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POHLKE PROJECTIONS IN THE HYPERBOLIC CASE II CIRCULAR AND DEGENERATE CASES

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Abstract We complete here the work started in [3]. Given three segments OP_1, OP_2, OP_3 in a plane, assuming $|OP_1| = |OP_2| = 1$ and $OP_1 \perp OP_2$ (circular case), we determine the hyperbolic Pohlke's projection(s). We also prove that if OP_1, OP_2, OP_3 are not contained in a line, but two of them are parallel (degenerate case), then there are infinite hyperbolic Pohlke's projections if these two segments are equal, none if they are different. Finally we give two examples of hyperbolic Pohlke's projections and conics.

1. Hyperbolic Pohlke's projections in the circular case

In this section we give the necessary and sufficient conditions for the existence of the hyperbolic Pohlke's projection(s) (according to Def. 2.4 of [3]) when the segments OP_1, OP_2, OP_3 are in the "circular case". More precisely, given OP_1, OP_2, OP_3 in a plane ω, we suppose that

$$OP_1 \perp OP_2 \quad \text{and} \quad |OP_1| = |OP_2| = 1. \tag{1}$$

In this situation we will prove the following:

Theorem 1.1. *Suppose condition (1) holds and let $\overrightarrow{OP_3} = h\overrightarrow{OP_1} + k\overrightarrow{OP_2}$. The following facts then apply:*

1) If $h = 0$ $(k = 0)$ then there are infinitely many hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 if $|k| = 1$ $(|h| = 1)$, none if $|k| \neq 1$ $(|h| \neq 1)$. If these projections exist, then $\rho = 1$, that is $\mathscr{H} = \mathscr{H}(1)$.

2) Let us suppose $h, k \neq 0$. Then there exists a hyperbolic Pohlke's projection $\Pi_{\mathbf{v}_\circ}$ for OP_1, OP_2, OP_3 if and only if

$$g(h,k) \stackrel{\text{def}}{=} (h+k+1)(h+k-1)(h-k+1)(h-k-1) < 0, \tag{2}$$

$$f(h,k) \stackrel{\text{def}}{=} (h^2+k^2-1)\left[(h^2-k^2)^2-1\right] \neq 0. \tag{3}$$

If (2), (3) are verified, $\Pi_{\mathbf{v}_\circ}$ is unique up to symmetry with respect to ω and the projection direction is represented by the vector

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$$\mathbf{v}_o = H\overrightarrow{OP_1} + K\overrightarrow{OP_2} \pm \sqrt{-g}\,\mathbf{k}\,, \tag{4}$$ [1]

with $H = h(h^2-k^2-1)$, $K = k(h^2-k^2+1)$ *and* $g = g(h,k)$.
Moreover, $\rho = 1$*, i.e.,* $\mathscr{H} = \mathscr{H}(1)$*; the conic* $\mathscr{C}_{\mathbf{v}_o} = \Pi_{\mathbf{v}_o}(\mathscr{H}(1)\cap\pi_{\mathbf{v}_o})$ *is unique and* $\mathscr{C}_{\mathbf{v}_o}$ *is an ellipse if* $f(h,k)<0$*, a hyperbola if* $f(h,k)>0$.

1.1. **Some preliminary facts.** Before proceeding, let's note some key facts. We assume that (1) holds and that $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is a given hyperbolic Pohlke's projection, in the sense of Def. 2.4 of [3], for OP_1, OP_2, OP_3.

Remark 1.2. *Considering the oblique symmetries* $\mathtt{S}_{\mathbf{v}}$ *with respect to* $\pi_{\mathbf{v}}$ *(Def. 3.3 of* [3]*), and* $\mathtt{S}_{\mathbf{k}}$*, with respect to the plane* $\pi_{\mathbf{k}} = \omega$ *(i.e. the usual symmetry with respect to* ω*), it is immediate to see that:*

i) If $Q_1, Q_2, Q_3 \in \mathscr{H}$ *satisfy (15), (16) of Def. 2.4 of* [3] *then, by Cor. 3.22 of* [3]*, also the points* $Q_1' = \mathtt{S}_{\mathbf{v}}(Q_1)$*,* $Q_2' = \mathtt{S}_{\mathbf{v}}(Q_2)$ $Q_3' = \mathtt{S}_{\mathbf{v}}(Q_3)$ *satisfy these conditions. This means that in Def. 2.4 of* [3] *the triads* Q_1*,* Q_2*,* Q_3 *and* Q_1'*,* Q_2'*,* Q_3' *are perfectly equivalent.*

ii) Let $\bar{\Pi}_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be the symmetric projection with respect to* ω*, i.e.,*

$$\bar{\Pi}_{\mathbf{v}}(P) = \Pi_{\mathbf{v}}(\mathtt{S}_{\mathbf{k}}(P)) \quad \text{for } P \in \mathbb{R}^3. \tag{5}$$

$\bar{\Pi}_{\mathbf{v}}$*, with the points* $\mathtt{S}_{\mathbf{k}}(Q_1)$*,* $\mathtt{S}_{\mathbf{k}}(Q_2)$ *and* $\mathtt{S}_{\mathbf{k}}(Q_3)$*, gives a hyperbolic Pohlke's projection for* OP_1, OP_2, OP_3*. Note also that if* $\bar{\mathbf{v}} = \mathtt{S}_{\mathbf{k}}(\mathbf{v})$ *then*

$$\bar{\Pi}_{\mathbf{v}} = \Pi_{\bar{\mathbf{v}}} \quad \text{and} \quad \Pi_{\bar{\mathbf{v}}}(\mathscr{H}\cap\pi_{\bar{\mathbf{v}}}) = \Pi_{\mathbf{v}}(\mathscr{H}\cap\pi_{\mathbf{v}}). \tag{6}$$

Remark 1.3. *Again assuming* $\Pi_{\mathbf{v}}$ *exists, from Claim 3.26 of* [3] *we deduce that* $\mathcal{E}_{P_1,P_2}$ *must be tangent to* $\mathscr{C}_{\mathbf{v}}$*. Noting that* $\mathcal{E}_{P_1,P_2}$ *is the circle with center* O *and radius* $r=1$*, taking into account Cor. 3.13 of* [3]*, we have:*

- *If* $\mathscr{C}_{\mathbf{v}}$ *is an ellipse (circle), having to be inscribed in the circle* $\mathcal{E}_{P_1,P_2}$*,* $\mathscr{C}_{\mathbf{v}}$ *must have semi-major axis* $a=1$ *(radius* $r=1$*).*
- *If* $\mathscr{C}_{\mathbf{v}}$ *is a hyperbola, having to circumscribe the circle* $\mathcal{E}_{P_1,P_2}$*,* $\mathscr{C}_{\mathbf{v}}$ *must have transverse semi-axis* $a=1$*.*

Then, from Claim 3.15 and Cor. 3.17 of [3], we conclude that:

Claim 1.4. *If* (1) *holds and if there is a hyperbolic Pohlke's projection for* OP_1, OP_2, OP_3 *then* $\rho = 1$*. That is, we have*

$$\mathscr{H} = \mathscr{H}(1) = \{(x,y,z)\in\mathbb{R}^3 \,|\, x^2+y^2-z^2=1\}. \tag{7}$$

After this, again assuming that a hyperbolic Pohlke's projection $\Pi_{\mathbf{v}}$ exists, we note that (1), (7) together imply that $P_1, P_2 \in \mathscr{H}$. Thus we must have:

$$P_1 = Q_1 \text{ or } Q_1' \quad \text{and} \quad P_2 = Q_2 \text{ or } Q_2'. \tag{8}$$ [2]

But to satisfy the conditions of Def. 2.4 of [3] it is necessary to set

$$Q_1 = P_1 \text{ and } Q_2 = P_2 \tag{9}$$

[1] As in [3], we fix a Cartesian system of coordinates x, y, z (with the corresponding orthonormal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$) such that $\omega = \{(x,y,z)\,|\,z=0\}$ and $O=(0,0,0)$.

[2] Given $Q \in \mathscr{H}$, we have $OP \parallel T_{\mathscr{H}}(Q) \Leftrightarrow x_Px_Q + y_Py_Q - z_Pz_Q = 0$. Therefore, if $P_1, P_2 \in \mathscr{H}\cap\omega$ are such that $OP_1 \perp OP_2$, then $OP_1 \parallel T_{\mathscr{H}}(P_2)$ and $OP_2 \parallel T_{\mathscr{H}}(P_1)$.

or, equivalently, $Q_1' = P_1$ and $Q_2' = P_2$. [3] In fact, if we set $Q_1 = P_1$ and $Q_2' = P_2$, applying Cor. 3.22 of [3], we find:

$$(10) \qquad OQ_3 \parallel T_{\mathscr{H}}(Q_1') \Leftrightarrow OQ_1 \parallel T_{\mathscr{H}}(Q_3') \Leftrightarrow OP_1 \parallel T_{\mathscr{H}}(Q_3'),$$

$$(11) \qquad OQ_2 \parallel T_{\mathscr{H}}(Q_3) \Leftrightarrow OQ_2' \parallel T_{\mathscr{H}}(Q_3') \Leftrightarrow OP_2 \parallel T_{\mathscr{H}}(Q_3').$$

Now, it is easy to see that

$$(12) \qquad OP_1 \parallel T_{\mathscr{H}}(Q_3') \text{ and } OP_2 \parallel T_{\mathscr{H}}(Q_3') \implies OQ_3' \perp \omega,$$ [4]

and the latter condition cannot be satisfied if $Q_3' \in \mathscr{H}$. Since the same argument works if we try to define $Q_1' = P_1$ and $Q_2 = P_2$, we are forced to assume (9). Besides, by choosing $Q_1 = P_1$ and $Q_2 = P_2$, we must have

$$(13) \qquad Q_3 \neq Q_3'.$$

Indeed, if $Q_3 = Q_3'$, from Cor. 3.22 of [3] and (16) of Def. 2.4 of [3], we easily deduce that $OP_1 \parallel T_{\mathscr{H}}(Q_3)$ and $OP_2 \parallel T_{\mathscr{H}}(Q_3)$. Hence, as in (12), we find $OQ_3 \perp \omega$ which cannot be satisfied. In conclusion, noting that (13) implies $Q_3Q_3' \parallel \mathbf{v}$, we can say that:

Conditions 1.5. *Having fixed the points $Q_1 = P_1, Q_2 = P_2$ as in (9), to obtain a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 as in (1), it is necessary and sufficient to determine $Q_3, Q_3' \in \mathscr{H}(1)$, $Q_3 \neq Q_3'$, such that the following conditions are true:*

- (a) *$OP_2 \parallel T_{\mathscr{H}}(Q_3)$ and $OP_1 \parallel T_{\mathscr{H}}(Q_3')$ (i.e., $OQ_3 \parallel T_{\mathscr{H}}(P_1')$, by Cor. 3.22 of [3]);*
- (b) *$Q_3Q_3' \nparallel \omega$, because Q_3Q_3' gives the direction of projection onto ω;*
- (c) *Q_3, Q_3', P_3 are collinear (i.e., $\Pi_{\mathbf{v}}(Q_3) = \Pi_{\mathbf{v}}(Q_3') = P_3$);*
- (d) *$\mathbf{v} = \overrightarrow{Q_3Q_3'}$ gives a non-degenerate projection direction.*

1.2. **Determination of $\Pi_{\mathbf{v}}$ in the circular case.** To proceed, we may suppose that the coordinate axes x, y are oriented in ω such that

$$(14) \qquad P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad P_3 = \begin{pmatrix} h \\ k \\ 0 \end{pmatrix}.$$

In this way we immediately have

$$(15) \qquad \overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}.$$

Then, taking into account (57) of [3], we see that (a) in Cond. 1.5 is satisfied if and only if $Q_3 \in \mathscr{H} \cap \{y = 0\}$ and $Q_3' \in \mathscr{H} \cap \{x = 0\}$. Thus we can

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[3] In the following will not distinguish between these two possibilities because, by Rem. 1.2, we know that the triads Q_1, Q_2, Q_3 and Q_1', Q_2', Q_3' are equivalent.

[4] Given $Q = (x_Q, y_Q, z_Q) \in \mathscr{H}$ and $P_1, P_2 \in \omega$ such that $OP_1 \nparallel OP_2$, we have that $OP_1, OP_2 \parallel T_{\mathscr{H}}(Q) \Leftrightarrow x_Q = y_Q = 0$. But the latter condition is equivalent to $OQ \perp \omega$.

express Q_3 and Q_3' in the form

$$Q_3 = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} \quad \text{and} \quad Q_3' = \begin{pmatrix} 0 \\ \cosh^*\beta \\ \sinh\beta \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}), \tag{16}$$

where, for simplicity, we have set

$$\cosh^* t \overset{\text{def}}{=} \pm \cosh t.\,^{5} \tag{17}$$

Having (16), it is clear that $Q_3 \neq Q_3'$ and that (b) in Cond. 1.5 holds iff

$$\sinh\alpha \neq \sinh\beta. \tag{18}$$

Then (c) of Cond. 1.5 is verified iff $P_3 = Q_3 + t\,\overrightarrow{Q_3Q_3'}$ for some $t \in \mathbb{R}$, i.e.,

$$\begin{pmatrix} h \\ k \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} + t \begin{pmatrix} -\cosh^*\alpha \\ \cosh^*\beta \\ \sinh\beta - \sinh\alpha \end{pmatrix} \quad \text{for some } t \in \mathbb{R}. \tag{19}$$

Now, assuming that (18) holds, we will first study the solvability of the system (19) and then we will verify if also (d) of in Cond. 1.5 is satisfied, i.e., if the projection direction found is non-degenerate. In this way we will obtain the proofs of part 1) and part 2) of Thm. 1.1.

1.3. **Proof of part 1) of Thm. 1.1.** Suppose first $h = 0$. Since $\cosh^*\alpha \neq 0$, the first equation of (19) gives $t = 1$. Then, considering also the third equation, we find $\sinh\beta = 0$. Thus $\cosh^*\beta = \pm 1$ and (by (18)) $\sinh\alpha \neq 0$. This means that when $h = 0$ system (19) is solvable iff

$$P_3 = \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{20}$$

If (20) holds, then we have

$$Q_3 = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} \quad \text{with } \alpha \neq 0, \qquad Q_3' = P_3. \tag{21}$$

The projection direction is parallel to $\overrightarrow{Q_3Q_3'} = -(\cosh^*\alpha)\,\mathbf{i} \pm \mathbf{j} - (\sinh\alpha)\,\mathbf{k}$ and condition (d) is verified because $(\cosh^*\alpha)^2 + 1 - \sinh^2\alpha = 2.$[6] In conclusion, when $h = 0$ there are no hyperbolic Pohlke's projections if (20) fails, infinitely many if (20) holds.

Next suppose $k = 0$. Reasoning as in the previous case, we find that when $k = 0$ (19) is solvable iff

$$P_3 = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

[5] Just to have a simple parametrization of the entire hyperbolas $\mathscr{H} \cap \{y = 0\}$ and $\mathscr{H} \cap \{x = 0\}$, suitable for subsequent calculations.

[6] Even if we assume (14), this calculation is justified because, given $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, the quantity $l^2 + m^2 - n^2$ is invariant for isometries that leave the z-axis fixed.

If (22) holds, then we have

$$Q_3 = P_3, \qquad Q_3' = \begin{pmatrix} 0 \\ \cosh^*\beta \\ \sinh\beta \end{pmatrix} \text{ with } \beta \neq 0. \tag{23}$$

As above, (d) of Cond. 1.5 is true because $\overrightarrow{Q_3Q_3'} = \pm\mathbf{i} + (\cosh^*\beta)\mathbf{j} + (\sinh\beta)\,\mathbf{k}$. Thus there are no hyperbolic Pohlke's projections if (22) fails, infinitely many if (22) holds.

Summing up, taking into account Cond.1.5, if (1) is verified and $h = 0$ $(k = 0)$ then there are infinitely many hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 if $|k| = 1$ $(|h| = 1)$, none if $|k| \neq 1$ ($|h| \neq 1$). Moreover, by Claim 1.4, we have $\rho = 1$.

1.4. **Case $h, k \neq 0$, that is $OP_3 \nparallel OP_1, OP_2$.** We note first that the condition $h, k \neq 0$ in (19) implies

$$\sinh\alpha, \ \sinh\beta \ \neq \ 0. \tag{24}$$

Indeed, if $\sinh\alpha = 0$, (18) and the third equation of (19) give $t = 0$. Then the second equation of (19) implies $k = 0$, contrary to our assumption. Similarly we find that $\sinh\beta \neq 0$.

Taking into account this fact, we deduce now a set of necessary conditions for the point P_3 to be collinear with Q_3, Q_3' (i.e., to satisfy (19) for some $t \in \mathbb{R}$) when (18) and (24) hold. After that, we will prove that these conditions are also sufficient.

Assuming (19) holds, by (18) and the third equation of (19), we have

$$t = \frac{\sinh\alpha}{\sinh\alpha - \sinh\beta}\,. \tag{25}$$

From (24) it follows that $t \neq 0, 1$ and that

$$h = \cosh^*\alpha - \frac{\cosh^*\alpha \ \sinh\alpha}{\sinh\alpha - \sinh\beta} \quad \Rightarrow \quad h \neq 0, \cosh^*\alpha; \tag{26}$$

$$k = \frac{\cosh^*\beta \ \sinh\alpha}{\sinh\alpha - \sinh\beta} \quad \Rightarrow \quad k \neq 0, \cosh^*\beta. \tag{27}$$

Then

$$\frac{h}{\cosh^*\alpha} + \frac{k}{\cosh^*\beta} = 1. \tag{28}$$

From (27), (28) we obtain

$$\begin{aligned} \cosh^*\alpha &= \frac{h\cosh^*\beta}{\cosh^*\beta - k}\,, \\ \sinh\alpha &= \frac{k\sinh\beta}{k - \cosh^*\beta}\,, \end{aligned} \tag{29}$$

because, by (27), we know that $k \neq \cosh^*\beta$. Next, since $(\cosh^*\alpha)^2 - \sinh^2\alpha = 1$, from (29) we have

$$h^2(\cosh^*\beta)^2 - k^2\sinh^2\beta = (k - \cosh^*\beta)^2. \tag{30}$$

Hence, simplifying the expression above, we find

$$\Big[(h^2 - k^2 - 1)\cosh^*\beta + 2k\Big]\cosh^*\beta \ = 0. \tag{31}$$

Since $\cosh^*\beta \neq 0$ and (by (27)) $k \neq 0$, we deduce that:

$$h^2 - k^2 - 1 \neq 0, \tag{32}$$

and then

$$\cosh^*\beta = \frac{-2k}{h^2 - k^2 - 1}\,. \tag{33}$$

Noting that $h \neq 0$, $\cosh^*\alpha$ (see (26)) by similar arguments we obtain

$$k^2 - h^2 - 1 \neq 0 \tag{34}$$

and

$$\cosh^*\alpha = \frac{-2h}{k^2 - h^2 - 1}\,. \tag{35}$$

Finally, since (24) is equivalent to $|\cosh^*\alpha| > 1$, $|\cosh^*\beta| > 1$, from the expressions (33), (35) we deduce the conditions:

$$(i)\ \left|\frac{2k}{h^2 - k^2 - 1}\right| > 1 \quad \text{and} \quad (ii)\ \left|\frac{2h}{k^2 - h^2 - 1}\right| > 1. \tag{36}$$

Summing up, we have:

Claim 1.6. *If (18), (24) are verified and if $P_3 = {}^t(h, k, 0)$ is given by formula (19), then the necessary conditions (32), (34) and (36) are satisfied.*

Definition 1.7. *We will denote with Σ the subset of $\mathbb{R}^2$ where (32), (34) hold, i.e.,*

$$\Sigma \overset{\text{def}}{=} \big\{(h, k) \big|\ h^2 - k^2 \neq \pm 1\big\}. \tag{37}$$

The solution region of (36) is given by the following:

Lemma 1.8. *A pair $(h, k) \in \Sigma$ satisfies the conditions (36) (i) and (ii) iff*

$$|h| + |k| > 1 \quad \text{and} \quad \big||h| - |k|\big| < 1 \tag{38}$$

or, equivalently,

$$(h + k + 1)(h + k - 1)(h - k + 1)(h - k - 1) < 0. \tag{39}$$

Proof. The inequalities of (36) are invariant under symmetry with respect to the coordinate axes, i.e., on replacing (h, k) with $(\pm h, \pm k)$. So it is sufficient to solve (36) for $h, k \geq 0$. Besides, we can obtain the first of (36) from the second, and vice versa, by permutation of the variables h, k. Hence it is sufficient to solve the second inequality of (36). To begin with, for $(h, k) \in \Sigma$ with $h, k \geq 0$, inequality (36) (ii) is equivalent to

$$-2h < k^2 - h^2 - 1 < 2h, \tag{40}$$

that is $(h - 1)^2 < k^2 < (h + 1)^2$, which, in turn, is equivalent to

$$|h - 1| < k < h + 1, \tag{41}$$

because $h + 1 \geq 0$ and $k \geq 0$. Next, it easy to see that

$$\big\{(h, k) \,\big|\, |h-1| < k < h+1\big\} = \big\{(h, k) \,\big|\, h+k > 1,\ |h-k| < 1\big\} \subset \big\{h, k \geq 0\big\}.$$

Thus, for $h, k \geq 0$, the solution region of (36) (ii) is given by

$$\Omega \,=\, \Sigma \cap \big\{(h, k) \,\big|\, h + k > 1,\ |h - k| < 1\big\}. \tag{42}$$

The set Ω in (42) is symmetric with respect to h, k. By the previous considerations, Ω gives also the solution region of (36) (i) for $h, k \geq 0$ and, taking into account the symmetry with respect to the axes, from this we immediately obtain (38). Finally, it is easy to verify the equivalence of (38) and (39), because they define the same subset of $\mathbb{R} \times \mathbb{R}$. □

Taking into account the definition (2) of $g(h,k)$, we have proved that:

Claim 1.9. *If the conditions* (18), (24) *are verified and if* $P = {}^t(h,k,0)$ *is given by* (19)*, then* $(h,k) \in \Sigma$ *and* $g(h,k) < 0$.[7]

The converse is also true:

Claim 1.10. *If* $(h,k) \in \Sigma$ *and* $g(h,k) < 0$*, then* $P = {}^t(h,k,0)$ *satisfies formula* (19) *for suitable* α, β *satisfying* (18), (24).

Proof. Let us suppose that $(h,k) \in \Sigma$ and that $g(h,k) < 0$. Then, by Lemma 1.8, there are (unique except for the sign) α, β such that

$$\cosh^*\alpha = \frac{-2h}{k^2 - h^2 - 1} \quad \text{and} \quad \cosh^*\beta = \frac{-2k}{h^2 - k^2 - 1}\,. \tag{43}$$

Since $|\cosh^* t| > 1 \Rightarrow \sinh t \neq 0$, condition (24) is certainly verified. With $\cosh^*\alpha$, $\cosh^*\beta$ such that (43) holds, the first two equations of (19) are satisfied by

$$t = \frac{k^2 - h^2 + 1}{2}\,. \tag{44}$$

Then, with t as in (44), the third equation of (19) is verified iff

$$\frac{\sinh\beta}{\sinh\alpha} = -\frac{k^2 - h^2 - 1}{h^2 - k^2 - 1}\,. \tag{45}$$

Now, introducing (43) inside the identity $\sinh^2 t = (\cosh^* t)^2 - 1$, we obtain

$$\sinh^2\alpha = -\frac{g(h,k)}{(k^2 - h^2 - 1)^2} \quad \text{and} \quad \sinh^2\beta = -\frac{g(h,k)}{(h^2 - k^2 - 1)^2}\,. \tag{46}$$

Since we are assuming $g(h,k) < 0$, we may conclude that (45) holds iff

$$(\sinh\alpha, \sinh\beta) = \pm\left(\frac{\sqrt{-g(h,k)}}{k^2 - h^2 - 1}, \frac{-\sqrt{-g(h,k)}}{h^2 - k^2 - 1}\right). \tag{47}$$

Finally, it remains to note that for $(h,k) \in \Sigma$ the relation (45) gives also the inequality $\sinh\alpha \neq \sinh\beta$, i.e., condition (18). In conclusion, we have proved that there are α, β such that both conditions (18), (24) hold and $P = {}^t(h,k,0)$ satisfies formula (19). □

Taking into account (16), (43) and (47), we may conclude the following:

Claim 1.11. *Let us suppose* $h, k \neq 0$. *Then, system* (19) *with condition* (18) *is solvable* $\Leftrightarrow$ $(h,k) \in \Sigma$ *and* $g(h,k) < 0$. *Moreover, if* $(h,k) \in \Sigma$ *and*

[7] Note that condition $g(h,k) < 0$ implies $h, k \neq 0$.

$g(h,k)<0$, *then*

$$(48)\qquad \begin{aligned} Q_3 &= \frac{1}{k^2-h^2-1}\begin{pmatrix} -2h \\ 0 \\ \delta\sqrt{-g(h,k)} \end{pmatrix} \\ Q_3' &= \frac{1}{k^2-h^2+1}\begin{pmatrix} 0 \\ 2k \\ \delta\sqrt{-g(x,y)} \end{pmatrix} \end{aligned} \qquad \textit{with} \quad \delta=\pm 1.$$

Proof. As we have already observed at the beginning of §1.4,

$$h,k\neq 0 \ \text{ and } \ (18),\,(19) \quad \Longrightarrow \quad (24).$$

Therefore, to demonstrate the equivalence of the two conditions it is sufficient to apply Claims 1.9 and 1.10. The expressions of (48) derive from fact that, if $(h,k)\in\Sigma$ and $g(h,k)<0$, then $\cosh^*\alpha, \cosh^*\beta$ are necessarily determined by (43) and $\sinh\alpha, \sinh\beta$ by (47). □

The previous statement gives the necessary and sufficient conditions for the existence of Q_3, Q_3' such that (a), (b), (c) of Cond. 1.5 hold, i.e., such that there is a projection $\Pi_{\mathbf{v}}$ satisfying the conditions (15), (16) of Def. 2.4 of [3], when (1) holds and $P_3={}^t(h,k,0)$ with $h,k\neq 0$.
So, in order to have a hyperbolic Pohlke's projection, it only remains to verify if (d) of Cond. 1.5 holds when Q_3, Q_3' are given by (48). To this end, noting (16), we write:

$$(49)\qquad \mathbf{v}=\overrightarrow{Q_3Q_3'}=-(\cosh^*\alpha)\mathbf{i}+(\cosh^*\beta)\mathbf{j}+(\sinh\beta-\sinh\alpha)\mathbf{k},$$

with $\cosh^*\alpha$, $\cosh^*\beta$ as in (43) and $\sinh\alpha$, $\sinh\beta$ as in (47). Then we have:

Claim 1.12. *Let $P_3={}^t(h,k,0)$ with $(h,k)\in\Sigma$ such that $g(h,k)<0$. Then the projection direction $\mathbf{v}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$ given by (49) satisfies*

$$(50)\qquad l^2+m^2-n^2=4\,\frac{h^2+k^2-1}{(h^2-k^2)^2-1}\,.$$

Proof. When $(h,k)\in\Sigma$ and $g(h,k)<0$, the expressions (43) and (47) are well defined real numbers. Writing $\mathbf{v}$ as in (49) and using (47), we find that

$$(51)\qquad \begin{aligned} l^2+m^2-n^2 &= (\cosh^*\alpha)^2+(\cosh^*\beta)^2-(\sinh\beta-\sinh\alpha)^2 \\ &= 2\big(1+\sinh\alpha\sinh\beta\big) \\ &= 2\left[1+\frac{g(h,k)}{(k^2-h^2-1)(h^2-k^2-1)}\right] \\ &= 2\left[1-\frac{g(h,k)}{(h^2-k^2)^2-1}\right] \\ &= 4\,\frac{h^2+k^2-1}{(h^2-k^2)^2-1}\,. \end{aligned}$$

□

1.5. **Proof of part 2) of Thm. 1.1.** Noting that

$$f(h,k) \neq 0 \quad \Leftrightarrow \quad (h,k) \in \Sigma \quad \text{and} \quad h^2 + k^2 \neq 1, \tag{52}$$

and summarizing up the results of the previous sections, we can now prove part 2) of Thm. 1.1.

$\Rightarrow$ Let $\Pi_{\mathbf{v}_o}$ be a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3. Having $h, k \neq 0$, taking into account Cond. 1.5 and the arguments at the beginning of Section 1.2, we deduce from Claim 1.11 that $(h,k) \in \Sigma$ and $g(h,k) < 0$. Furthermore, the points Q_3, Q_3' are necessarily given by (48). Since $\Pi_{\mathbf{v}_o}$ is non-degenerate, (50) of Claim. 1.12 gives $h^2 + k^2 \neq 1$. By (52) we see that $f(h,k) \neq 0$. Therefore (2) and (3) are verified.

$\Leftarrow$ Conversely, let us suppose that (2), (3) are true. Then $(h,k) \in \Sigma$ and $g(h,k) < 0$. Thus the existence of a projection $\Pi_{\mathbf{v}_o} : \mathbb{R}^3 \to \omega$ satisfying (15) and (16) of Def. 2.4 of [3] follows from Claim 1.11. Thanks to Claim 1.12 and (52), the condition $f(h,k) \neq 0$ also implies that (d) of Cond. 1.5 is true, i.e., $\Pi_{\mathbf{v}_o}$ is non-degenerate. Hence $\Pi_{\mathbf{v}_o}$ is a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3.

As for the uniqueness of the hyperbolic Pohlke's projection $\Pi_{\mathbf{v}_o}$, assuming that exists, i.e., (2) and (3) hold, we recall that by Claim 1.4 we necessarily have $\rho = 1$, that is $\mathscr{H} = \mathscr{H}(1)$. Furthermore, the vector $\overrightarrow{Q_3Q_3'}$, given by Claim 1.11, is uniquely determined up to choosing the plus and minus sign in formula (48). This means that we can obtain only two projections, say $\Pi_{\mathbf{v}_o}$ and $\Pi_{\bar{\mathbf{v}}_o}$, which are symmetric with respect to the plane ω (according to the second part of Rem. 1.2, we have $\Pi_{\bar{\mathbf{v}}_o} = \bar{\Pi}_{\mathbf{v}_o}$). Hence, taking into account that $\mathscr{H} = \mathscr{H}(1)$, from (6) we may conclude that $\mathscr{C}_{\mathbf{v}_o} = \Pi_{\mathbf{v}_o}(\mathscr{H} \cap \pi_{\mathbf{v}_o})$ is unique. Furthermore, by Cor. 3.11 of [3] and (50) above, the conic $\mathscr{C}_{\mathbf{v}_o}$ is an ellipse or a hyperbola depending on whether it is $f(h,k) < 0$ or $f(h,k) > 0$. Finally, the expression (4) of $\mathbf{v}_o$ follows immediately from (48). In fact, it is sufficient to write explicitly $\overrightarrow{Q_3Q_3'}$ taking into account the representation (14) of the points P_1, P_2, i.e., $\overrightarrow{OP_1} = \mathbf{i}$ and $\overrightarrow{OP_2} = \mathbf{j}$.

2. The degenerate case

In the "degenerate case" we suppose the segments OP_1, OP_2, OP_3 are not contained in a line but two of them are parallel. To investigate the existence of hyperbolic Pohlke's projections also in this situation, we introduce degenerate ellipses. [8]

Definition 2.1. *If OP, OQ do not both vanish and $OP \parallel OQ$, the degenerate ellipse $\mathcal{E}_{P,Q}$ is the segment MN parallel to OP, OQ such that*

$$|MN|^2 = 4\big(|OP|^2 + |OQ|^2\big) \quad \text{and} \quad \frac{M+N}{2} = O. \tag{53}$$

Given a central conic $\mathcal{C}$, with center O, we say that $\mathcal{C}$ circumscribes the degenerate ellipse $\mathcal{E}_{P,Q}$ (or that $\mathcal{E}_{P,Q}$ is inscribed in $\mathcal{C}$) if $M, N \in \mathcal{C}$.

Now we can reformulate Def. 2.7 of [3] just saying that:

[8] We assume, by convention, that the null segment is parallel to any other segment. Degenerate ellipses were introduced in [1, pp. 372-373]. See also Defs. 3.1, 3.3 of [2].

Definition 2.2. *If OP_1, OP_2, OP_3 are not contained in a line but two of them are parallel, a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3 is a hyperbola, with center O, circumscribing the three (eventually degenerate) ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.*

Using the Def. 2.2, instead of Def. 2.7 of [3], we can state the following:

Theorem 2.3. *Suppose OP_1, OP_2, OP_3 are not contained in a line. If two of them are parallel, then there are infinitely many, distinct hyperbolic Pohlke's projections (conics) if these two segments are equal (i.e., congruent), none if they are different.*

2.1. **Some preliminary lemmas.** In Claim 3.19 of [3] we have assumed that $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ is an admissible conic, tangent to $\mathscr{C}_{\mathbf{v}}$. But in view of the proof of Thm. 2.3 we need also to consider what happen if $\mathcal{C}$ is a degenerate ellipse (in the sense of Def. 2.1) inscribed in $\mathscr{C}_{\mathbf{v}}$, when $\mathscr{C}_{\mathbf{v}}$ is a hyperbola.

Claim 2.4. *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate and let $\ell \subset \omega$ be a line through O such that $\ell \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$. Let ζ be the plane through ℓ and parallel to $\mathbf{v}$. Then $\mathscr{H} \cap \zeta$ is an ellipse (hyperbola) iff $\mathscr{C}_{\mathbf{v}}$ is a hyperbola (ellipse).*

Proof. As in the proof of Claim 3.15 of [3], it suffices to prove the result for

$$\mathbf{v} = \lambda \mathbf{j} + n\mathbf{k} \quad \text{with} \quad \lambda^2 - n^2 \neq 0, \tag{54}$$

because $\mathscr{H}$ is invariant under rotations about the z-axis. Therefore, taking into account formula (43) of [3], $\mathscr{C}_{\mathbf{v}} \subset \omega$ has equation

$$x^2 + \left(\frac{n^2}{n^2 - \lambda^2}\right) y^2 = \rho^2 \quad \text{with } \rho > 0. \tag{55}$$

Now, by hypothesis, there exits a point $L = L(x_L, y_L, 0) \in \ell \cap \mathscr{C}_{\mathbf{v}}$. By (55) the coordinates of L must then satisfy the relation

$$n^2\left(x_L^2 + y_L^2\right) - \lambda^2 x_L^2 = \left(n^2 - \lambda^2\right)\rho^2. \tag{56}$$

On the other hand, ζ is the plane through OL and parallel to $\mathbf{v}$. Thus ζ has equation

$$\zeta : \, (ny_L)x - (nx_L)y + (\lambda x_L)z = 0. \tag{57}$$

Noting (56), by Claim 3.5 of [3], we deduce that:

- $\mathscr{H} \cap \zeta$ is an ellipse $\Leftrightarrow$ $n^2 - \lambda^2 < 0$ $\Leftrightarrow$ $\mathscr{C}_{\mathbf{v}}$ is a hyperbola;
- $\mathscr{H} \cap \zeta$ is a hyperbola $\Leftrightarrow$ $n^2 - \lambda^2 > 0$ $\Leftrightarrow$ $\mathscr{C}_{\mathbf{v}}$ is an ellipse.

Claim 2.5. *Given $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$, let $\zeta \parallel \mathbf{v}$ be a plane through O. Let $\mathcal{E} \subset \zeta$ be an ellipse with center O and let $\Pi_{\mathbf{v}}(\mathcal{E}) = MN$, for suitable $M, N \in \omega \cap \zeta$.*

1) Let (OQ_1, OQ_2) be a pair of conjugate semi-diameters for $\mathcal{E}$. If $P_1 = \Pi_{\mathbf{v}}(Q_1)$ and $P_2 = \Pi_{\mathbf{v}}(Q_2)$, then we have

$$|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big). \tag{58}$$

2) If $P_1, P_2 \in \omega \cap \zeta$ satisfy (58), then there are $Q_1, \widehat{Q}_1, Q_2, \widehat{Q}_2 \in \mathcal{E}$ such that $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathcal{E} = \{Q_1, \widehat{Q}_1\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathcal{E} = \{Q_2, \widehat{Q}_2\}$ and (OQ_1, OQ_2), $(O\widehat{Q}_1, O\widehat{Q}_2)$ are distinct pairs of conjugate semi-diameters for $\mathcal{E}$.

Proof. 1) To begin with, we introduce orthogonal coordinates $\mathbf{x}, \mathbf{y}$ in the plane ζ such that $O = (0,0)$ and

$$\mathcal{E} : \frac{\mathbf{x}^2}{a^2} + \frac{\mathbf{y}^2}{b^2} = 1 \quad \text{with} \quad a, b > 0. \tag{59}$$

In this situation it is well known (see [4], p. 39) that OQ_1, OQ_2 are conjugate semi-diameters for $\mathcal{E}$ if and only if there is $\theta \in [0, 2\pi)$ such that

$$Q_1 = \big(a\cos\theta, b\sin\theta\big) \quad \text{and} \quad Q_2 = \pm\big(a\sin\theta, -b\cos\theta\big). \tag{60}$$

Since $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is linear, given a unit vector $\mathbf{u}$, $\mathbf{u} \parallel \omega \cap \zeta$, there are $\alpha, \beta \in \mathbb{R}$ (not both zero) such that, if $P_1 = \Pi_{\mathbf{v}}(Q_1)$ and $P_2 = \Pi_{\mathbf{v}}(Q_2)$,

$$\overrightarrow{OP_1} = \big(a\alpha\cos\theta + b\beta\sin\theta\big)\mathbf{u} \quad \text{and} \quad \overrightarrow{OP_2} = \pm\big(a\alpha\sin\theta - b\beta\cos\theta\big)\mathbf{u}, \tag{61}$$

for all $\theta \in [0, 2\pi)$. From (61) we immediately have

$$|OP_1|^2 + |OP_2|^2 = (a\alpha)^2 + (b\beta)^2 \quad \text{for all} \quad \theta \in [0, 2\pi) \tag{62}$$

and, in particular,

$$|OM|^2 = |ON|^2 = (a\alpha)^2 + (b\beta)^2, \tag{63}$$

because $\Pi(Q_1) = M$ or N when $\Pi(Q_2) = O$, that is, when OQ_2 is parallel to the projection direction $\mathbf{v}$. Hence, from (63) we deduce that $|MN|^2 = 4(a\alpha)^2 + 4(b\beta)^2$, because $O = \frac{M+N}{2}$.

2) Conversely, let $P_1, P_2 \in \omega \cap \zeta$ such that the relation (58) is true. Before proceeding, let's remember that the ellipse $\mathcal{E}$ has oblique symmetry, in the direction of $\mathbf{v}$, with respect to the line, say $l_{\mathbf{v}}$, through O and parallel to the direction conjugate to that of $\mathbf{v}$. Thus, if

$$\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathcal{E} = \{R_1, \widehat{R}_1\} \quad \text{and} \quad \Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathcal{E} = \{R_2, \widehat{R}_2\},$$

it is clear that the points R_1 and R_2 are obliquely symmetrical (in the direction of $\mathbf{v}$ and with respect to $l_{\mathbf{v}}$) to $\widehat{R}_1$ and $\widehat{R}_2$, respectively. In addition, we know that $R_1 = \widehat{R}_1 \Leftrightarrow R_1 \in l_{\mathbf{v}} \Leftrightarrow P_1 = M$ or N (i.e., $P_2 = O$, by (58)) and, similarly, for the couple $R_2, \widehat{R}_2$. Now, starting for instance from R_1, we certainly have

$$R_1 = (a\cos\theta_1, b\sin\theta_1) \quad \text{for a suitable} \quad \theta_1 \in [0, 2\pi). \tag{64}$$

By (58) and noting (61) and (62), one of the following must hold:

$$P_2 = \Pi_{\mathbf{v}}\big(a\sin\theta_1, -b\cos\theta_1\big) \quad \text{or} \quad P_2 = \Pi_{\mathbf{v}}\big(-a\sin\theta_1, b\cos\theta_1\big).^{9} \tag{65}$$

If, for example, the second of (65) holds, we define

$$Q_1 = R_1 = (a\cos\theta_1, b\sin\theta_1) \quad \text{and} \quad Q_2 = (-a\sin\theta_1, b\cos\theta_1).^{10} \tag{66}$$

By (60) above, (OQ_1, OQ_2) is a pair of conjugate semi-diameters such that $\Pi_{\mathbf{v}}(Q_1) = P_1$ and $\Pi_{\mathbf{v}}(Q_2) = P_2$. Finally, denoting with $\widehat{Q}_1$ and $\widehat{Q}_2$ the symmetric to Q_1 and Q_2, respectively, we can easily see that

$$(O\widehat{Q}_1, O\widehat{Q}_2) \tag{67}$$

[9] There are only two points at a distance of $\frac{1}{2}\sqrt{|MN|^2 - 4|OP_1|^2}$ from O in $\omega \cap \zeta$.

[10] Clearly, we have $Q_2 = R_2$ or $\widehat{R}_2$.

is a pair of conjugate semi-diameters such that $(O\widehat{Q}_1, O\widehat{Q}_2) \neq (OQ_1, OQ_2)$. In fact, if for example $\widehat{Q}_1 = Q_1$, then (as we observed above) $P_1 = M$ or N and $P_2 = O$. But, in turn, $P_2 = O$ implies $\widehat{Q}_2 \neq Q_2$. $\square$

Next, we assume that $OP_1, OP_2 \subset \omega$ do not both vanish and that $OP_1 \parallel OP_2$. Then we consider the degenerate ellipse $\mathcal{E}_{P_1,P_2} = MN$, according to Def. 2.1. Applying Claims 2.4 and 2.5, we deduce the following:

Claim 2.6. *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate and such that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola. Besides, let $\mathcal{E}_{P_1,P_2} = MN$ be a degenerate ellipse inscribed in $\mathscr{C}_{\mathbf{v}}$ and let ζ be the plane through MN and parallel to $\mathbf{v}$.*

Then $\mathscr{H} \cap \zeta$ is an ellipse, with center O, such that $\Pi_{\mathbf{v}}(\mathscr{H} \cap \zeta) = \mathcal{E}_{P_1,P_2}$. Furthermore, there are $Q_1, Q_1', Q_2, Q_2' \in \mathscr{H} \cap \zeta$ such that $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathscr{H} = \{Q_1, Q_1'\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathscr{H} = \{Q_2, Q_2'\}$ and (OQ_1, OQ_2), (OQ_1', OQ_2') are distinct pairs of conjugate semi-diameters of $\mathscr{H} \cap \zeta$.

Proof. Since MN is a segment through O and $M, N \in \mathscr{C}_{\mathbf{v}}$, from Claim 2.4 we know that $\mathcal{E} = \mathscr{H} \cap \zeta$ is an ellipse, with center O, because $\mathscr{C}_{\mathbf{v}}$ is a hyperbola. Then we can see that

$$\Pi_{\mathbf{v}}(\mathcal{E}) = \mathcal{E}_{P_1,P_2}. \tag{68}$$

Indeed, assuming $\mathcal{E}_{P_1,P_2} = MN$ inscribed in the hyperbola $\mathscr{C}_{\mathbf{v}}$, we have: $\mathcal{E}_{P_1,P_2} \subset \Pi_{\mathbf{v}}(\mathcal{E})$, because $\mathcal{E}_{P_1,P_2} \subset \mathbf{int}(\mathscr{C}_{\mathbf{v}})$, and also $\mathcal{E}_{P_1,P_2} \supset \Pi_{\mathbf{v}}(\mathcal{E})$ because $M, N \in \mathscr{C}_{\mathbf{v}}$. To proceed, we recall that $\mathcal{E}_{P_1,P_2} = MN$ implies

$$|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big). \tag{69}$$

Moreover,

$$\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{H} = \Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathcal{E}, \quad \text{for } i = 1, 2,$$

and $\mathcal{E}$ has oblique symmetry, in the direction of $\mathbf{v}$, with respect to the line $l_{\mathbf{v}} = \zeta \cap \pi_{\mathbf{v}}$.[11] It is therefore sufficient to apply part 2) of Claim 2.5 with $\mathcal{E} = \mathscr{H} \cap \zeta$ to obtain the thesis immediately. $\square$

To conclude we prove the following result which extends to the degenerate case the result of Claim 3.26 of [3].

Claim 2.7. *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be a parallel projection. Let $Q_1, Q_2 \in \mathscr{H}$ such that $OQ_1 \parallel T_{\mathscr{H}}(Q_2)$ and let $P_1 = \Pi_{\mathbf{v}}(Q_1)$, $P_2 = \Pi_{\mathbf{v}}(Q_2)$. If $OP_1 \parallel OP_2$, then $\Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2))$ is the degenerate ellipse $\mathcal{E}_{P_1,P_2}$ determined by the segments OP_1, OP_2. If we further assume that $\Pi_{\mathbf{v}}$ is non-degenerate, then $\mathscr{C}_{\mathbf{v}}$ is necessarily a hyperbola and $\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$ in the sense Def. 2.1.*

Proof. By Claim 3.25 of [3], $OQ_1 \nparallel OQ_2$ and $\mathscr{C}(Q_1, Q_2)$ is an ellipse with conjugate semi-diameters OQ_1, OQ_2. Having $\Pi_{\mathbf{v}}(Q_1) = P_1$, $\Pi_{\mathbf{v}}(Q_2) = P_2$ with $OQ_1 \nparallel OQ_2$, the segments OP_1, OP_2 cannot both vanish. Assuming $OP_1 \parallel OP_2$, it follows that $\zeta = \langle O, Q_1, Q_2 \rangle$ is the plane through O, P_1, P_2 and parallel to the vector $\mathbf{v}$. Next, we consider the degenerate ellipse $\mathcal{E}_{P_1,P_2}$ and then we apply part 1) of Claim 2.5 with $\mathcal{E} = \mathscr{C}(Q_1, Q_2)$. We have that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2)) = MN = \mathcal{E}_{P_1,P_2}, \tag{70}$$

[11] It turns out that $l_{\mathbf{v}} = \zeta \cap \pi_{\mathbf{v}}$ is the diameter of $\mathcal{E}$ conjugate to the diameter parallel to $\mathbf{v}$. In other words, in the plane ζ the $\pi_{\mathbf{v}}-$symmetry is equivalent to the oblique symmetry, in the direction of $\mathbf{v}$, of the conjugate diameters of $\mathcal{E}$.

because OQ_1, OQ_2 are conjugate semi-diameters of $\mathscr{C}(Q_1, Q_2)$ and, by (58), we have $|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big)$. Furthermore, since $\mathscr{C}(Q_1, Q_2)$ is an ellipse in $\zeta = \langle O, Q_1, Q_2\rangle$, we can see as in the proof of Claim 3.26 of [3] that $\mathscr{C}(Q_1, Q_2) \cap \pi_{\mathbf{v}} \neq \emptyset$. Hence, $\mathcal{E}_{P_1,P_2} \cap \Pi_{\mathbf{v}}\big(\mathscr{H} \cap \pi_{\mathbf{v}}\big) \neq \emptyset$.

If we now suppose $\Pi_{\mathbf{v}}$ is non-degenerate, we have

$$MN \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset. \tag{71}$$

This means that the line ℓ through M, N is a line through O such that $\ell \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$. Then, applying Claim 2.4, we see that $\mathscr{C}_{\mathbf{v}}$ must be a hyperbola, because $\mathscr{C}(Q_1, Q_2) = \mathscr{H} \cap \zeta$ is an ellipse. [12] Finally, $\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$. In fact, we have shown above that $MN \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$ and, by Cor. 3.13 of [3], we know that $MN \subset \mathbf{int}(\mathscr{C}_{\mathbf{v}})$. So we have $M, N \in \mathscr{C}_{\mathbf{v}}$, since M, N (as well $\mathscr{C}_{\mathbf{v}}$) are symmetrical with respect to the origin O. □

2.2. **Proof of Theorem 2.3.** We will first show that the equivalence (1) ⇔ (2) of Thm. 2.8 of [3] remains valid under the hypotheses of Thm. 2.3, if we allow degenerate ellipses, in the sense of Def. 2.1, and if we replace Def. 2.7 of [3] with Def. 2.2 as hyperbolic Pohlke's conic definition.
According to the hypotheses, we will assume that OP_1, OP_2, OP_3 are not contained in a line, but two of them are parallel to each other. More precisely, in the following we will suppose that

$$OP_1 \nparallel OP_2 \quad \text{and} \quad OP_2 \parallel OP_3. \tag{72}$$

(1) ⇒ (2). We apply Claim 3.26 of [3] (as in the proof of Thm. 2.8 of [3]) if $OP_i \nparallel OP_j$, and Claim 2.7 if $OP_i \parallel OP_j$. To begin with, since we suppose $OP_1 \nparallel OP_2$, by the conditions (15), (16) of Def. 2.4 of [3] and Claim 3.26 of [3] we deduce that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2)) = \mathcal{E}_{P_1,P_2} \quad \text{and } \mathscr{C}_{\mathbf{v}} \text{ is tangent to } \mathcal{E}_{P_1,P_2}. \tag{73}$$

To proceed, we consider then the pair OP_2, OP_3. In this case $OP_2 \parallel OP_3$, thus $\mathcal{E}_{P_2,P_3}$ is a degenerate ellipse in the sense of Def. 2.1. Hence, by Claim 2.7, we deduce that $\Pi_{\mathbf{v}}(\mathscr{C}(Q_2, Q_3)) = \mathcal{E}_{P_2,P_3}$ and that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola circumscribing $\mathcal{E}_{P_2,P_3}$. Knowing that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola, from (73) and Cor. 3.13 of [3] it also follows that $\mathscr{C}_{\mathbf{v}}$ circumscribes the ellipse $\mathcal{E}_{P_1,P_2}$. Finally, we consider the pair OP_3, OP_1. Applying as above Claim 3.26 of [3] (if $OP_3 \nparallel OP_1$) or Claim 2.7 (if $P_3 = O$), we find that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_3, Q_1')) = \mathcal{E}_{P_3,P_1} \quad \text{and } \mathscr{C}_{\mathbf{v}} \text{ circumscribes } \mathcal{E}_{P_3,P_1}. \tag{74}$$

In conclusion, $\mathscr{C}_{\mathbf{v}}$ is a hyperbola circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Hence $\mathscr{C}_{\mathbf{v}}$ is a hyperbolic Pohlke's conic, in the sense of Def. 2.2, for OP_1, OP_2, OP_3.

(2) ⇒ (1). Let $\mathcal{C}$ be a hyperbolic Pohlke's conic in the sense of Def. 2.2. By applying Claim 3.15 and Cor. 3.17 of [3] (as in (2) ⇒ (1), Step 1, of the proof of Thm. 2.8 of [3]) we can determine the hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$ and the projection direction, represented by $\mathbf{v}$, up to symmetry with respect to the

[12] Noting (70), we may deduce directly from Cor. 3.13 of [3] that $\mathscr{C}_{\mathbf{v}}$ must be a hyperbola. In fact, $O \in MN \subset \Pi_{\mathbf{v}}(\mathscr{H})$ and this means that $\mathscr{C}_{\mathbf{v}}$ cannot be an ellipse.

plane ω. It automatically follows that $\mathbf{v}$ is non-degenerate (by Claim 3.18 of [3]) and that

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) \stackrel{\text{def}}{=} \mathscr{C}_{\mathbf{v}}.$$

After this we consider the (eventually degenerate) ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Using 1) and 2) of Claim 3.19 of [3] (if $OP_i \nparallel OP_j$) or Claim 2.6 (if $OP_i \parallel OP_j$) and then Claim A.3 of [3], we can show (as in (2) $\Rightarrow$ (1), Step 2, of the proof of Thm. 2.8 of [3]) that there are $Q_1, Q_2, Q_3 \in \mathscr{H}$ such that the conditions (15), (16) of Def. 2.4 of [3] are verified. In this way we prove that $\Pi_{\mathbf{v}}$ is a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3.

Conclusion of the proof. We can now demonstrate that under the assumptions (72) the are infinite, distinct hyperbolic Pohlke's projections (conics) if $|OP_2| = |OP_3|$, none if $|OP_2| \neq |OP_3|$.

To this end, we resort to the circular case as in the proof of Thm. 2.8 of [3]. Namely, since we assume $OP_1 \nparallel OP_2$, we may consider the affine transformation $\Phi : \omega \to \omega$ defined in (79) of [3]. In this case we have $\Phi(P_i) = N_i$, for $1 \leq i \leq 3$, with

$$ON_1 \perp ON_2, \quad |ON_1| = |ON_2| = 1 \quad \text{and} \quad ON_2 \parallel ON_3. \tag{75}$$

We note also that Claim 4.2 of [3] continues to hold even though we apply Def. 2.2 instead of Def. 2.7 of [3]. So we still have that $\mathcal{C}$ is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3 if and only if $\Phi(\mathcal{C})$ is a hyperbolic Pohlke's conic for ON_1, ON_2, ON_3. [13]

Now, having $ON_2 \parallel ON_3$, by part 1) of Thm. 1.1 there are infinite, distinct hyperbolic Pohlke's projections for ON_1, ON_2, ON_3 if $|ON_3| = 1$, none if $|ON_3| \neq 1$. By the equivalence (1) $\Leftrightarrow$ (2) proved above, it follows that there are infinite, distinct hyperbolic Pohlke's conics for ON_1, ON_2, ON_3 if $|ON_3| = 1$, none if $|ON_3| \neq 1$. Since

$$|ON_3| = 1 \quad \Leftrightarrow \quad |OP_3| = |OP_2|, \tag{76}$$

we deduce that, under assumption (72), there are infinite, distinct hyperbolic Pohlke's conics for OP_1, OP_2, OP_3 if $|OP_3| = |OP_2|$, none if $|OP_3| \neq |OP_2|$. Finally, again by the equivalence (1) $\Leftrightarrow$ (2), the same holds for the hyperbolic Pohlke's projections for OP_1, OP_2, OP_3.

3. Examples of hyperbolic Pohlke's projections and conics

We conclude by building two examples of hyperbolic Pohlke's projections in which Thm. 2.8 of [3] applies. In the first the hyperbolic Pohlke's conic $\mathcal{C}$ is an ellipse; in the second $\mathcal{C}$ is a hyperbola. We indicate now the main steps by referring to §3.1, §3.2 below for further details.

To proceed, it is easier to set ρ right away (we take $\rho = 1$) and then work backwards choosing Q_1, Q_2, Q_3 and Q'_1 (or, equivalently for (78) below, Q'_3) in $\mathscr{H} = \mathscr{H}(\rho)$ so that (16) of [3] holds. For this, we have only to take into account Claims 3.1, 3.21 and Cor. 3.22 of [3]. Having done this, and verified that $Q_3Q'_3 \nparallel \omega$, the projection direction, i.e., $\mathbf{v} \parallel Q_3Q'_3$, and the points $P_1, P_2, P_3 \in \omega$ are uniquely determined. We recall the following facts: [14]

[13] It is worth noting that if $\mathcal{E}_{P,Q}$ is a degenerate ellipse, then $\Phi(\mathcal{E}_{P,Q}) = \mathcal{E}_{\Phi(P),\Phi(Q)}$.

[14] See Claim 3.21 and Cor. 3.22 of [3].

(i) given $R(x_R, y_R, z_R), S(x_S, y_S, z_S) \in \mathscr{H}$,

$$OR \parallel T_{\mathscr{H}}(S) \quad \Leftrightarrow \quad OS \parallel T_{\mathscr{H}}(R) \quad \Leftrightarrow \quad x_R x_S + y_R y_S - z_R z_S = 0; \tag{77}$$

(ii) if $R', S' \in \mathscr{H}$ are $\pi_{\mathbf{v}}$–symmetric to $R, S \in \mathscr{H}$ respectively, then

$$OR' \parallel T_{\mathscr{H}}(S) \quad \Leftrightarrow \quad OR \parallel T_{\mathscr{H}}(S'). \tag{78}$$

Given $Q_1 \in \mathscr{H}$, taking into account (77), we choose $Q_2, Q_3^* \in \mathscr{H}$ such that $OQ_2, OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$,[15] and $Q_3 \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. For **a**), **b**) of §3.2 below, we always get $Q_3 \neq Q_3^*$ and we can choose Q_3, Q_3^* such that $Q_3Q_3^* \nparallel \omega$ gives the direction of a non-degenerate projection onto the plane ω. For **c**) of §3.2, to prevent two of the segments OP_1, OP_2, OP_3 from being parallel, we also require $OQ_3 \nparallel OQ_1$ and, after choosing Q_2 and Q_3,

$$OQ_3^* \nparallel OQ_2 \quad \text{and} \quad \overrightarrow{OQ_3^*} \neq \overrightarrow{OQ_3} - \bar{\lambda}\,\overrightarrow{OQ_1} \pm \bar{\lambda}\,\overrightarrow{OQ_2}, \tag{79}$$

where $\bar{\lambda} \overset{\text{def}}{=} x_1x_3 + y_1y_3 - z_1z_3$ with $Q_1(x_1, y_1, z_1)$, $Q_3(x_3, y_3, z_3)$. In this way we will be in the hypotheses of Thm. 2.8 of [3].

Then, taking $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \parallel Q_3Q_3^*$, it follows $\mathbf{v}$ is non-degenerate and that Q_3 and Q_3^* are $\pi_{\mathbf{v}}$–symmetric, i.e., $Q_3^* = Q_3'$.[16] By (78), we have also $OQ_3 \parallel T_{\mathscr{H}}(OQ_1')$, because $OQ_3' = OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$. The conditions of (16) of [3] are therefore satisfied and we can define $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \leq i \leq 3)$. By (79) the segments OP_1, OP_2, OP_3 are non-parallel, so we can draw the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Finally we trace the contour of the projection of $\mathscr{H}$ into ω, that is the conic:

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}). \tag{80}$$

By Thm. 2.8 of [3], $\mathcal{C}$ is the unique hyperbolic Pohlke's conic for OP_1, OP_2, OP_3. From Cor. 3.11 of [3], we also know $\mathcal{C}$ is an ellipse (hyperbola) if $l^2 + m^2 - n^2 < 0$ (> 0).

Example 3.1. *We fix* $\rho = 1$ *and* $Q_1 = (2, 0, -\sqrt{3})$. *Taking into account* (77)–(79) *we choose* $Q_2 = (-\sqrt{3}, \sqrt{2}, 2)$, $Q_3 = (0, \sqrt{2}, 1)$ *and* $Q_3' = (\frac{3\sqrt{3}}{4}, \frac{5}{4}, -\frac{3}{2})$. *In this way the projection direction is given by*

$$\mathbf{v} = \tfrac{3\sqrt{3}}{4}\mathbf{i} + \left(\tfrac{5}{4} - \sqrt{2}\right)\mathbf{j} - \tfrac{5}{2}\mathbf{k}. \tag{81}$$

Since $\left(\frac{3\sqrt{3}}{4}\right)^2 + \left(\frac{5}{4} - \sqrt{2}\right)^2 - \left(\frac{5}{2}\right)^2 < 0$, $\mathbf{v}$ *is non-degenerate.*

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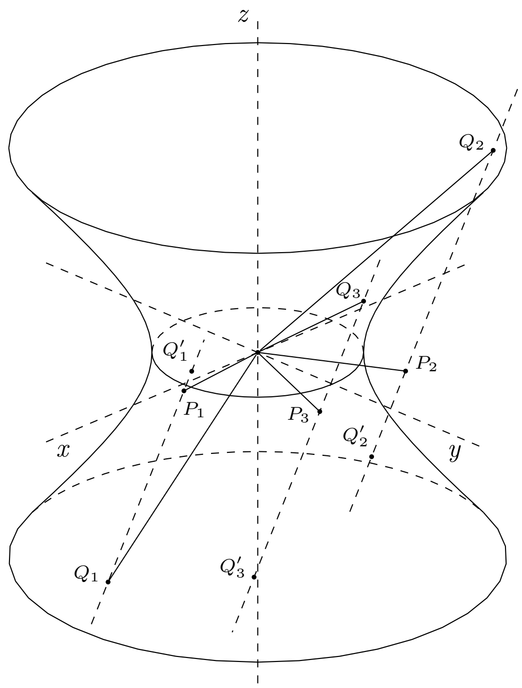
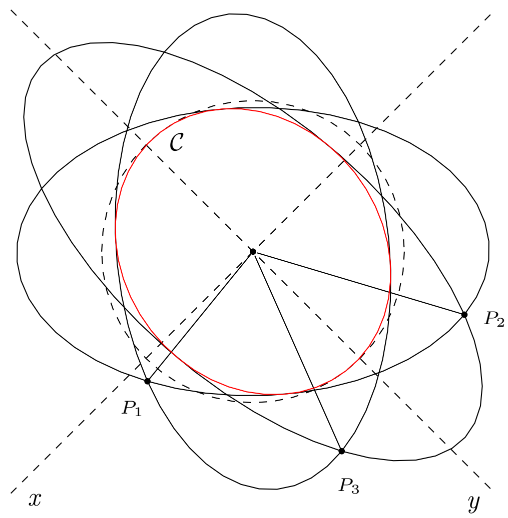
15 We use Q_3^* just to proceed, because we don't have $\mathbf{v}$ yet. We will soon set $Q_3' = Q_3^*$.

16 By Claim 3.1 of [3].



Hyperbolic Pohlke's projection of Ex. 3.1 (scaled 0.7).

We find $P_1 = (\frac{11}{10}, \frac{4\sqrt{6}-5\sqrt{3}}{10}, 0)$, $P_2 = (-\frac{2\sqrt{3}}{5}, \frac{5+\sqrt{2}}{5}, 0)$ *and* $P_3 = (\frac{3\sqrt{3}}{10}, \frac{6\sqrt{2}+5}{10}, 0)$. *The segments* OP_1, OP_2, OP_3 *are non-parallel and the hyperbolic Pohlke's conic* $\mathcal{C}$ *is an ellipse.*

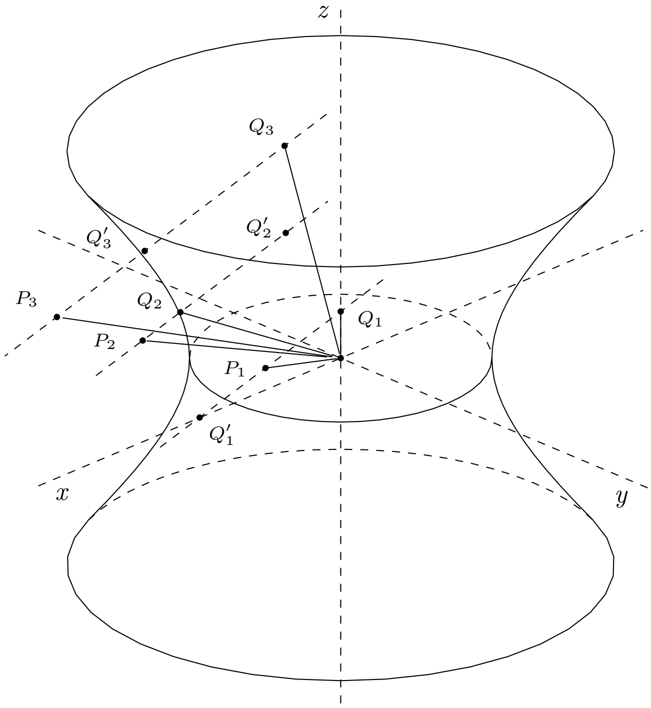


In plane $\{z = 0\}$, *the ellipse* $\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ *inscribed in* $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

Example 3.2. *We fix* $\rho = 1$ *and* $Q_1 = (1, 1, 1)$. *Taking into account* (77)–(79) *we choose* $Q_2 = (1, -\frac{1}{2}, \frac{1}{2})$, $Q_3 = (\frac{5}{2}, \frac{79}{40}, \frac{121}{40})$ *and* $Q'_3 = (\frac{3}{2}, -\frac{1}{3}, \frac{7}{6})$. *In this way the projection direction is given by*

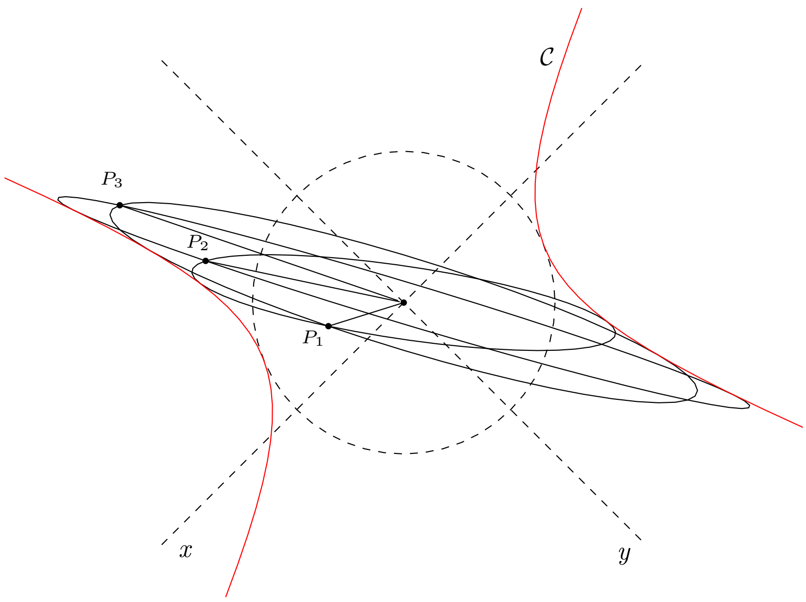
$$\mathbf{v} = \mathbf{i} + \frac{277}{120}\mathbf{j} + \frac{223}{120}\mathbf{k}. \tag{82}$$

Since $1+\left(\frac{277}{120}\right)^2-\left(\frac{223}{120}\right)^2>0$, $\mathbf{v}$ *is non-degenerate.*



Hyperbolic Pohlke's projection of Ex. 3.2.

We find $P_1=(\frac{103}{223},-\frac{54}{223},0)$, $P_2=(\frac{163}{223},-\frac{250}{223},0)$ *and* $P_3=(\frac{389}{446},-\frac{795}{446},0)$. *The segments* OP_1, OP_2, OP_3 *are non-parallel and the hyperbolic Pohlke's conic* $\mathcal{C}$ *is a hyperbola.*



In plane $\{z=0\}$, *the hyperbola* $\mathcal{C}=\Pi_{\mathbf{v}}(\mathscr{H}\cap\pi_{\mathbf{v}})$ *circumscribing* $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

3.1. **Detailed construction of Examples 3.1 and 3.2.** For completeness we report the calculations necessary to obtain Ex. 3.1 and Ex. 3.2. After which, in §3.2 below, we will make some general considerations.

Ex. 3.1 (calculations). To begin with, we set $\rho = 1$ and $Q_1(2, 0, -\sqrt{3})$.

Choice of Q_2. We look for $Q_2(x_2, y_2, z_2) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$. With $Q_1(2, 0, -\sqrt{3})$, this is equivalent to require:

$$x_2^2 + y_2^2 - z_2^2 = 1 \quad \text{and} \quad 2x_2 + \sqrt{3}z_2 = 0, \tag{83}$$

that is, $4y_2^2 = 4 + z_2^2$ and $2x_2 = -\sqrt{3}z_2$. We can take $z_2 = 2$, $y_2 = \sqrt{2}$ and $x_2 = -\sqrt{3}$. So we have the point $Q_2(-\sqrt{3}, \sqrt{2}, 2)$.

Choice of Q_3^*. We impose the same conditions on $Q_3^*(x_3^*, y_3^*, z_3^*)$ as on Q_2:

$$4{y_3^*}^2 = 4 + {z_3^*}^2, \quad 2x_3^* = -\sqrt{3}z_3^*. \tag{84}$$

This time we can choose $z_3^* = -\frac{3}{2}$, $y_3^* = \frac{5}{4}$ and $x_3^* = -\frac{3\sqrt{3}}{4}$. Thus, we find the point $Q_3^*(\frac{3\sqrt{3}}{4}, \frac{5}{4}, -\frac{3}{2})$.

Choice of Q_3. After Q_2, Q_3^* we need $Q_3(x_3, y_3, z_3) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. Having $Q_2(-\sqrt{3}, \sqrt{2}, 2)$, this means that

$$x_3^2 + y_3^2 - z_3^2 = 1 \quad \text{and} \quad -\sqrt{3}x_3 + \sqrt{2}y_3 - 2z_3 = 0. \tag{85}$$

Here we set $x_3 = 0$. Then $y_3 = \sqrt{2}z_3$ and $z_3^2 = 1$. We choose $z_3 = 1$ and we finally get the point $Q_3(0, \sqrt{2}, 1)$.

Determination of $\mathbf{v}$. Since $Q_3^* \neq Q_3$ and $Q_3Q_3^* \nparallel \omega$, $Q_3Q_3^*$ gives a projection direction onto the plane ω. Furthermore, this direction is non-degenerate. Thus, we define $\mathbf{v}$ as in (81). In this way we get $Q_3' = Q_3^*$ (i.e., Q_3^* is $\pi_{\mathbf{v}}$–symmetric to Q_3) and $OQ_3 \parallel T_{\mathscr{H}}(Q_1')$, by (78).

Determination of P_1, P_2, P_3. Having fixed $\mathbf{v}$, we can project the points Q_1, Q_2, Q_3 into the plane ω. Setting $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \le i \le 3)$ we find:

$$P_1 = \Big(\tfrac{11}{10}, \tfrac{4\sqrt{6}-5\sqrt{3}}{10}, 0\Big), \quad P_2 = \Big(-\tfrac{2\sqrt{3}}{5}, \tfrac{5+\sqrt{2}}{5}, 0\Big), \quad P_3 = \Big(\tfrac{3\sqrt{3}}{10}, \tfrac{6\sqrt{2}+5}{10}, 0\Big)$$

and we can see that OP_1, OP_2, OP_3 are non-parallel. Finally, we note that

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with} \quad h = \tfrac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}},\ k = \tfrac{10+3\sqrt{2}}{5+7\sqrt{2}}. \tag{86}$$

As observed above in Ex. 3.1, the hyperbolic Pohlke's conic is an ellipse. In fact, according to Thm. 2.8 of [3], we find: $g\big(\frac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}}, \frac{10+3\sqrt{2}}{5+7\sqrt{2}}\big) < 0$ and $f\big(\frac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}}, \frac{10+3\sqrt{2}}{5+7\sqrt{2}}\big) < 0$.

Ex. 3.2 (calculations). To begin with, we set $\rho = 1$ and $Q_1(1, 1, 1)$.

Choice of Q_2. We look for $Q_2(x_2, y_2, z_2) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$. Since $Q_1(1, 1, 1)$, this is equivalent to require:

$$x_2^2 + y_2^2 - z_2^2 = 1 \quad \text{and} \quad x_2 + y_2 - z_2 = 0, \tag{87}$$

that is, $-2x_2y_2 = 1$ and $z_2 = x_2 + y_2$. For example, we can take $x_2 = 1$ and $y_2 = -\frac{1}{2}$. So we have the point $Q_2(1, -\frac{1}{2}, \frac{1}{2})$.

Choice of Q_3^*. We impose the same conditions on $Q_3^*(x_3^*, y_3^*, z_3^*)$ as on Q_2:

(88) $$-2x_3^* y_3^* = 1, \quad z_3^* = x_3^* + y_3^*.$$

This time we can choose $x_3^* = \frac{3}{2}$ and $y_3^* = -\frac{1}{3}$. Thus, we have $Q_3^*(\frac{3}{2}, -\frac{1}{3}, \frac{7}{6})$.

Choice of Q_3. After Q_2, Q_3^* we need $Q_3(x_3, y_3, z_3) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. Having $Q_2(1, -\frac{1}{2}, \frac{1}{2})$, this means that

(89) $$x_3^2 + y_3^2 - z_3^2 = 1 \quad \text{and} \quad x_3 - \tfrac{1}{2}y_3 - \tfrac{1}{2}z_3 = 0.$$

Equivalently, $-3x_3^2 + 4x_3y_3 = 1$ and $z_3 = 2x_3 - y_3$. Now, we choose $x_3 = \frac{5}{2}$. Then, $y_3 = \frac{79}{40}$ and we finally get $Q_3(\frac{5}{2}, \frac{79}{40}, \frac{121}{40})$.

Determination of $\mathbf{v}$. Since $Q_3^* \neq Q_3$ and $Q_3Q_3^* \nparallel \omega$, $Q_3Q_3^*$ gives a projection direction onto the plane ω. Furthermore, this direction is non-degenerate. Thus, we define $\mathbf{v}$ as in (82). In this way we get $Q_3' = Q_3^*$ (i.e., Q_3^* is $\pi_{\mathbf{v}}$–symmetric to Q_3) and $OQ_3 \parallel T_{\mathscr{H}}(Q_1')$, by (78).

Determination of P_1, P_2, P_3. Having fixed $\mathbf{v}$, we can project the points Q_1, Q_2, Q_3 into the plane ω. Setting $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \le i \le 3)$ we find:

$$P_1 = \left(\tfrac{103}{223}, -\tfrac{54}{223}, 0\right), \quad P_2 = \left(\tfrac{163}{223}, -\tfrac{250}{223}, 0\right), \quad P_3 = \left(\tfrac{389}{446}, -\tfrac{795}{446}, 0\right)$$

and we can see that OP_1, OP_2, OP_3 are non-parallel. Finally, we note that

(90) $$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with} \quad h = -\tfrac{145}{152},\ k = \tfrac{273}{152}.$$

As observed in Ex. 3.2, the hyperbolic Pohlke's conic is a hyperbola. Indeed, according to Thm. 2.8 of [3], we find $g\left(-\frac{145}{152}, \frac{273}{152}\right) < 0$ and $f\left(-\frac{145}{152}, \frac{273}{152}\right) > 0$.

3.2. **Some remarks on the previous examples.** After fixing $Q_1, Q_2 \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$, in the choice of points $Q_3^*, Q_3 \in \mathscr{H}$ of Ex. 3.1 and Ex. 3.2 the following facts are observed:

a) By Claim A.3 of [3], we have $OQ_3 \nparallel OQ_3^*$. In fact, if $OQ_3 \parallel OQ_3^*$, then

$$OQ_3^* \parallel T_{\mathscr{H}}(Q_1) \quad \Rightarrow \quad OQ_3 \parallel T_{\mathscr{H}}(Q_1).$$

Having by construction $OQ_2, OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$ and $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$, applying (78) we immediately get the sequence

$$OQ_1 \parallel T_{\mathscr{H}}(Q_2), \quad OQ_2 \parallel T_{\mathscr{H}}(Q_3), \quad OQ_3 \parallel T_{\mathscr{H}}(Q_1)$$

which contradicts Claim A.3 of [3].

b) After fixing $Q_1(x_1, y_1, z_1)$, $Q_2(x_2, y_2, z_2)$, we find that

(91) $$Q_3^* \in \mathcal{H}_1 = \mathscr{H} \cap \pi_1 \quad \text{with} \quad \pi_1 : x_1x + y_1y - z_1z = 0$$

and

(92) $$Q_3 \in \mathcal{H}_2 = \mathscr{H} \cap \pi_2 \quad \text{with} \quad \pi_2 : x_2x + y_2y - z_2z = 0.$$ [17]

Noting that $x_i^2 + y_i^2 - z_i^2 = 1$ $(i = 1, 2)$, from Claim 3.5 of [3] and a) we deduce that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two disjoint hyperbolas with center at O. In particular, since $\pi_1, \pi_2 \nparallel \omega$, there is no problem getting $Q_3Q_3^* \nparallel \omega$.

[17] Note that $\pi_1 \nparallel \pi_2$ because $OQ_1 \nparallel OQ_2$.

Furthermore, assuming $Q_3Q_3^* \nparallel \omega$ and setting $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} = \overrightarrow{Q_3Q_3^*}$, it is easy to see that

$$l^2 + m^2 - n^2 = 2 - 2\big(x_3x_3^* + y_3y_3^* - z_3z_3^*\big). \tag{93}$$

Hence, $\mathbf{v}$ is degenerate if and only if $x_3x_3^* + y_3y_3^* - z_3z_3^* = 1$.

If $OQ_3 \nparallel OQ_1$, i.e., $Q_3 \neq \pm Q_1$, the plane

$$\pi(\lambda)\colon\ x_3x + y_3y - z_3z = \lambda \quad (\lambda \in \mathbb{R}) \tag{94}$$

is not parallel to π_1 and $\pi(\lambda)\cap\mathcal{H}_1 \neq \emptyset$ if $|\lambda|$ is large enough. So, if we already have Q_3 and $OQ_3 \nparallel OQ_1$, we can choose $Q_3^* \in \mathcal{H}_1$ so that $l^2 + m^2 - n^2$ takes any large enough positive or negative value. [18]

c) Since $R, S \in \mathscr{H}$ and $OR \parallel T_{\mathscr{H}}(S) \Rightarrow OR \nparallel OS$, we necessarily have

$$OQ_2 \nparallel OQ_1, \quad OQ_3^* \nparallel OQ_1 \quad \text{and} \quad OQ_3 \nparallel OQ_2. \tag{95}$$

On the other hand, we can even choose $Q_3 = \pm Q_1$ and $Q_3^* = \pm Q_2$.

To prevent two of the segments OP_1, OP_2, OP_3 from being parallel, it is clear we must have $OQ_3 \nparallel OQ_1$, because $OQ_3 \parallel OQ_1 \Rightarrow OP_3 \parallel OP_1$. Then, after choosing Q_2 and Q_3, with $OQ_3 \nparallel OQ_1$, taking into account that $\mathbf{v} \parallel Q_3Q_3^*$ it is easy to see that:

$$OP_1 \parallel OP_2 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_1, Q_2\rangle, \tag{96}$$

$$OP_1 \parallel OP_3 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_1, Q_3\rangle, \tag{97}$$

$$OP_2 \parallel OP_3 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_2, Q_3\rangle, \tag{98}$$

where with $\langle O, Q_i, Q_j\rangle$ we indicate the plane through O, Q_i, Q_j.

Starting from the last, we see that the second of (98) holds iff

$$\overrightarrow{OQ_3^*} = \lambda\overrightarrow{OQ_2} + \mu\overrightarrow{OQ_3}, \tag{99}$$

for suitable λ, μ. Having $OQ_2, OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$ and, by Claim A.3 of [3], $OQ_3 \nparallel T_{\mathscr{H}}(Q_1)$, it follows that $\mu = 0$. Therefore, we deduce that the second condition of (98) holds iff $OQ_3^* \parallel OQ_2$, i.e., $Q_3^* = \pm Q_2$.

Similarly, we see that the second of (97) holds iff

$$\overrightarrow{OQ_3^*} = \lambda\overrightarrow{OQ_1} + \mu\overrightarrow{OQ_3}, \tag{100}$$

for suitable λ, μ. Here $OQ_1, OQ_3 \parallel T_{\mathscr{H}}(Q_2)$ and (100) immediately give $OQ_3^* \parallel T_{\mathscr{H}}(Q_2)$, i.e., $OQ_2 \parallel T_{\mathscr{H}}(Q_3^*)$. This contradicts Claim A.3 of [3] because $OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$. We conclude that the second of (97) never holds.

Finally, the second of (96) holds iff, for suitable λ, μ,

$$\overrightarrow{OQ_3^*} = \overrightarrow{OQ_3} + \lambda\overrightarrow{OQ_1} + \mu\overrightarrow{OQ_2}. \tag{101}$$

Now, $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$ and (101) imply that $OQ_3^* \parallel T_{\mathscr{H}}(Q_1) \Leftrightarrow \lambda + x_1x_3 + y_1y_3 - z_1z_3 = 0$. Furthermore, since $OQ_1, OQ_3 \parallel T_{\mathscr{H}}(Q_2)$, if (101) holds

[18] Just note that we can take coordinates $\bar{x}, \bar{y}$ in π_1 such that $\mathcal{H}_1$ has equation $\bar{x}\bar{y} = 1$. In the same coordinates the straight line $\pi_1 \cap \pi(\lambda)$ has equation $a\bar{x} + b\bar{y} = \alpha\lambda$ for suitable a, b (not both zero) and $\alpha \neq 0$.

then $Q_3^* \in \mathscr{H}$ iff

$$1 + \lambda^2 + \mu^2 + 2\lambda\big(x_1x_3 + y_1y_3 - z_1z_3\big) = 1. \tag{102}$$

Hence, we easily see that (101) holds iff

$$\overrightarrow{OQ_3^*} = \overrightarrow{OQ_3} - \bar{\lambda}\,\overrightarrow{OQ_1} \pm \bar{\lambda}\,\overrightarrow{OQ_2}, \tag{103}$$

with $\bar{\lambda} \overset{\text{def}}{=} x_1x_3 + y_1y_3 - z_1z_3$.

Summarizing up, after fixing Q_1, Q_2 and Q_3, with $OQ_3 \nparallel OQ_1$, the segments OP_1, OP_2, OP_3 are non-parallel if and only if Q_3^* satisfies the conditions indicated in (79).

d) We can see that O, Q_1, Q_2, Q_3 are coplanar if and only if $Q_3 = \pm Q_1$. Indeed, if $\overrightarrow{OQ_3} = \lambda\,\overrightarrow{OQ_1} + \mu\,\overrightarrow{OQ_2}$, then the condition $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$ gives

$$x_2(\lambda x_1 + \mu x_2) + y_2(\lambda y_1 + \mu y_2) - z_2(\lambda z_1 + \mu z_2) = 0. \tag{104}$$

Since $x_1x_2 + y_1y_2 - z_1z_2 = 0$ and $x_2^2 + y_2^2 - z_2^2 = 1$, (104) implies $\mu = 0$. Then, we have that

$$Q_3 \in \mathscr{H} \quad \text{and} \quad \overrightarrow{OQ_3} = \lambda\,\overrightarrow{OQ_1} \quad \Rightarrow \quad \lambda = \pm 1. \tag{105}$$

Similarly, the points O, Q_1, Q_2, Q_3^* are coplanar if and only if $Q_3^* = \pm Q_2$.

We may conclude that the points O, Q_1, Q_2, Q_3, Q_3^* are coplanar (i.e., O, P_1, P_2, P_3 are ultimately collinear) if and only if we choose Q_3 and Q_3^* such that $Q_3 = \pm Q_1$ and $Q_3^* = \pm Q_2$.

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