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INSCRIBED EQUILATERAL TRIANGLES IN GENERAL TRIANGLES

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Abstract. We study the possible positions of an inscribed equilateral triangle with vertices on the sides of a generic triangle. We show that infinitely many such configurations exist and examine some extremal properties.

1. Introduction

A Jordan curve contained in the plane is a simple closed curve, that is the image of an injective continuous map of a circle. Clearly, the model of Jordan curves is the circle, considered the perfect figure in geometry, which contains regular polygons with any number of sides. The polygon P is said to be inscribed in the Jordan curve J (not necessarily in the interior), if J contains all the vertices of P [1].

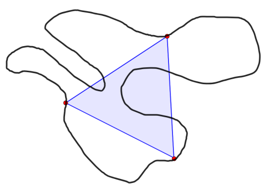

Figure 1. Inscribed equilateral triangle in a Jordan curve.

While Jordan curves are actually complicated, they can be proved to satisfy some regular properties. For example, Meyerson [7] showed that in every Jordan curve one can inscribe an equilateral triangle. Also, Neilsen proved that [8, Theorem 1.1]: *Let $J \subset \mathbb{R}^2$ be a Jordan curve and let Δ be any triangle. Then infinitely many triangles similar to Δ can be inscribed in J.* Such results also exist for Jordan curves in $\mathbb{R}^n$ [4].

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Interestingly, Toeplitz' statement from 1911 that *every Jordan curve admits an inscribed square* is still a conjecture in the general setting. It was only solved recently when the curve is convex or piecewise smooth, while extensions exist for rectangles, curves, and Klein bottles (see, [10], [11]).

The simplest example of non-smooth, piecewise linear Jordan curve is the triangle. While the equilateral triangle appears to be a simple configuration, it can generate very interesting properties and applications [6].

In the sense of the above definition, an inscribed equilateral triangle MNP can have two vertices on the same side of an arbitrary triangle ABC, a situation that is not very interesting from the geometric point of view. This is why here it is considered the special case when $M \in [BC]$, $N \in [CA]$, and $P \in [AB]$ (that is, the triangle MNP is nested with respect to the triangle ABC, as seen in Fig. 2). Similarly to Nielsen's result, there are infinitely many such triangles, generating interesting properties linked to triangle geometry [2], [5], [9], or [12].

In this paper we explore all possible such configurations, ratios of points on the sides, side lengths, and extremal properties. The structure of the paper is as follows. Section 2 presents some key formulas regarding the inscribed equilateral triangles, proving that they can all be indexed by a single parameter. Section 3 is devoted to exact formulae for the lengths of the sides of inscribed equilateral triangles, as function of this unique parameter, and to extremal properties related to the side length. Finally, section 4 provides some illustrative examples.

2. Inscribed equilateral triangles

Let ΔABC be a triangle in the Euclidean plane, for which measures of the angles from vertices A, B, and C are denoted by A, B and C, respectively. We may assume without loss of generality that $A \geq B \geq C$. Clearly, in this case we have $C \leq 60° \leq A$.

In the example in Fig. 2, the initial triangle is chosen with the coordinates $A(0,7)$, $B(-3,0)$, $C(7,0)$, for which the angles in degrees measure $A = 68.1986°$, $B = 66.8014°$ and $C = 45°$, while $M = N = P = 60°$.

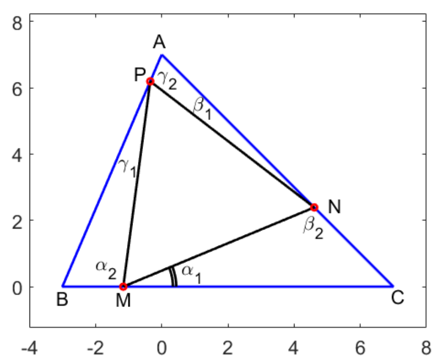

FIGURE 2. Inscribed equilateral triangle.

Using the notations from Fig. 2 one can write the equations

$$\text{(1)} \qquad \begin{cases} \alpha_1 + \alpha_2 = \frac{2\pi}{3} \\ \beta_1 + \beta_2 = \frac{2\pi}{3} \\ \gamma_1 + \gamma_2 = \frac{2\pi}{3} \\ \beta_1 + \gamma_2 = \pi - A \\ \gamma_1 + \alpha_2 = \pi - B \\ \alpha_1 + \beta_2 = \pi - C \end{cases} .$$

In matrix form the system can be written as

$$\text{(2)} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \frac{2\pi}{3} \\ \frac{2\pi}{3} \\ \frac{2\pi}{3} \\ \pi - A \\ \pi - B \\ \pi - C \end{pmatrix} .$$

Clearly, this system is compatible and indeterminate having infinitely many solutions, and it is fully determined by one single variable (since the rank of the matrix is 5). Substituting α_2, β_2 and γ_2 from the first three equations into the last three, one obtains the simplified system

$$\text{(3)} \qquad \begin{cases} \gamma_1 - \beta_1 = \frac{\pi}{3} - A \\ \alpha_1 - \gamma_1 = \frac{\pi}{3} - B \\ \beta_1 - \alpha_1 = \frac{\pi}{3} - C \end{cases} .$$

In matrix form, the system can be written as

$$\text{(4)} \qquad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{3} - A \\ \frac{\pi}{3} - B \\ \frac{\pi}{3} - C \end{pmatrix} .$$

One can easily show that the associated determinants are

$$\text{(5)} \qquad \begin{vmatrix} \frac{\pi}{3} - A & -1 & 1 \\ \frac{\pi}{3} - B & 0 & -1 \\ \frac{\pi}{3} - C & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \frac{\pi}{3} - A & 1 \\ 1 & \frac{\pi}{3} - B & -1 \\ -1 & \frac{\pi}{3} - C & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 & \frac{\pi}{3} - A \\ 1 & 0 & \frac{\pi}{3} - B \\ -1 & 1 & \frac{\pi}{3} - C \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = 0.$$

This shows that the solutions of the system (3) are determined by the parameter $\alpha_1 = \alpha \in [0, 120°] = m(\angle NMC)$, with

$$\beta_1 = \alpha + C - 60°, \quad \gamma_1 = \alpha + 60° - B.$$

From the conditions $0 \le \beta_1, \gamma_1 \le 120°$ one obtains $\alpha + 60° - B \le 120°$. The constraints

$$\text{(6)} \qquad 60° - C \le \alpha \le \min\{60° + B, 120°\},$$

shows that there are infinitely many possible configurations. We analyze them in detail. Standard arguments are given in [2], [3], [9], and [12].

3. Extremal properties

Consider the equilateral triangle ΔMNP inscribed in the triangle ΔABC. Denote by $l = MN = MP = PN$. For a fixed value of $\alpha_1 = \alpha$ with $60^\circ - C \leq \alpha \leq \min\{60^\circ + B, 120^\circ\}$ we compute $M(\alpha)$, $N(\alpha)$, $P(\alpha)$ and $l(\alpha)$, as well as the ratios in which the M, N, P divide the segments.

3.1. **The length of the equilateral triangle sides.** From the Law of Sines in the triangle ABC one obtains

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{CA}{\sin B} = 2R, \tag{7}$$

where R is the radius of the circumscribed circle of triangle ABC.

By the Law of Sines in the triangles ΔMNC, ΔMPB, ΔPNA we get

$$\begin{aligned}\frac{l}{\sin C} &= \frac{CN}{\sin \alpha} = \frac{CM}{\sin \beta_2}\\ \frac{l}{\sin B} &= \frac{BP}{\sin \alpha_2} = \frac{BM}{\sin \gamma_1}\\ \frac{l}{\sin A} &= \frac{AP}{\sin \beta_1} = \frac{AN}{\sin \gamma_2}.\end{aligned}$$

For a fixed value of $\alpha_1 = \alpha$ we have

$$\begin{aligned}\alpha_2 &= 120^\circ - \alpha\\ \beta_2 &= 180^\circ - (\alpha + C)\\ \beta_1 &= 120^\circ - \beta_2 = 120^\circ - (180^\circ - \alpha - C) = \alpha + C - 60^\circ,\\ \gamma_1 &= 180^\circ - (B + \alpha_2) = \alpha + 60^\circ - B\\ \gamma_2 &= 120^\circ - \gamma_1 = 120^\circ - (\alpha + 60^\circ - B) = 60^\circ + B - \alpha,\end{aligned}$$

hence we obtain

$$\frac{l}{\sin C} = \frac{CN}{\sin \alpha} = \frac{CM}{\sin(\alpha + C)} \tag{8}$$

$$\frac{l}{\sin B} = \frac{BP}{\sin(\alpha + 60^\circ)} = \frac{BM}{\sin(\alpha + 60^\circ - B)} \tag{9}$$

$$\frac{l}{\sin A} = \frac{AP}{\sin(\alpha + C - 60^\circ)} = \frac{AN}{\sin(60^\circ + B - \alpha)}. \tag{10}$$

From the identities (8), (9), and (10) above we obtain

$$BM = \frac{\sin(\alpha + 60^\circ - B)}{\sin B} \cdot l, \qquad CM = \frac{\sin(\alpha + C)}{\sin C} \cdot l, \tag{11}$$

$$CN = \frac{\sin \alpha}{\sin C} \cdot l, \qquad AN = \frac{\sin(60^\circ + B - \alpha)}{\sin A} \cdot l, \tag{12}$$

$$AP = \frac{\sin(\alpha + C - 60^\circ)}{\sin A} \cdot l, \qquad BP = \frac{(\alpha + 60^\circ)}{\sin B} \cdot l. \tag{13}$$

We can now find the ratios in which the points M, N, and P divide the sides $M \in (BC)$, $N \in (AC)$, $P \in (AB)$, denoted by

$$0 \leq k_A(\alpha) = \frac{BM}{BC}, \quad k_B(\alpha) = \frac{CN}{CA}, \quad k_C(\alpha) = \frac{AP}{AB} \leq 1. \tag{14}$$

One can first eliminate l in (11), (12) and (13) to obtain

$$\frac{BM}{CM} = \frac{\sin C}{\sin B} \cdot \frac{\sin(\alpha + 60^\circ - B)}{\sin(\alpha + C)}$$
$$\frac{CN}{AN} = \frac{\sin A}{\sin C} \cdot \frac{\sin\alpha}{\sin(60^\circ + B - \alpha)}$$
$$\frac{AP}{BP} = \frac{\sin B}{\sin A} \cdot \frac{\sin(\alpha + C - 60^\circ)}{\sin(\alpha + 60^\circ)}.$$

Since $BM + CM = BC$, $CN + AN = AC$, and $AP + BP = AB$, one gets

$$k_A(\alpha) = \frac{BM}{BC} = \frac{\sin C \cdot \sin(\alpha + 60^\circ - B)}{\sin C \cdot \sin(\alpha + 60^\circ - B) + \sin B \cdot \sin(\alpha + C)},$$
$$k_B(\alpha) = \frac{CN}{CA} = \frac{\sin A \cdot \sin\alpha}{\sin A \cdot \sin\alpha + \sin C \cdot \sin(60^\circ + B - \alpha)}$$
$$k_C(\alpha) = \frac{AP}{AB} = \frac{\sin B \cdot \sin(\alpha + C - 60^\circ)}{\sin B \cdot \sin(\alpha + C - 60^\circ) + \sin A \cdot \sin(\alpha + 60^\circ)}.$$

From these relations and using (7), it follows that

$$BM = 2R \cdot \frac{\sin A \cdot \sin C \cdot \sin(\alpha + 60^\circ - B)}{\sin C \cdot \sin(\alpha + 60^\circ - B) + \sin B \cdot \sin(\alpha + C)},$$
$$CN = 2R \cdot \frac{\sin B \cdot \sin A \cdot \sin\alpha}{\sin A \cdot \sin\alpha + \sin C \cdot \sin(60^\circ + B - \alpha)}$$
$$AP = 2R \cdot \frac{\sin C \cdot \sin B \cdot \sin(\alpha + C - 60^\circ)}{\sin B \cdot \sin(\alpha + C - 60^\circ) + \sin A \cdot \sin(\alpha + 60^\circ)}.$$

Replacing in formulae (11), (12) and (13), the following formulae are obtained for the side of the inscribed equilateral triangle, as a function of α:

$$l = \frac{BM \cdot \sin B}{\sin(\alpha + 60^\circ - B)} = \frac{2R \cdot \sin A \cdot \sin B \cdot \sin C}{\sin C \cdot \sin(\alpha + 60^\circ - B) + \sin B \cdot \sin(\alpha + C)}$$
$$l = \frac{CN \cdot \sin C}{\sin(\alpha)} = \frac{2R \cdot \sin A \cdot \sin B \cdot \sin C}{\sin A \cdot \sin\alpha + \sin C \cdot \sin(60^\circ + B - \alpha)}$$
$$l = \frac{AP \cdot \sin A}{\sin(\alpha + C - 60^\circ)} = \frac{2R \cdot \sin A \cdot \sin B \cdot \sin C}{\sin B \cdot \sin(\alpha + C - 60^\circ) + \sin A \cdot \sin(\alpha + 60^\circ)},$$

which also proves the identity

$$\begin{aligned} &\sin B \cdot \sin(\alpha + C - 60^\circ) + \sin A \cdot \sin(\alpha + 60^\circ) \\ = &\sin C \cdot \sin(\alpha + 60^\circ - B) + \sin B \cdot \sin(\alpha + C) \\ = &\sin A \cdot \sin\alpha + \sin C \cdot \sin(60^\circ + B - \alpha). \end{aligned}$$

Denoting the sides of triangle ABC by a, b, c, its area by $K[ABC]$, and using the relation $K[ABC] = \frac{abc}{4R^2}$, and the Law of Sines, one obtains

$$\begin{aligned} l &= \frac{2K[ABC]}{c \cdot \sin(\alpha + 60^\circ - B) + b \cdot \sin(\alpha + C)} \\ &= \frac{2K[ABC]}{a \cdot \sin\alpha + c \cdot \sin(60^\circ + B - \alpha)} \\ &= \frac{2K[ABC]}{b \cdot \sin(\alpha + C - 60^\circ) + a \cdot \sin(\alpha + 60^\circ)}. \end{aligned} \tag{15}$$

3.2. **Extremal configurations.** One can find the largest and smallest side of an inscribed equilateral triangle, and the values of the angle α for which these are obtained. The numerator is constant, so when the function

$$f(\alpha) = \sin B \cdot \sin(\alpha + C - 60^\circ) + \sin A \cdot \sin(\alpha + 60^\circ) \tag{16}$$

is minimal (or maximal), the inscribed triangle has maximal (or minimal) side. Recall that

$$\alpha_{\min} = C - 60^\circ \le \alpha \le \min\{B + 60^\circ, 120^\circ\} = \alpha_{\max}.$$

Clearly, we have $\sin B > 0$, $\sin A > 0$, while $\sin(\alpha + C - 60^\circ) \ge 0$ and $\sin(\alpha + 60^\circ) \ge 0$ cannot vanish simultaneously, therefore one obtains that $f(\alpha) > 0$, for all feasible values of α. Simple calculations show that

$$f'(\alpha) = \sin B \cdot \cos(\alpha + C - 60^\circ) + \sin A \cdot \cos(\alpha + 60^\circ)$$
$$f''(\alpha) = -\sin B \cdot \sin(\alpha + C - 60^\circ) - \sin A \cdot \sin(\alpha + 60^\circ) = -f(\alpha) < 0,$$

therefore function f is strictly concave on the interval $[\alpha_{\min}, \alpha_{\max}]$, hence has a unique local maximum. If there exists a point $\alpha^* \in (\alpha_{\min}, \alpha_{\max})$ with $f'(\alpha^*) = 0$, then $f(\alpha^*)$ is the maximal value. Otherwise, the function f is strictly monotonic on the interval $[\alpha_{\min}, \alpha_{\max}]$ and takes the maximum and minimum values at the ends of the interval (this does not seem to happen).

Hence, the function $l(\alpha)$ is convex, taking its minimum value at $l(\alpha^*)$.

By expanding the expression in (16) one obtains

$$f(\alpha) = \sin\alpha \left(\frac{\sqrt{3}}{2} \sin B \cdot \sin C + \frac{1}{2} \cos B \sin C + \sin B \cos C \right) + \cos\alpha \left(\frac{\sqrt{3}}{2} \cos B \sin C + \frac{1}{2} \sin B \sin C \right).$$

By differentiation we obtain

$$f'(\alpha) = \cos\alpha \left[\frac{\sqrt{3}}{2} \sin B \cdot \sin C + \frac{1}{2} \cos B \sin C + \sin B \cos C \right] - \sin\alpha \left[\frac{\sqrt{3}}{2} \cos B \sin C + \frac{1}{2} \sin B \sin C \right],$$

hence the critical point of f is given by

$$\alpha^* = \arctan \frac{\frac{\sqrt{3}}{2} \sin B \cdot \sin C + \frac{1}{2} \cos B \sin C + \sin B \cos C}{\frac{\sqrt{3}}{2} \cos B \sin C + \frac{1}{2} \sin B \sin C}.$$

The numerical examples in Section 4 suggest that $\alpha^* \in [60^\circ, 90^\circ]$.

Remark. For certain triangles ABC and parameter values α, the points $M(\alpha)$, $N(\alpha)$, and $P(\alpha)$ describe segments strictly included within the sides $[BC]$, $[CA]$, and $[AB]$, respectively. This does not contradict Meyerson's result [7, Theorem 1.1]: *Let J be any Jordan loop. Then all but at most 2 points of J are vertices of inscribed equilateral triangles*, in which the inscribed triangles can have multiple points on the same side.

4. Some illustrative examples

4.1. **Equilateral case:** $A = 60°$**,** $B = 60°$**,** $C = 60°$**.** In this case, we have

$$\alpha_{min} = 60° - C = 0°, \alpha_{max} = \min\{60° + B, 120°\} = 120°, \tag{17}$$

hence $\alpha \in [0°, 120°]$, and $\alpha^* = 60°$. In Fig. 3 we plot some of the inscribed triangles obtained for key values of the angle $\alpha_{min} \le \alpha \le \alpha_{max}$. For convenience, we will denote

$$\begin{aligned} M_{\text{min}} &= M(\alpha_{\text{min}}), \quad N_{\text{min}} = N(\alpha_{\text{min}}), \quad P_{\text{min}} = P(\alpha_{\text{min}}), \\ M_{\text{max}} &= M(\alpha_{\text{max}}), \quad N_{\text{max}} = N(\alpha_{\text{max}}), \quad P_{\text{max}} = P(\alpha_{\text{max}}). \end{aligned} \tag{18}$$

First, in Fig. 3 (a) we show three equilateral triangles obtained for the angles α_{min}, α_{max} and α^* where $f'(\alpha^*) = 0$. One may notice that we have

$$\begin{aligned} M_{\text{min}} = M(0°) = B, \quad N_{\text{min}} = N(0°) = C, \quad P_{\text{min}} = P(0°) = A \\ M_{\text{max}} = M(120°) = C, \quad N_{\text{max}} = N(120°) = A, \quad P_{\text{max}} = P(120°) = B, \end{aligned}$$

while the triangle $M^*N^*P^*$ computed for the critical angle α^* obtained by solving $f'(\alpha^*) = 0$ is the median triangle of ΔABC. We notice that this corresponds to $\alpha^* = 60°$. In Fig. 3 (b) we plot the segments $[M_{\text{min}}, M_{\text{max}}]$, $[N_{\text{min}}, N_{\text{max}}]$, and $[P_{\text{min}}, P_{\text{max}}]$. The function f defined in (16) is plotted in Fig. 3 (c), while the side length $l(\alpha)$ in Fig. 3 (d).

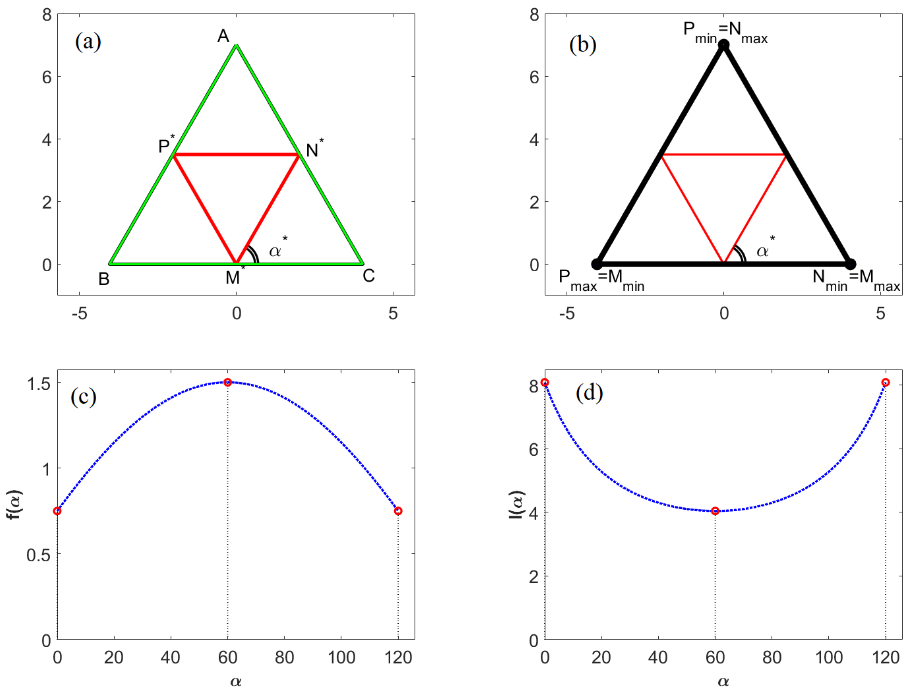

Figure 3. (a) Inscribed equilateral triangles obtained for α_{min}, α_{max} in (17) and α^*; (b) Segments $M(\alpha)$, $N(\alpha)$, $P(\alpha)$, when $\alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]$; (c) $f(\alpha)$ by (16); (d) $l(\alpha)$ by (15).

4.2. **Acute case $A = 70°$, $B = 65°$, $C = 45°$.** In this case we have

$$\alpha_{min} = 60° - C = 15°, \quad \alpha_{max} = \min\{60° + B, 120°\} = 120°, \tag{19}$$

hence $\alpha \in [15°, 120°]$, and $\alpha^* = 66.70°$.

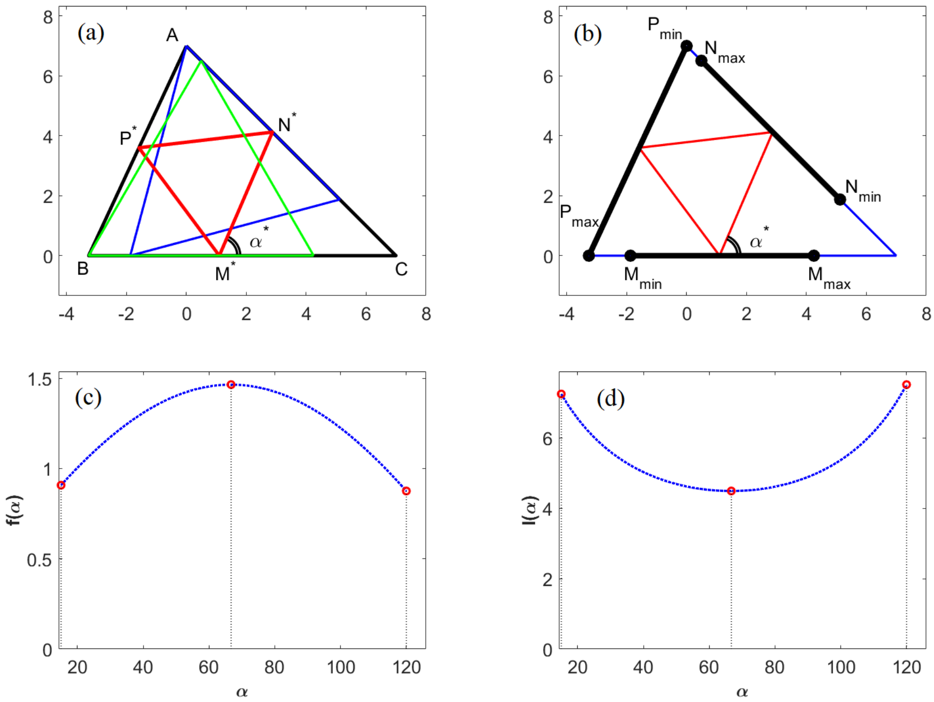

FIGURE 4. (a) Inscribed equilateral triangles obtained for α_{min}, α_{max} in (19) and α^*; (b) Segments $M(\alpha)$, $N(\alpha)$, $P(\alpha)$, when $\alpha \in [\alpha_{\min}, \alpha_{\max}]$; (c) $f(\alpha)$ by (16); (d) $l(\alpha)$ by by (15).

In Fig. 4 (a) we show three equilateral triangles obtained for the angles α_{min} (blue line), α_{max} (green line) and α^* (red line) obtained for $f'(\alpha^*) = 0$. One may notice that in this case we have

$$\begin{aligned} M_{\min} &= M(15°), \\ N_{\min} &= N(15°), \\ P_{\min} &= P(15°) = A \\ M_{\max} &= M(120°), \\ N_{\max} &= N(120°), \\ P_{\max} &= P(120°) = B, \end{aligned}$$

with the triangle $M^*N^*P^*$ computed for the critical angle $\alpha^* = 66.70°$.

In Fig. 4 (b) we plot the segments $[M_{\min}, M_{\max}]$, $[N_{\min}, N_{\max}]$, and $[P_{\min}, P_{\max}]$. The function f defined in (16) is plotted in Fig. 4 (c), while the corresponding side length $l(\alpha)$ is shown in Fig. 4 (d).

4.3. **Acute case** $A = 80°$**,** $B = 55°$**,** $C = 45°$**.** In this case

$$\alpha_{min} = 60° - C = 15°, \quad \alpha_{max} = \min\{60° + B, 120°\} = 115°, \tag{20}$$

hence $\alpha \in [15°, 115°]$, and $\alpha^* = 63.46°$.

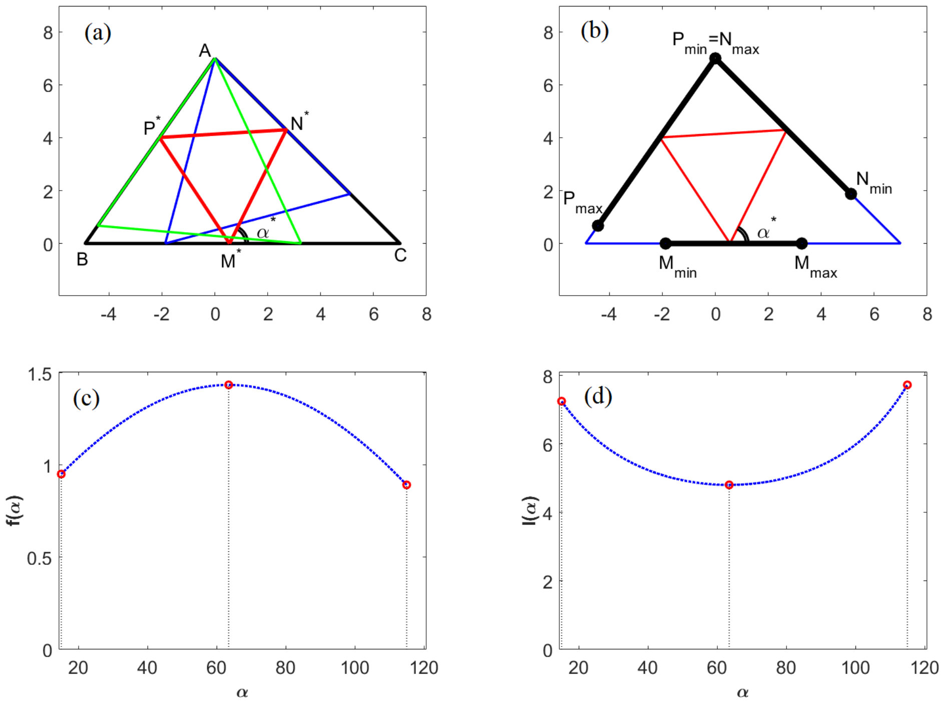

FIGURE 5. (a) Inscribed equilateral triangles obtained for α_{min}, α_{max} in (20) and α^*; (b) Segments $M(\alpha)$, $N(\alpha)$, $P(\alpha)$, when $\alpha \in [\alpha_{\min}, \alpha_{\max}]$; (c) $f(\alpha)$ by (16); (d) $l(\alpha)$ by by (15).

In Fig. 5 (a) we show three equilateral triangles obtained for the angles α_{min} (blue line), α_{max} (green line) and α^* (red line) obtained for $f'(\alpha^*) = 0$. One may notice that we have

$$\begin{aligned} M_{\min} &= M(15°), \\ N_{\min} &= N(15°), \\ P_{\min} &= P(15°) = A \\ M_{\max} &= M(115°), \\ N_{\max} &= N(115°) = A, \\ P_{\max} &= P(115°), \end{aligned}$$

with the triangle $M^*N^*P^*$ computed for the critical angle $\alpha^* = 63.46°$.

In Fig. 5 (b) we plot the segments $[M_{\min}, M_{\max}]$, $[N_{\min}, N_{\max}]$, and $[P_{\min}, P_{\max}]$. The function f defined in (16) is plotted in Fig. 5 (c), while the corresponding side length $l(\alpha)$ is shown in Fig. 5 (d).

4.4. **Obtuse case** $A = 105°$**,** $B = 45°$**,** $C = 30°$**.** In this case

$$\alpha_{min} = 60° - C = 30°, \quad \alpha_{max} = \min\{60° + B, 120°\} = 105°, \tag{21}$$

hence $\alpha \in [30°, 105°]$, and $\alpha^* = 66.20°$.

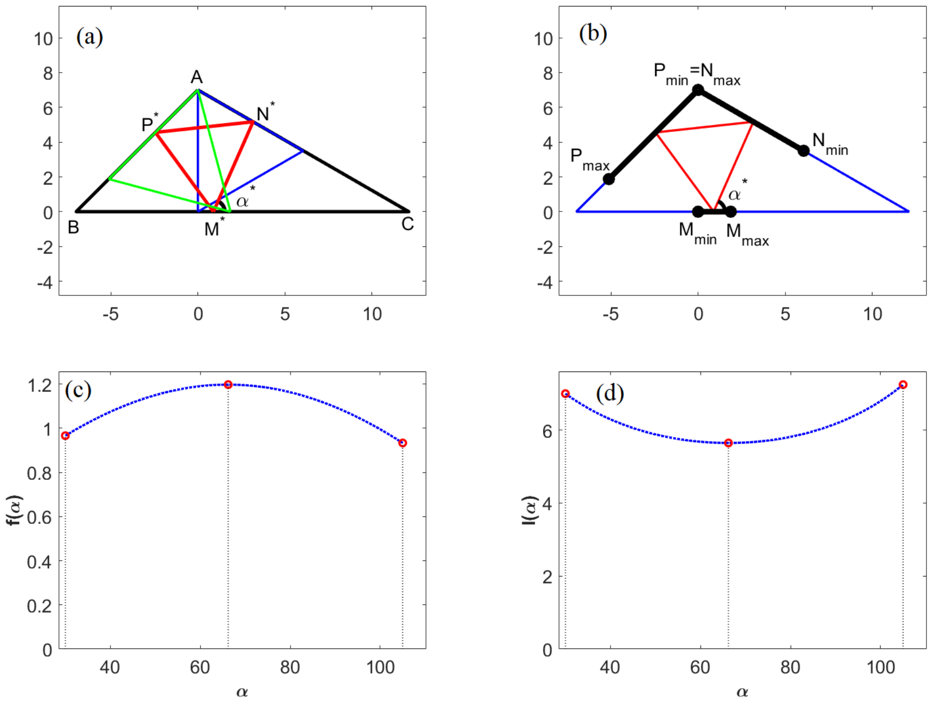

FIGURE 6. (a) Inscribed equilateral triangles obtained for α_{min}, α_{max} in (21) and α^*; (b) Segments $M(\alpha)$, $N(\alpha)$, $P(\alpha)$, when $\alpha \in [\alpha_{\min}, \alpha_{\max}]$; (c) $f(\alpha)$ by (16); (d) $l(\alpha)$ by by (15).

In Fig. 6 (a) we show three equilateral triangles obtained for the angles α_{min} (blue line), α_{max} (green line) and α^* (red line) obtained for $f'(\alpha^*) = 0$. One may notice that we have

$$\begin{aligned} M_{\min} &= M(30°), \\ N_{\min} &= N(30°), \\ P_{\min} &= P(30°) = A \\ M_{\max} &= M(105°), \\ N_{\max} &= N(105°) = A, \\ P_{\max} &= P(105°), \end{aligned}$$

with the triangle $M^*N^*P^*$ computed for the critical angle $\alpha^* = 66.20°$.

In Fig. 6 (b) we plot the segments $[M_{\min}, M_{\max}]$, $[N_{\min}, N_{\max}]$, and $[P_{\min}, P_{\max}]$. The function f defined in (16) is plotted in Fig. 6 (c), while the corresponding side length $l(\alpha)$ is shown in Fig. 6 (d).

4.5. **Obtuse case $A = 135°$, $B = 30°$, $C = 15°$.** In this case

$$\alpha_{min} = 60° - C = 45°, \quad \alpha_{max} = \min\{60° + B, 120°\} = 90°, \tag{22}$$

hence $\alpha \in [45°, 90°]$, and $\alpha^* = 69.89°$.

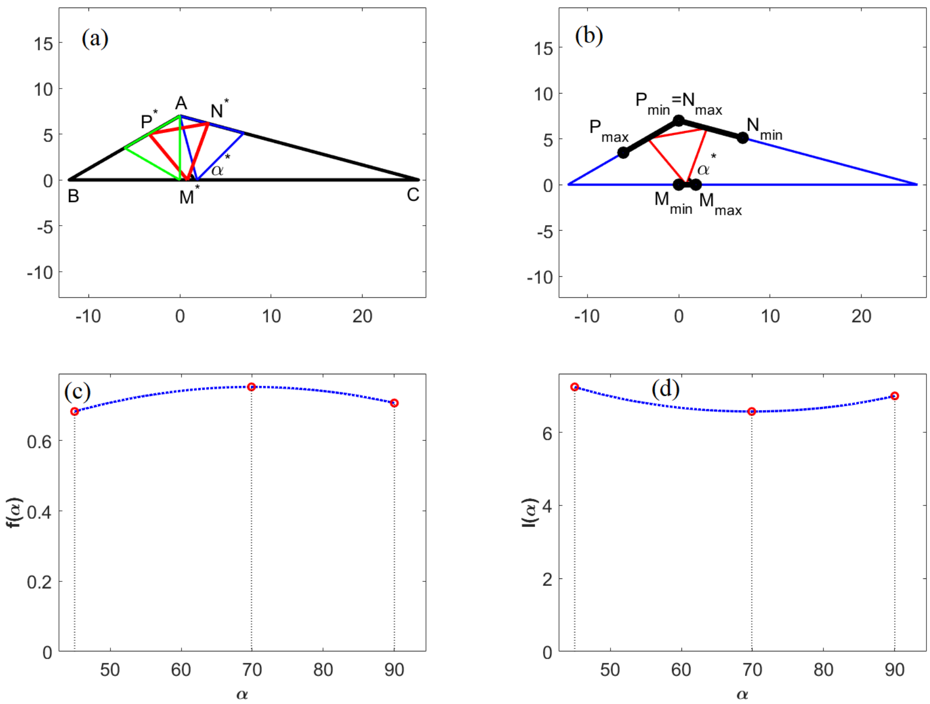

FIGURE 7. (a) Inscribed equilateral triangles obtained for $\alpha_{min}, \alpha_{max}$ in (22) and α^*; (b) Segments $M(\alpha)$, $N(\alpha)$, $P(\alpha)$, when $\alpha \in [\alpha_{\min}, \alpha_{\max}]$; (c) $f(\alpha)$ by (16); (d) $l(\alpha)$ by by (15).

In Fig. 7 (a) we show three equilateral triangles obtained for the angles α_{min} (blue line), α_{max} (green line) and α^* (red line) obtained for $f'(\alpha^*) = 0$. One may notice that we have

$$\begin{aligned} M_{\min} &= M(45°), \\ N_{\min} &= N(45°), \\ P_{\min} &= P(45°) = A \\ M_{\max} &= M(90°), \\ N_{\max} &= N(90°) = A, \\ P_{\max} &= P(90°), \end{aligned}$$

with the triangle $M^*N^*P^*$ computed for the critical angle $\alpha^* = 69.89°$.

In Fig. 7 (b) we plot the segments $[M_{\min}, M_{\max}]$, $[N_{\min}, N_{\max}]$, and $[P_{\min}, P_{\max}]$. The function f defined in (16) is plotted in Fig. 7 (c), while the corresponding side length $l(\alpha)$ is shown in Fig. 7 (d).

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