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ON THE EXTREMA OF SUMS AND PRODUCTS FOR POINTS ON THE CIRCUMCIRCLE OF A TRIANGLE

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Abstract. Given triangle ABC inscribed in a circle and a variable point P on this circle, we find the extrema of the sum $PA+PB+PC$ and product $PA \cdot PB \cdot PC$ and the point(s) at which these extrema are achieved.

1. INTRODUCTION

Most metric relations involving the distances from a variable point on a circle to the vertices of a triangle establish the constancy of certain expressions depending on these distances. For example (see [1]), given a triangle and a circle centered at the triangle's centroid, the sum of the squared distances from a random point on this circle to the vertices of the triangle is constant. The same expression remains constant if the circle above is replaced by the incircle of the triangle ([3]). The case when the variable point lies on the circumcircle of the triangle seems to have received very little attention aside from the situation when the triangle is equilateral.

This article was motivated mostly by problem 5 on the 2022 edition of the SUNYMathics competition ([2]), in which one is given an equilateral triangle inscribed in a circle and a point P on this circle. It is then required to find all points P maximizing the product $PA \cdot PB \cdot PC$. As it turns out, the maximum value is achieved when P is one of the three midpoints of arcs AB, AC, or BC (see Problem 1). On the other hand, one has the classical problem of minimizing the sum $PA + PB + PC$, where P is a point in the plane. As is well-known, the solution is the Fermat point of the triangle. What if P is restricted to lying on the circumcenter of the triangle?

Inspired by the problems above, we consider the general case of a scalene triangle inscribed in a circle and study the extreme values for both the product $PA \cdot PB \cdot PC$ as well as the sum $PA + PB + PC$. In section 2, we first show that the points realizing the maximum value for the sum/product are restricted to lying on certain arcs.

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In sections 3 and 4, we use the results of section 2 to determine the extreme values for both the sum and the product as well as the points realizing these extrema (see Theorems 3.1, 4.1, Remark 2.2, and Propositions 3.1, 4.1, and 4.2). In addition, we determine when the maximum value of the product/sum is achieved at one or more of the three arc midpoints, thus providing an answer to the full as well as the partial/modified converses of the original competition question (see Propositions 3.2, 3.3, 3.4 for the product and Propositions 4.3, 4.4, 4.5 for the sum).

In the consideration below, given a triangle ABC inscribed in a circle, by the arc AB we mean the arc obtained as the intersection of the circle with the interior of $\angle ACB$. Also, by abuse of notation, we will use the same notation for an angle as well as its measure. Furthermore, without loss of generality, we will assume that $\angle A \geq \angle B \geq \angle C$.

We end this introductory section with a simple proof of the competition problem mentioned above (reworded for convenience) and note that the solution is different from the four solutions given in [2].

Problem 1. *Given an equilateral triangle ABC inscribed in a circle, if P is a variable point on this circle, show that the maximum value of the product $PA \cdot PB \cdot PC$ is achieved when P is the midpoint of one of the three arcs AB, AC, BC.*

Proof. Without loss of generality, we may assume that P is on arc AB. With x denoting the measure of $\angle ACP, 0 \leq x \leq 60$, as in the figure below, by the Law of Sines, we get $AP = 2R\sin x, BP = 2R\sin(60-x), CP =$

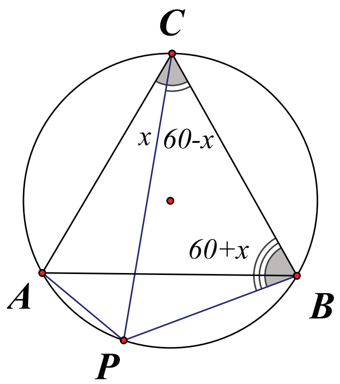

FIGURE 1. Maximum product for an equilateral triangle

$2R\sin(60+x)$, where R is the radius of the circle. Thus,

$$AP \cdot BP \cdot CP = 8R^3 \sin x \sin(60-x)\sin(60+x) = 8R^3\sin(3x).$$

As the maximum value of $\sin(3x)$ on $[0, 60]$ is achieved when $x = 30$, the conclusion follows.

2. PRELIMINARIES

In this section we show that, under the assumption that $\angle A \geq \angle B \geq \angle C$, the point(s) realizing the maximum values for the sum and product must lie on the arc determined by the midpoint of the largest of the three arcs BC, AC, AB and the antipodal point associated to the triangle vertex opposite the largest angle. First, we show that the point achieving the maximum product/sum must lie on the largest of the aforementioned three arcs.

Lemma 2.1. *If $\angle A > \angle B \geq \angle C$ and if P is a point on the circle containing $A, B,$ and C, then the maximum value of product(sum) $PA \cdot PB \cdot PC (PA + PB + PC)$is achieved at some point(s) on arc BC. If $\angle A = \angle B > \angle C$, then, the maximum value is obtained at two (or more) points, (at least) one on arc BC and (at least) another on arc AC.*

Proof. It is enough to show that if $\angle A > \angle B$, then the maximum of the product/sum for P on arc BC is greater than the maximum for P on arc AC. To do so, note that if Q is any point on arc BC, then, with P denoting

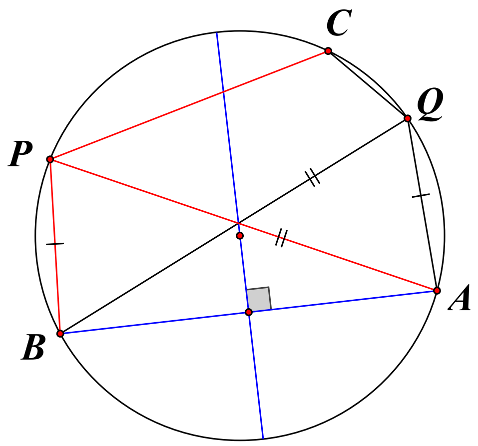

FIGURE 2. Maximum sum/product symmetry

the reflection of Q in the perpendicular bisector of segment AB, we have $AP = BQ$ and $AQ = BP$ (see Figure 2 above). By the hypothesis, we also have $\angle ABC = \angle CBQ + \angle QBA < \angle CAB = \angle CAP + \angle PAB$. As $\angle PAB = \angle QBA$, we have $\angle CBQ < \angle CAP$. This implies $\angle CPQ < CQP$, thus showing that $CQ < CP$. Hence, we get

$$AP \cdot BP \cdot CP > AQ \cdot BQ \cdot CQ$$

and

$$AP + BP + CP > AQ + BQ + CQ$$

The second claim from the lemma follows by using the same type of argument as above for $\angle A > \angle C$ to show that the maximum cannot be achieved on arc AB. Symmetry considerations imply that the maximum is achieved at points on both arcs BC and AC.

Next, we refine the location of the point(s) of maximum by studying several cases depending on whether or not we have equality between the angles of the triangle.

Lemma 2.2. *If $\angle A > \angle B > \angle C$ and if M be the midpoint of arc BC, then the maximum value of the product(sum) $PA \cdot PB \cdot PC(PA + PB + PC)$ is obtained at some point on arc CM.*

Proof.

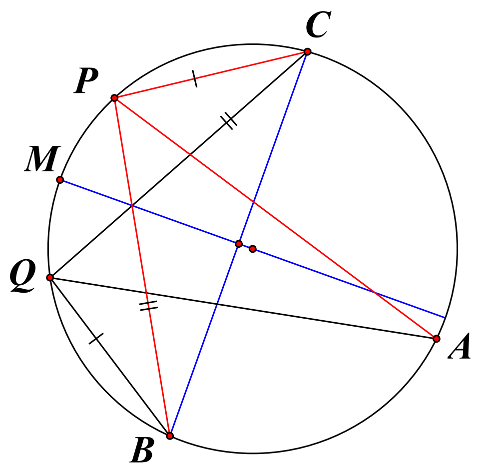

FIGURE 3. Maximum value and arc midpoints

If Q is any point on arc MB, we let P be the reflection of Q in the perpendicular bisector of segment BC, and note that P is on arc MC. Observe that segments PQ and BC have the same perpendicular bisector (see Figure 3). Clearly, $PB = CQ$ and $PC = BQ$. By hypothesis, $AC > AB$, which implies that A is on the $C-$side of the perpendicular bisector. But C and P are on the same side of the perpendicular bisector and so are B and Q. This implies that $AP > AQ$ and, thus, we have

$$PA \cdot PB \cdot PC > QA \cdot QB \cdot QC.$$

Remark 2.1. *If $\angle A = \angle B > \angle C$, by following the same argument as in the proof of the previous lemma, we can conclude that the maximum value of the product/sum is achieved at some point on arc CM and at some point on arc CN, where M and N are the midpoints of arcs AB and AC, respectively.*

Lemma 2.3. *If $\angle A > \angle B = \angle C$, then the maximum value of the product/sum is achieved at the midpoint M of arc BC.*

Proof. Due to the symmetry, it is enough to show that the product/sum is maximal at M if P is restricted to lying on the closed arc CM. For the sum, by Ptolemy's Theorem, we have

$$(PB + PC)AB = PA \cdot BC.$$

Thus,

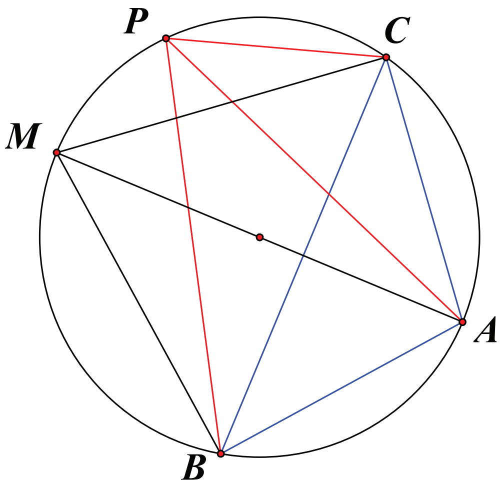

FIGURE 4. Maximum for an isosceles triangle

$$PA + PB + PC = PA\left(1 + \frac{BC}{AB}\right)$$

depends only on PA. Clearly, PA is maximal when $P = M$.

For the product, we note that

$$PB \cdot PC \sin A = 2S_{PBC} = 2hBC,$$

where h denotes the length of the altitude from P for triangle PBC. This way

$$PA \cdot PB \cdot PC = \frac{2BC}{\sin A} hPA.$$

As both h and PA clearly increase as P traverses arc CM from C to M, the conclusion follows.

Remark 2.2. *Since for $\angle A > \angle B = \angle C$ the maximum product/sum is achieved at the midpoint of arc BC, it is straightforward to check that*

$$AM = 2R, BM = CM = 2R\sin(A/2), \sin(A/2) = a/2b,$$

and

$$R = \frac{b^2}{2\sqrt{b^2 - \frac{a^2}{4}}},$$

where $BC = a, AC = b, AB = c$, and R is the circumradius of the triangle. This way, the maximum value of the product is

$$\frac{a^2b^4}{4\sqrt{(b^2 - \frac{a^2}{4})^3}}$$

and the maximum value of the sum is

$$\frac{b^2}{\sqrt{b^2 - \frac{a^2}{4}}}\left(1 + \frac{a}{b}\right).$$

Finally, we further refine the location of the points where the maximum is achieved. Since the $\angle A > \angle B = \angle C$ has already been settled, we are only left with the two cases $\angle A > \angle B > \angle C$ and $\angle A = \angle B > \angle C$.

Lemma 2.4. *If $\angle A > \angle B > \angle C$, if M be the midpoint of arc BC and A' is the antipodal of A, then the maximum value of the product(sum) $PA \cdot PB \cdot PC(PA + PB + PC)$ is obtained at some point(s) on arc $A'M$.*

Proof. By Lemma 2.2, it is enough to show that if P lies on the open arc CA', then the product (sum) $PA \cdot PB \cdot PC(PA + PB + PC)$ is smaller than the value corresponding to $P = A'$. Observe that A' is on (the minor) arc CM since $B > C$.

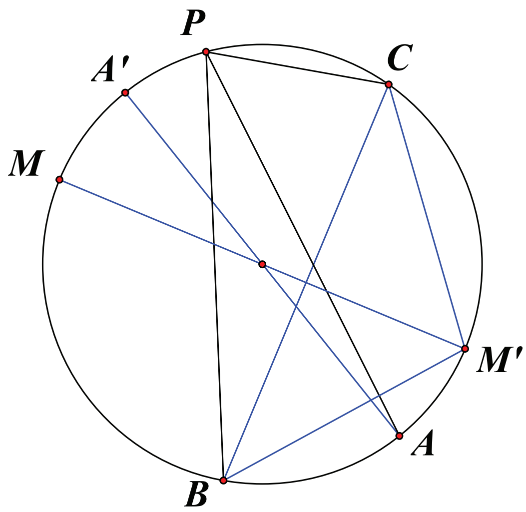

FIGURE 5. Maximum sum/product achieved on arc A'M

It is easy to see from Figure 5 that PA increases as P traverses arc CA' from C to A'. We will show the same type of increasing behavior for $PB \cdot PC$ and $PB + PC$. For the product, note that

$$PB \cdot PC = \frac{2S_{PBC}}{\sin A}.$$

As S_{PBC} increases as P traverses arc CA' from C to A' (as it, in fact, increases until P reaches M,) this finishes the proof for the product.

For the sum, we let M' be the antipodal of M. Note that M' is on arc AC since $B > C$. By Ptolemy's Theorem in quadrilateral $PCM'B$, we obtain $PC + PB = BC \cdot M'P/M'C$. As $M'P$ increases when P traverses arc CA', we obtain the maximality on arc MA' for the sum, as well.

Remark 2.3. *The same argument above shows that if $A = B > C$, then there are (at least) two points realizing the maximum of the product/sum, (at least)one located on arc MA' and (at least) another on arc NB', where M and N are the midpoint of arcs BC and AC, respectively, and A' and B' is the antipodals of A and B, respectively.*

In closing this section, we show that, for a scalene triangle, the maximum product/sum cannot occur at the midpoint of any arc.

Lemma 2.5. *If $\angle A > \angle B > \angle C$ and if M is the midpoint of arc BC, then the maximum value of the product(sum) $PA \cdot PB \cdot PC(PA + PB + PC)$ is not achieved at M (or at the midpoints of arcs AC or AB).*

Proof. By Lemma 2.1, we may assume P is on arc BC. Letting x denote the measure of $\angle PAB, 0 \leq x \leq A$, as in Figure 6 below, we have that the measures of $\angle CAP$ and $\angle ACP$ are $A - x$ and $C + x$, respectively. This way, assuming a unit radius for the circumscribing circle, by the Extended Law of Sines in $\triangle ABP$ and $\triangle ACP$, we have $BP = 2\sin x, CP = 2\sin(A - x)$, and $AP = 2\sin(C + x)$. Thus, we need to maximize the functions

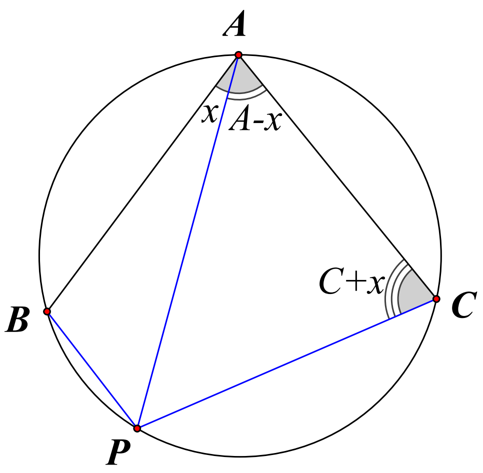

FIGURE 6. No maxima at arc midpoints

$$f_1(x) = 2\sin x + 2\sin(A - x) + 2\sin(C + x)$$

and

$$f_2(x) = 8\sin x \sin(A - x)\sin(C + x)$$

on the interval $[0, A]$. Here, we will only show that the maximum value is not obtained when $x = A/2$ by checking that the derivatives of f_1 and f_2 at $A/2$ cannot vanish. Indeed,

$$f_1'(x) = 2(\cos x - \cos(A - x) + \cos(C + x))$$

and

$$(1) \quad f_2'(x) = 8(\cos x \sin(A - x)\sin(C + x) - \sin x \cos(A - x)\sin(C + x)+$$
$$(2) \qquad + \sin x \sin(A - x)\cos(C + x)).$$

Since $f_1'(A/2) = 2\cos(C + A/2)$ and $f_2'(A/2) = 8\sin^2(A/2)\cos(C + A/2)$, in order for either one of the derivatives to vanish, we must have $C + A/2 = \pi = (A + B + C)/2$. However, this implies $B = C$, which contradicts the hypothesis.

3. MAXIMIZING $PA \cdot PB \cdot PC$

Our approach to determining the maximum value of the product and the point realizing it is a rather analytic one, supported by results from the previous section. Let us denote the sides the triangle by $a = BC, b = AC, c = AB$ and first investigate the case $a > b > c$. By Lemma 2.1 and Ptolemy's theorem, we have $PA = \frac{b}{a}PB + \frac{c}{a}PC$. Further denoting $PC = x, PB = y$, the problem of finding the maximum value of the product $PA \cdot PB \cdot PC$ becomes equivalent to investigating the maximum of the function $(x, y) \to \frac{b}{a}xy^2 + \frac{c}{a}yx^2$, subject to the constraints $x^2 + y^2 + 2xy\cos(A) = a^2$ (given by the Law of Cosines in triangle PBC), $x \geq y$ given by Lemma 2.2, and $x > 0$ since, when $x = 0, PC$ and, thus, the product $PA \cdot PB \cdot PC$ are zero. If we let $t = x/y$, we have $0 < t \leq 1$ and $y^2(t^2 + 2t\cos A + 1) = a^2$, by the constraints. With

$$y = \frac{a}{\sqrt{t^2 + 2t\cos A + 1}},$$

the function to be maximized on $[0, 1]$ becomes

$$f(t) = \frac{a^2(t^2c + tb)}{\sqrt{(t^2 + 2t\cos A + 1)^3}}.$$

Lemma 3.1. *With function f defined above, there exists a unique $t_0 \in (0, 1)$ such that $f'(t_0) = 0$. Moreover $f(t) \leq f(t_0)$ for $t \in [0, 1]$.*

Proof. Following straightforward computations, we obtain

$$f'(t) = \frac{a^2(-ct^3 + t^2(c\cos A - 2b) + t(2c - b\cos A) + b)}{\sqrt{(t^2 + 2t\cos A + 1)^5}}$$

Let $g(t) = -ct^3 + t^2(c\cos A - 2b) + t(2c - b\cos A) + b$. As $g(0) = b > 0$ and $g(1) = (\cos A + 1)(c - b) < 0$ (given that $c < b$) it follows that f' has at least one root in $(0, 1)$. Next, we will show that this root is unique. Indeed, if f' (and hence g) has two roots in $(0, 1)$, then the quadratic polynomial g' must also have two roots in $(0, 1)$. Indeed, if we denote the roots of g by r_1 and r_2, then g' will have at least one root s in (r_1, r_2) which is either a local minimum or a local maximum for g. If g has a local maximum at s, then since $g(0) > g(r_1) = 0$ and $g(s) > g(r_1) = 0$, then g must have a local minimum on $(0, s)$, thus giving a second root for g'. Similar considerations apply if g has a local maximum at s.

Now, note that

$$g'(t) = -3ct^2 + 2t(c\cos A - 2b) + 2c - b\cos A$$

In order for both roots of g' to be positive, one must have that their product is also positive. However, this product equals

$$\frac{2c - b\cos A}{-3c} = \frac{b^2 - a^2 - 3c^2}{6c^2} < 0.$$

The inequality above holds since $b < a$.Finally, to show that f has a maximum at t_0, it is enough to note that $f'(t) > 0$ on $(0, t_0)$ since $g(0) = f'(0) > 0$ and $f'(t) < 0$ on $(t_0, 1)$ since $g(1) = f'(1) < 0$.

Summarizing the results above and Lemma 2.1, we have the following

Theorem 3.1. *Given a circle and three points A, B, C on it such that $\angle A > \angle B > \angle C$, if P is a variable point on the circle, then the maximum value of $PA \cdot PB \cdot PC$ is*

$$\frac{a^2(t_0^2 c + t_0 b)}{\sqrt{(t_0^2 + 2t_0 \cos A + 1)^3}},$$

where t_0 is the unique solution in $(0, 1)$ of the equation

$$-ct^3 + t^2(c \cos A - 2b) + t(2c - b \cos A) + b = 0,$$

and $a = BC, b = AC, c = AB$. With M denoting the midpoint of arc BC, the maximum value is achieved at a unique point P_0 on the open arc CM determined by the condition

$$CP_0 = \frac{at_0}{\sqrt{t_0^2 + 2t_0 \cos A + 1}}.$$

Remark 3.1. *For a scalene triangle ABC, the point P_0 realizing the maximum of the product $PA \cdot PB \cdot PC$ is, in general, not constructible by straightedge and compass. Indeed, the location of this point on the circumscribing circle is determined by CP_0 which is, generally, not a constructible number since it depends on t_0, the root of a cubic equation.*

Remark 3.2. *As we set out to solve the maximum product problem, we had hoped to find a purely geometric solution, perhaps by determining the point P_0 realizing the maximum of the product as the intersection of the circumscribing circle of triangle ABC with a line determined by important points in the triangle and the vertices of the triangle/other important points. If such a solution does exist, by the previous observation, it must involve important points that are not constructible with straightedge and compass.*

Let us now analyze the case of two or more congruent angles. Clearly, if all three angles of triangle ABC are congruent, we can apply Problem 1. It only remains to investigate the case $\angle A = \angle B > \angle C$ since the case $\angle A > \angle B = \angle C$ has been settled in Remark 2.2.

Remark 3.3. *If $\angle A = \angle B > \angle C$ $(a = b > c)$, based on symmetry and Remark 2.1, we can restrict ourselves to the case when point P lies on arc CM, where M is the midpoint of arc AB. But then, except for one instance, the entire argument from Lemma 3.1 still holds since we have only used $c < b$ (which still holds) in the proof. The aforementioned instance involves showing that the product of the roots of g' is negative and was based on $b < a$. However, the same conclusion (negative product of roots) can be reached by using $b \leq a$ and $c > 0$, which does hold in our case $(a = b > c)$.*

Thus, by also relying on Lemma 2.1, we can conclude that if two of the angles are congruent, the maximum product occurs at one of two points located on each of the two longest arcs, between the midpoint of the arc and the vertex having the smallest angle measure.

To summarize, based on the remark above and straightforward computations, we have

Proposition 3.1. *Given a circle and three points A, B, C on it such that $\angle A = \angle B > \angle C$, if P is a variable point on the circle, then the maximum* *value of* $PA \cdot PB \cdot PC$ *is*

$$\frac{a^2(t_0^2 c + t_0 a)}{\sqrt{(t_0^2 + 2t_0 \cos A + 1)^3}},$$

where t_0 *is the unique solution in* $(0,1)$ *of the equation*

$$-ct^3 + \frac{c^2 - 4a^2}{2a} t^2 + \frac{3c}{2} t + a = 0$$

and $a = BC = AC$ *and* $c = AB$*. The maximum value of the product is achieved at some point* P_0 *on the open arcs* CM *or* CN *determined by the condition*

$$CP_0 = \frac{at_0}{\sqrt{t_0^2 + 2t_0 \cos A + 1}},$$

where M *and* N *are the midpoints of arcs* CB *and* CA*, respectively.*

Finally, we return to the converse(s) of Problem 1 and we have the following.

Proposition 3.2. *Given a triangle* ABC *inscribed in a circle, if the maximum value of the product* $PA \cdot PB \cdot PC$ *is achieved at each of the three midpoints of arcs* AB, BC, AC*, then the triangle is equilateral.*

Proof. Indeed, working under the assumption $\angle A \geq \angle B \geq \angle C$, if we assume that $\angle A > \angle C$, then, by Lemma 2.1, this would exclude the midpoint of arc AB from being a point where maximum is achieved.

Proposition 3.3. *Given a triangle* ABC *inscribed in a circle, it is impossible for the maximum value of the product* $PA \cdot PB \cdot PC$ *to be achieved at exactly two of the three midpoints of arcs* AB, BC, AC*.*

Proof. First, we note that the triangle cannot be equilateral. By the assumption and Lemma 2.1, we must have that the larger angles (A and B) are congruent and the maximum occurs on arcs CA and CB, at their midpoints. But, by Proposition 3.1, we see that these points of maximum cannot be the midpoints of the arcs.

Proposition 3.4. *Given a triangle* ABC *inscribed in a circle, if the maximum value of the product* $PA \cdot PB \cdot PC$ *is achieved at exactly one of the three midpoints of arcs* AB, BC, AC*, say at the midpoint of arc* BC*, then the triangle is isosceles with* $\angle A > \angle B = \angle C$*.*

Proof. By Lemma 2.1, we must have $\angle A > \angle B (\geq \angle C)$. If $\angle B \neq \angle C$, then $\angle B > \angle C$. But then, by Theorem 3.1, we conclude that the maximum cannot occur at the midpoint of arc BC. Thus, we must have $\angle B = \angle C$

4. THE EXTREMA FOR $PA + PB + PC$

Consider three points A, B, C on a circle $\mathcal{C}$. Given some point $P \in \mathcal{C}$, we study the minimum and maximum values of the sum $PA + PB + PC$. We begin by investigating the maximum value of the sum. First, we recall Lemma 2.1 restricting the location of the point(s) realizing the maximum to the largest arc(s) when $\angle A > \angle B \geq \angle C$ or $\angle A = \angle B > \angle C$. We will have a closer look at these two cases below. Before we do, we note that the case $\angle A > \angle B = \angle C$ has been settled in Remark 2.2 and that, in this case, the maximum sum is attained at the midpoint of arc BC. The case of an equilateral triangle will be dealt with last.

Theorem 4.1. *Let A, B, C be three points on a circle such that $\angle A > \angle B > \angle C$. If P is any point on the circle and if R denotes the radius of the circumscribing circle, then we have*

$$PA + PB + PC \leq 2R\sqrt{3 + 2(\cos B + \cos C - \cos A)}.$$

Equality holds iff P is the point on arc CM characterized by

$$\cos(\angle BAP) = \frac{\sin C + \sin A}{\sqrt{3 + 2(\cos C - \cos A + \cos B)}},$$

where M is the midpoint of arc BC.

Proof. Since the inequality is invariant under homotheties, we may assume

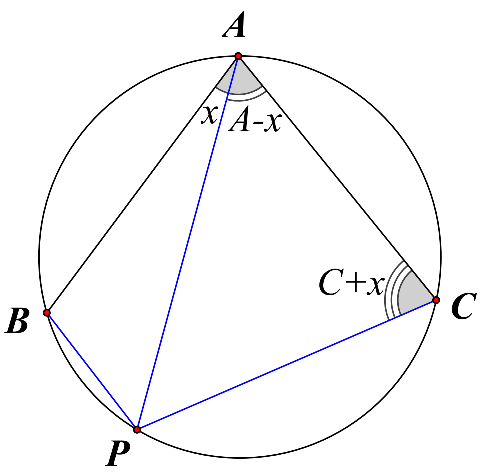

Figure 7. Largest sum

that the radius of the circle is one. By the lemma, we may also assume that P is on the arc BC. Letting x denote the measure of $\angle PAB, 0 \leq x \leq A$, we have that the measures of $\angle CAP$ and $\angle ACP$ are $A - x$ and $C + x$, respectively (see Figure 7). This way, by the Extended Law of Sines in $\triangle ABP$ and $\triangle ACP$, we have $BP = 2\sin x, CP = 2\sin(A - x)$, and $AP = 2\sin(C + x)$. Thus, we need to maximize the function

$$f(x) = 2\sin(C + x) + 2\sin x + 2\sin(A - x)$$

on the interval $[0, A]$.

By using well-known trigonometric formulas, we can rewrite the function as

$$f(x) = 2(\sin C\cos x + \sin x\cos C + \sin x + \sin A\cos x - \sin x\cos A) =$$

$$= 2(\sin x(\cos C + 1 - \cos A) + \cos x(\sin C + \sin A))$$

The maximum value of the function $\alpha\sin x+\beta\cos x$, with x *unrestricted* is $\sqrt{\alpha^2+\beta^2}$. Indeed,

$$\alpha\sin x+\beta\cos x=\sqrt{\alpha^2+\beta^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin x+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\cos x\right)=$$

$$=\sqrt{\alpha^2+\beta^2}\cos(x-\theta)\le\sqrt{\alpha^2+\beta^2},$$

where θ is defined by

$$\cos\theta=\frac{\beta}{\sqrt{\alpha^2+\beta^2}},\sin\theta=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}.$$

Note that equality holds when $x=\theta$.

For the moment, we will leave the restriction aside and calculate $\sqrt{\alpha^2+\beta^2}$ in our case:

$$2\sqrt{(\cos C+1-\cos A)^2+(\sin C+\sin A)^2}=$$

$$=2\sqrt{3+2(\cos C-\cos A+\sin A\sin C-\cos A\cos C)}=$$

$$=2\sqrt{3+2(\cos C-\cos A-\cos(A+C))}=2\sqrt{3+2(\cos C-\cos A+\cos B)}$$

Next we show this maximum is actually achieved and has the same value even if x is restricted to $[0,A]$. Note that in our case,

$$\cos\theta=\frac{\sin C+\sin A}{\sqrt{3+2(\cos C-\cos A+\cos B)}}$$

and

$$\sin\theta=\frac{\cos C+1-\cos A}{\sqrt{3+2(\cos C-\cos A+\cos B)}}.$$

Given that the cosine function is decreasing on $[0,\pi]$, it is enough to show that

$$\cos A\le\frac{\sin C+\sin A}{\sqrt{3+2(\cos C-\cos A+\cos B)}}\le 1.$$

Observe that the inequality on the left side is clearly satisfied if $A>\pi/2$ since the left side is negative while the right side is positive. So we may assume that $0\le A\le\pi/2$. Moreover, since A is the largest angle, we must also have $A\ge\pi/3$. Thus, we have $\pi/3\le A\le\pi/2$, which implies $0\le\cos A\le 1/2$. This way, the inequality is equivalent to

$$\cos^2 A(3+2\cos C-2\cos A+2\cos B)\le 1-\cos^2 A+1-\cos^2 C+2\sin A\sin C,$$

which, in turn, can be written as

$$4\cos^2 A+\cos^2 C+2\cos^2 A\cos C-2\cos^3 A+2\cos^2 A\cos B-2\sin A\sin C\le 2.$$

Note that $4\cos^2 A\le 1$ and $\cos^2 C\le 1$. So it will be enough to show that

$$\cos^2 A\cos C-\cos^3 A+\cos^2 A\cos B-\sin A\sin C\le 0.$$

To do so, we rewrite the inequality as

$$-\cos^3 A+\cos^2 A(\cos B+\cos C)-\cos A\cos C-\cos B\le 0,$$

or

$$-\cos^3 A+\cos B(\cos^2 A-1)+\cos A\cos C(\cos A-1)\le 0.$$

As $0\le\cos A\le 1,\cos B>0,\cos C>0$, each of the three terms above are negative and the inequality follows.

The inequality on the right can be written as follows:

$$(\sin A + \sin C)^2 \leq 3 + 2\cos C - 2\cos A + 2\sin A \sin C - 2\cos A \cos C \Leftrightarrow$$
$$\Leftrightarrow 1 - \cos^2 A + 1 - \cos^2 C \leq 3 + 2\cos C - 2\cos A - 2\cos A \cos C.$$

In turn, this is equivalent to

$$0 \leq \cos^2 A + \cos^2 C + 1 + 2\cos C - 2\cos A - 2\cos A \cos C \Leftrightarrow$$
$$\Leftrightarrow 0 \leq (1 + \cos C - \cos A)^2$$

For the case when $\angle A = \angle B > \angle C$, we first recall that, based on Lemma 2.1, the maximum is achieved at two points, one on arc AC, the other on arc BC. Thus, we can set up two modified versions of the function f introduced in the proof of the previous theorem and study their maximum. Tedious but straightforward computations yield the following:

Proposition 4.1. *Let A, B, C be three points on a circle such that $\angle A = \angle B > \angle C$. If P is any point on the circle and if R denotes the radius of the circumscribing circle, then we have*

$$PA + PB + PC \leq 2R\sqrt{3 - 2\cos(2A)}.$$

Equality holds iff P is either the point on arc BM characterized by

$$\cos(\angle BAP) = \frac{\sin(2A) + \sin A}{\sqrt{3 - 2\cos(2A)}},$$

where M is the midpoint of arc BC or the point on arc CN characterized by

$$\cos(\angle ABP) = \frac{\sin(2A) + \sin A}{\sqrt{3 - 2\cos(2A)}},$$

where N is the midpoint of arc AC.

Finally, for an equilateral triangle, we have

Proposition 4.2. *If A, B, C are the vertices of an equilateral triangle, then*

$$PA + PB + PC \leq 4R.$$

Equality holds iff P is the midpoint of one of the three arcs determined by the three vertices of the triangle on the circumscribing circle.

Proof. After applying Ptolemy's theorem, we obtain $PA+PB+PC$ equals either $2PA$, or $2PB$, or $2PC$, depending on whether P belongs to arcs BC, AC, or AB. Since the largest value of, say PA, is obtained when P is the midpoint of arc BC, we conclude that in the case of an equilateral triangle, the largest sum is $4R$ and is achieved at each of the three arc midpoints.

As a direct consequence of the three results above, we obtain the following converse of Proposition 4.2.

Proposition 4.3. *Given a triangle ABC inscribed in a circle, If P lies on the circumcircle of the triangle and the maximum value of the sum $PA + PB + PC$ is achieved at each of the three midpoints of arcs AB, BC, AC, then the triangle is equilateral.*

In terms of the maximum occurring at fewer than three points, we have the following two results.

Proposition 4.4. *Given a triangle ABC inscribed in a circle, if P lies on the circumcircle of the triangle, then it is impossible for maximum the value of the sum $PA + PB + PC$ to be achieved at exactly two of the midpoints of arcs AB, BC, AC.*

Proof. Based on Lemma 2.3 and Propositions 4.1, and 4.2, the only way in which the maximum could be achieved at two points is if, up to relabeling the vertices, $\angle A = \angle B > \angle C$ and

$$\frac{\sin(2A) + \sin A}{\sqrt{3 - 2\cos(2A)}} = \cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

By expressing the left side in terms of $\cos A$ and squaring, the equation above is equivalent to

$$(2\cos A - 1)(8\cos^3 A + 12\cos^2 A - 6\cos A - 11) = 0.$$

Letting $f(x) = 8x^3 + 12x^2 - 6x - 11$, we note that f has a local maximum at $1/2 - \sqrt{2}/2$ and a local minimum at $1/2 + \sqrt{2}/2$. As $f(1/2 - \sqrt{2}/2) < 0$, it follows that f has a single real root. Moreover, since $f(1/2) < 0$ and $f(1) > 0$, this root must be greater than $1/2$. The same conclusion can be reached by noting that the actual roots are

$$x = \frac{-1 + \sqrt{3}2 + \sqrt[3]{4}}{2} > 1/2,$$

$$x = \frac{-2 - \sqrt[3]{4}(1 + i\sqrt{3}) + i\sqrt[3]{2}(\sqrt{3} + i)}{4},$$

$$x = \frac{-2 - \sqrt[3]{2}(1 + i\sqrt{3}) + i\sqrt[3]{4}(\sqrt{3} + i)}{4}$$

However, since A is the largest angle, we have $A > \pi/3$, thus $\cos A < 1/2$. Thus, the first factor must be zero, implying $A = \pi/3$. But then the triangle is equilateral, which is impossible.

Finally, in the case of a single point of maximum, we have the following:

Proposition 4.5. *Given a triangle ABC inscribed in a circle, if P lies on the circumcircle of the triangle and the maximum value of the sum $PA + PB + PC$ is achieved at exactly one of the midpoints of arcs AB, BC, AC, say BC, then triangle ABC is isosceles and $\angle A > \angle B = \angle C$.*

Proof. First, we can rule out the equilateral triangle and the $\angle A = \angle B > \angle C$ since they correspond to either three midpoints as three points of maximum in the first case or no midpoints as points of maximum (see also the previous proposition). Thus, we only need to rule out the case $A > B > C$. The fact that the single point of maximum (guaranteed by Theorem 4.1) cannot be a midpoint is a consequence of Lemma 2.5.

We end by analyzing the minimum value of the sum $PA + PB + PC$.

Proposition 4.6. *Let A, B, C be three points on a unit circle. If P is any point on the circle, then*

$$PA + PB + PC \geq min\{AB + AC, AB + BC, AC + BC\}.$$

The minimum is achieved when P is one of the vertices of the triangle.

Proof. If we assume that P is on the arc BC determined by the sides of angle BAC, then, on one hand, we have $PB+PB \geq BC$ by triangle inequality. On the other hand, since angles ABP and ACP are supplementary, one of them must be obtuse or right, implying that either $AP \geq AB$ or $AP \geq AC$. Thus, $AP+BP+PC \geq \min\{AB+BC, AC+BC\}$. Clearly, equality holds if and only if $P = B$ or $P = C$, depending on whether $AB > AC$ or $AC > AB$. Similar arguments apply if P belongs to arcs AC or AB.

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