we shall prove that in a parallelogram, the equipartitioning point Q coincide with the area centroid G_2. From *(b)* we know that $G_2 = P$ in a parallelogram. But the diagonals partition a parallelogram into four subtriangles of equal area according to Theorem 8.1 *(h)* in part 1, so we have that the equipartitioning point $Q = P$. Hence $G_2 = Q$.

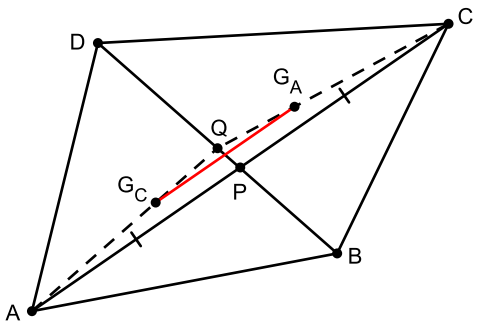

FIGURE 23. Triangle centroids G_A and G_C

Conversely, we shall prove that if $G_2 = Q$ in a bisect-diagonal quadrilateral, then it must be a parallelogram. Again by [12, pp. 68–70] and [38, pp. 17–18], we may assume without loss of generality that $AP = CP$ and $BQ = DQ$. Now suppose that AQC is not a straight line (see Figure 23). Denote by G_A and G_C the area centroids of triangles CBD and ABD respectively. Then G_A and G_C lie on CQ and AQ respectively and G_AQG_C is a triangle. But it's well-known that the area centroid G_2 in a quadrilateral lies on the line segment G_AG_C (see [3, p. 7]). Thus we have a contradiction: $G_2 = Q$ and $G_2 \neq Q$. Hence the assumption that AQC is not a straight line was wrong, so Q lies on AC (i.e. $Q = P$). Then AC and BD bisect each other, so $ABCD$ is a parallelogram according to Theorem 3.1 *(a)* in part 1. □

What about quadrilaterals satisfying $G_1 = G_2$? This equality holds in parallelograms, but the converse is not true. In [30] it's proved that there exist quadrilaterals without symmetry where these two centroids coincide.

5. Chronological compilation

Here we summarize all 103 characterizations of parallelograms we have studied in the two parts of this paper. They are given in chronological order with respect to the oldest source for the *sufficient* condition that we know of, but some of them have surely been published earlier.

If it's not specified which type of quadrilateral a characterization is about, it concerns an arbitrary convex quadrilateral $ABCD$ (abbreviated quad), with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$, diagonals $AC = p$, $BD = q$ and diagonal intersection P. The area of triangles ABP, BCP, CDP, DAP are denoted by T_a, T_b, T_c, T_d respectively. For the meaning of other notations, see the corresponding theorems.

BriMO, CroMO, DutMO, LenMO, MedMO, MexMO, and MolMO, are used as abbreviations for the British, Croatian, Dutch, Leningrad, Mediterranean, Mexican, and Moldovan Mathematical Olympiad respectively, and BDQ stands for bisect-diagonal quadrilateral.

When there are at least two possible cases (for instance in characterization number 2, where *one pair of opposite parallel equal sides* can be $AB \parallel DC$ and $AB = DC$ or $BC \parallel AD$ and $BC = AD$), only one case is given, and this is marked with the symbol ‡ in the third column.

#	Year	Source	Short description	Ref.
1	-300	Euclid	$AB \parallel DC$ and $BC \parallel AD$ (def.)	[29]
2	-300	Euclid‡	$AB \parallel DC$ and $AB = DC$	[29]
3	1750	Euler	$a^2 + b^2 + c^2 + d^2 = p^2 + q^2$	[41]
4	1794	Legendre	$AB = DC$ and $BC = AD$	[34]
5	1833	Young‡	$\triangle ABC \cong \triangle CDA$	[57]
6	1833	Young‡	$\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$	[57]
7	1833	Young	$\angle A = \angle C$ and $\angle B = \angle D$	[57]
8	1833	Young	$AP = CP$ and $BP = DP$	[57]
9	1833	Young	Any two adjacent angles are supplementary	[57]
10	1833	Young	$\triangle ABP \cong \triangle CDP$ and $\triangle BCP \cong \triangle DAP$	[57]
11	1871	Todhunter	Each diagonal bisects the area	[47]
12	1879	Henrici	It has a center of symmetry	[17]
13	1896	Halsted‡	$AB = DC > AC$ and $\angle ACB = \angle CAD$	[16]
14	1896	Halsted‡	$AB = DC < AC$ and $\angle B = \angle D$	[16]
15	1964	Spiegel	$A + C = B + D$ for $\{A, B, C, D\} \in \mathbb{C}$	[46]
16	1967	Keedy	Trapezoid with $AB \parallel DC$ and $BC \parallel AD$	[51]
17	1972	Eperson	$\mathbf{OA} + \mathbf{OC} = \mathbf{OB} + \mathbf{OD}$ for arbitrary pt O	[10]
18	1973	Sagar‡	$\mathbf{AB} + \mathbf{CD} = \mathbf{0}$	[52]
19	1977	Milenkovic	$T_{ABY} + T_{CDY} = T_{BCY} + T_{DAY}$ $\forall Y \in$ quad	[42]
20	1980	[LenMO]	Sum of bimedians equals semiperimeter	[31]
21	1981	[LenMO]	Sum of dist. fr. any int. pt to sides is const.	[33]
22	1985	Wagner	$T_a = T_c$ and $T_b = T_d$	[32]
23	1988	Okumura	$\frac{\angle CAD}{\angle CAB} = \frac{\angle ACB}{\angle ACD}$ and $\frac{\angle DBA}{\angle DBC} = \frac{\angle BDC}{\angle BDA}$	[35]
24	1993	Hoehn	$da\cos A + ab\cos B + bc\cos C + cd\cos D = 0$	[18]
25	1993	Seimiya	$PA\sin A = PC\sin C$ & $PB\sin B = PD\sin D$	[43]
26	1994	Toumasis‡	Trapezoid with two opposite equal angles	[48]
27	1994	Toumasis‡	Trapezoid where one diag. bisects the other	[48]

28	1994	Toumasis‡	$\angle B = \angle D$ and $AP = CP$	[48]
29	1994	Toumasis‡	$\angle B = \angle D$ and $BP = DP$ and $BD \not\perp AC$	[48]
30	1996	[MexMO]‡	E and F are the midpoints of BC and CD	[55]
31	1996	[CroMO]‡	$T_b^2 = T_a T_c$ and $2T_d = T_a + T_c$	[21]
32	1997	Pompe	$h_a = h_c$ and $h_b = h_d$	[39]
33	2000	[MedMO]	$a^2 + b^2 + c^2 + d^2 = \frac{4}{5}(m_A^2 + m_B^2 + m_C^2 + m_D^2)$	[4]
34	2001	[MolMO]‡	Trapezoid ($AD \parallel BC$) with $AO = 4OM_c$	[20]
35	2005	Tydd	2 bimedians and 2 diagonals are concurrent	[49]
36	2006	[BriMO]	Any line thru X bisects the area of the quad	[7]
37	2006	Iurea‡	$BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$	[13]
38	2006	[Cono Sur]‡	$DR = RQ = QC$	[1]
39	2008	Usiskin	Rotational symmetry of order 2	[51]
40	2009	Al-Sharif	$\mathbf{PA + PB + PC + PD = 0}$	[2]
41	2009	Al-Sharif‡	Each diagonal bisects the perimeter	[2]
42	2009	Al-Sharif	$G_0 = P$	[2]
43	2009	Al-Sharif	$G_2 = P$	[2]
44	2009	Al-Sharif	$G_0 = G_2$	[2]
45	2009	Al-Sharif	$G_0 = G_1$	[2]
46	2009	Al-Sharif	$G_1 = P$	[2]
47	2009	Webb	Bimedians have constant length	[9]
48	2010	Pillay‡	Bisectdiag. quad w. equipart. point $= G_2$	[38]
49	2013	Pop	$T_{ABC} \le T_{CDA} \le T_{BCD} \le T_{DAB}$	[40]
50	2013	Pop‡	Trapezoid w. 1 diag. bisecting the perimeter	[40]
51	2013	Fritsch	External van Aubel quadrilateral is a square	[11]
52	2016	Ufnarovski‡	Two pairs of opp. parallel angle bisectors	[50]
53	2016	Ufnarovski‡	1 angle bisect. is perp. to 2 adj. angle bisect.	[50]
54	2016	Ufnarovski‡	Eq. dist. from opp. vertices to corresp. diag.	[50]
55	2016	Ufnarovski	$T_a = T_b = T_c = T_d$	[50]
56	2016	[DutMO]‡	$PX = PY$	[37]
57	2016	[brainly]	All four angle bisectors form a rectangle	[53]
58	2017	Josefsson‡	BDQ ($AP = CP$) with $a^2 + b^2 = c^2 + d^2$	[23]
59	2020	Alsina	Trapezoid ($AB \parallel DC$) w. $T_a + T_c = T_b + T_d$	[3]
60	?	?	Bimedians quadrisect the area	[8]
61	2022	[JMAP]‡	$AH_B = CH_D$ and $\angle ABH_D = \angle CDH_B$	[22]
62	2023	Hajja‡	Each bimedian bisects the area	[14]
63	2023	Hajja‡	Each bimedian bisects the perimeter	[14]
64	?	?‡	$T_b = T_d = \frac{1}{4}K$	[14]
65	2023	Hajja	$UVWX$ is a square	[15]
66	2023	Hajja‡	Triangle CIJ is equilateral	[15]
67	2024	Josefsson	$w + x = y + z$ and $z + w = x + y$	[27]
68	2024	Josefsson‡	$T_{DAX} = T_{ABX}$ and $T_{BCX} = T_{CDX}$, $X \in AC$	[27]
69	2024	Josefsson‡	Sum of any 2 adjacent angles is a constant	[27]
70	2024	Josefsson	Angles btw. extensions of opp. sides are zero	[27]
71	2024	Josefsson	Bimedians quadrisect the perimeter	[27]
72	2024	Josefsson	Bimedians intersect at a diagonal midpoint	[27]
73	2024	Josefsson‡	Trapezoid w. 1 diagonal bisecting the area	[27]
74	2024	Josefsson‡	Trapezoid ($AB \parallel DC$) with $QI = RI$	[27]
75	2024	Josefsson‡	Trapezoid ($AB \parallel DC$) with $BJ = \frac{1}{3}BD$	[27]
76	2024	Josefsson‡	Trapezoid ($AB \parallel DC$) w. $AE = \sqrt{EF \cdot EG}$	[27]
77	2024	Josefsson‡	Trapezoid ($AB \parallel DC$) with $T_a = T_c$	[27]
78	2024	Josefsson‡	BDQ ($AP = PC$) with $IP = PJ$	[27]
79	2024	Josefsson‡	BDQ ($AP = PC$) with $BK = DL$	[27]
80	2024	Josefsson‡	BDQ ($AP = PC$) with $\angle MAP = \angle NCP$	[27]
81	2024	Josefsson	$ab = cd$ and $da = bc$	[27]
82	2024	Josefsson	$(a - c)^2 + (b - d)^2 = 0$	[27]
83	2024	Josefsson	$(a + c)^2 + (b + d)^2 = 2(p^2 + q^2)$	[27]
84	2024	Josefsson	$a^2 + b^2 + c^2 + d^2 = 2(m^2 + n^2)$	[27]

85	2024	Josefsson	$K = \frac{1}{2}\sqrt{(s^2-p^2)(s^2-q^2)}$	[27]
86	2024	Josefsson‡	Eq. dist. fr. any 2 adj. vertices to opp. side	[28]
87	2024	Josefsson‡	$AM_c = CM_a$ and $AM_c \parallel CM_a$	[28]
88	2024	Josefsson	$AH = CF$ and $BG = DE$	[28]
89	2024	Josefsson	Internal tang. to opp. incircles bisect diag.	[28]
90	2024	Josefsson	$R_1 = R_3$ and $R_2 = R_4$	[28]
91	2024	Josefsson	$r_A = r_C$ and $r_B = r_D$	[28]
92	2024	Josefsson	$m_A = m_C$ and $m_B = m_D$	[28]
93	2024	Josefsson	$R_a = R_c$ and $R_b = R_d$	[28]
94	2024	Josefsson	$t_a = t_c$ and $t_b = t_d$	[28]
95	2024	Josefsson	$m_a = m_c$ and $m_b = m_d$	[28]
96	2024	Josefsson	P bisects MN and UV	[28]
97	2024	Josefsson	Any line segment thru P is bisected by P	[28]
98	2024	Josefsson	$T_{AYC}\cot AYC - T_{BYD}\cot BYD$ indep. of Y	[28]
99	2024	Josefsson	$\mathbf{PA} = -\mathbf{PC}$ and $\mathbf{PB} = -\mathbf{PD}$	[28]
100	2024	Josefsson‡	$BE = \frac{1}{3}BD$ and $EN = \frac{1}{3}AN$	[28]
101	2024	Josefsson	Internal van Aubel quadrilateral is a square	[28]
102	2024	Josefsson	$U'V'W'X'$ is a square	[28]
103	2024	Josefsson‡	Triangle $CI'J'$ is equilateral	[28]

TABLE 1. Characterizations of parallelograms

6. Concluding remarks

We have studied 103 characterizations of parallelograms in the two parts of this paper, but how many of these were used in the proofs of other characterizations? After doing a count of how the sufficient conditions in each theorem were proved, we can conclude that 19 different characterizations have been used for that purpose. It will probably not come as a surprise for the reader which conditions that have been the most useful: they were listed on the first page of part 1 including the definition, that is, the most well-known. In Table 2 we list how frequently these were used in other proofs.

Characterization	**Number of proofs**
Bisecting diagonals	30
Opposite parallel sides	20
One pair of opposite parallel and equal sides	12
Opposite equal sides	12
$A + C = B + D$	6
Opposite equal angles	4

TABLE 2. Most frequently used characterizations

Besides the six conditions in Table 2, Theorem 4.1 *(d)* in part 1 and Theorem 3.1 *(d)* in this part were used three times each, and for the rest, eleven other characterizations were used a total of twelve times. Adding these numbers we get 102, and what is missing is the definition, which of course cannot be proved.

At the beginning of the recent paper [14], it's referred to an e-mail conversation between me and one of the authors of that paper. There it's claimed that I would have expressed the opinion "that nothing interesting can be added" to the knowledge about characterizations of parallelograms,

which for me was the nine characterizations listed at Wikipedia at that time (the fall of 2015). However, I never wrote that this topic was exhausted, I merely encouraged that author to write a paper about his findings and that I was involved in other projects as answer to his question on a possible collaboration. The parallelogram was not a topic on my mind at that time, but I certainly realized that the subject of characterizations of any class of quadrilaterals was far from fully explored.

One of the reasons for writing this two part paper was to open the eyes of people engaged in mathematics education, textbook writing, and mathematical research to the important fact that there are *not* just a handful or even a dozen of characterizations of such a famous and well-studied object as the parallelogram. I am convinced this opinion has been the general view among most of these scholars for a long time, but it's simply not the case. In fact I am sure the complete opposite is true: the majority of properties are not only necessary but also sufficient conditions. To disprove this you would have to be able to list more than 103 properties of parallelograms where the converses are not true, and also provide counter examples that these converses really are false. Are you up for that challenge, or are you ready to be converted from the old view to the more modern view that true converses are not the exception but rather the rule?!

References

[1] 2006 Cono Sur Olympiad, *AoPS Online*, n.d., `https://artofproblemsolving.com/community/c5479`

[2] Al-Sharif, A. I.; Hajja, M.; Krasopoulos, P. T., Coincidences of Centers of Plane Quadrilaterals, *Results Math.*, **55**(**3**) (Nov. **2009**) 231–247.

[3] Alsina, C. and Nelsen, R. B., *A Cornucopia of Quadrilaterals*, MAA Press, **2020**.

[4] Andreescu, T.; Feng, Z.; Lee Jr., G., *Mathematical Olympiads 2000–2001. Problems and Solutions From Around the World*, MAA, **2003**.

[5] Andreescu, T. and Andrica, D., *Complex Numbers from A to... Z*, Second Edition, Birkhäuser, **2014**.

[6] *BMO1 2005/2006 Solutions*, United Kingdom Mathematics Trust, n.d., available at: `https://bmos.ukmt.org.uk/solutions/bmo1-2006/q5.flv`

[7] *British Mathematical Olympiad*, United Kingdom Mathematics Trust, **2022**, available at: `https://bmos.ukmt.org.uk/home/bmolot.pdf`

[8] Cola, C., Given a quadrilateral with 4 equal areas, prove that it is a parallelogram, *Mathematics Stack Exchange*, **2021**, `https://math.stackexchange.com/questions/4057138/`

[9] De Villiers, M., *Some Adventures in Euclidean Geometry*, Lulu Press, **2009**.

[10] Eperson, D. B., Puzzles, Pastimes and Problems, *Mathematics in School*, **1**(**5**) (July **1972**), 12–13 & 24–25.

[11] Fritsch, R. and Pickert, G., A Quadrangle's Centroid of Vertices and van Aubel's Square Theorem, *Crux Math.*, **39**(**8**) (**2013**) 362–367.

[12] Gilbert, G. T.; Krusemeyer, M.; Larson, L. C., *The Wohascum County problem book*, Dolciani Mathematical Expositions No. 14, MAA, Washington DC, **1993**.

[13] Gologan, R. (editor), *Romanian Mathematical Competitions. RMC 2006*, Romanian Mathematical Society, Theta Foundation, Bucharest, **2006**.

[14] Hajja, M. and Krasopoulos, P. T., More characterisations of parallelograms, *Math. Gaz.*, **107** (March **2023**) 76–83.

[15] Hajja, M. and Krasopoulos, P. T., Yet more characterisations of parallelograms, *Math. Gaz.*, **107** (July **2023**) 225–233.

[16] Halsted, G. B., *Elementary Synthetic Geometry*, John Wiley & Sons, New York, **1896**.
[17] Henrici, O., *Elementary Geometry. Congruent Figures*, Spottiswoode and Co., London, **1879**.
[18] Hoehn, L., Problem 515, *The College Mathematics Journal*, **24**(**5**) (Nov. **1993**) 473; Solution ibid, **25**(**5**) (Nov. **1994**) 467–468.
[19] Humenberger, H., Centroids of Quadrilaterals and a Peculiarity of Parallelograms, *At Right Angles*, Issue 17 (Nov. **2023**) 90–98, available at: `https://azimpremjiuniversity.edu.in/at-right-angles`
[20] IMOmath, 45-th Moldova Mathematical Olympiad 2001, n.d., available at: `https://imomath.com/othercomp/Mda/MdaMO01.pdf`
[21] jasperE3 (username), area chasing iff parallelogram, *AoPS Online*, **2021**, `https://artofproblemsolving.com/community/c6h2728683p23760535`
[22] *JMAP*, G.SRT.B.5: Quadrilateral Proofs, **2022**, available at: `https://www.jmap.org/Worksheets/G.SRT.B.5.QuadrilateralProofs.pdf`
[23] Josefsson, M., Properties of bisect-diagonal quadrilaterals, *Math. Gaz.*, **101** (July **2017**) 214–226.
[24] Josefsson, M., Properties of tilted kites, *Int. J. Geo.*, **7**(**1**) (**2018**) 87–104.
[25] Josefsson, M., On the heights of a convex quadrilateral, *Math. Gaz.*, **103** (March **2019**) 145–150.
[26] Josefsson, M., Great compilation of characterizations of squares, *Int. J. Geo.*, **12**(**3**) (**2023**) 13–37.
[27] Josefsson, M., Characterizations of parallelograms part 1, *Int. J. Geo.*, **13**(**2**) (**2024**) 86–112.
[28] Josefsson, M., Characterizations of parallelograms part 2, *Int. J. Geo.*, **13**(**3**) (**2024**) 65–93.
[29] Joyce, D. E., *Euclid's Elements*. Book I, **1996**, `https://mathcs.clarku.edu/~djoyce/java/elements/bookI/bookI.html`
[30] Kim, I. and Kim, D.-S., Various Centroids of Quadrilaterals Without Symmetry, *Journal of the Chungcheong Mathematical Society*, **33**(**4**) (Nov. **2020**) 429–444, available at: `http://ccms.or.kr/data/pdfpaper/jcms33_4/33_4_429.pdf`
[31] Klamkin, M. S., The Olympiad Corner, *Crux Math.*, **10**(**9**) (Nov. **1984**) 289.
[32] Klamkin, M. S., The Olympiad Corner, *Crux Math.*, **11**(**1**) (Jan. **1985**) 12–13.
[33] Klamkin, M. S., The Olympiad Corner, *Crux Math.*, **17**(**2**) (Feb. **1991**) 37–38.
[34] Legendre, A.-M., *Éléments de géométrie, avec des notes*, Chez Firmin Didot, Paris, **1794**.
[35] Okumura, H., A Characterisation of Parallelograms, *Math. Gaz.*, **72** (June **1988**) 117–118.
[36] Pamfilos, P., A Characterization of the Parallelogram, *Mathematical Reflections*, No. 3 **2016**, available at: `https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2016-03/article2_characterization_of_the_parallelogram.pdf`
[37] parmenides51 (username), orthocenters inside a quadrilateral, parallelogram wanted, *AoPS Online*, **2019**, `https://artofproblemsolving.com/community/c6h1911583p13092077`
[38] Pillay, S. and Pillay, P., Equipartitioning and Balancing Points of Polygons, *Pythagoras*, **71** (**2010**), available at: `https://pythagoras.org.za/index.php/pythagoras/article/view/2/9`
[39] Pompe, W., Problem 2257, *Crux Math.*, **23**(**5**) (**1997**) 300.
[40] Pop, O. T.; Minculete, N.; Bencze, M., *An introduction to quadrilateral geometry*, Editura Didactică şi Pedagogică, Bucharest, Romania, **2013**.
[41] Sandifer, C. E., *How Euler Did It*, MAA, **2007**.
[42] Sauvé, L., Editor's comment, *Crux Math.*, **3**(**7**) (Aug.-Sep. **1977**) 193–195.
[43] Seimiya, T., Problem 1841, *Crux Math.*, **20**(**4**) (April **1994**) 111–112.
[44] Sharygin, I. F., *Problems in Plane Geometry*, Mir publishers, Moscow, **1988**.
[45] Shirali, S., How to prove it, *At Right Angles*, Issue 5 (Nov. **2019**) 51–57, available at: `https://azimpremjiuniversity.edu.in/at-right-angles`

[46] Spiegel, M. R., *Theory and Problems of Complex Variables*, McGraw-Hill, **1964**.
[47] Todhunter, I., *The elements of Euclid for the use of schools and colleges*, Macmillan & Co., London, **1871**.
[48] Toumasis, C., When is a Quadrilateral a Parallelogram, *Math. Teacher*, **87**(**3**) (March **1994**) 208–211.
[49] Tydd, M., Flying Two Kites: Part 2, *AMESA KZN Mathematics Journal*, **9** (**2005**) 51–54, available at:
`http://dynamicmathematicslearning.com/matthew-tydd-two-kites.pdf`
[50] Ufnarovski, V.; Wikström, F.; Madjarova, J., *Mathematical Buffet. Problem solving at the Olympiad level*, Studentlitteratur, Lund, Sweden, **2016**.
[51] Usiskin, Z. and Griffin, J., *The Classification of Quadrilaterals. A Study of Definition*, Information Age Publishing, **2008**.
[52] Vasishtha, A. R., *Krishna's Vector Algebra*, Krishna Prakashan Media, India, **1981**.
[53] vinod19 (username), If the bisectors of angles of a quadrilateral enclose a rectangle, then show it is a parallelogram, *brainly*, **2016**,
`https://brainly.in/question/1003966`
[54] Wolstenholme, J., *A Book of Mathematical Problems*, Macmillan and Co., London and Cambridge, **1867**.
[55] Woodrow, R. E., 10^{th} Mexican Mathematics Olympiad National Contest November 1996, *Crux Math.*, **26**(**2**) (**2000**) 71.
[56] Xiong, B. and Lee, P. Y. (Editors), *Mathematical Olympiad in China. Problems and Solutions*, East China Normal University Press, **2007**.
[57] Young, J. R., *Elements of Geometry, with Notes*, Carey, Lea & Blanchard, **1833**.

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