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CHARACTERIZATIONS OF PARALLELOGRAMS PART 1

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Abstract. We collect, categorize and prove 62 necessary and sufficient conditions for when a quadrilateral is a parallelogram.

1. Introduction

The parallelogram is one of the most fundamental and well-known quadrilaterals, probably due to its symmetric nature. It has an abundance of interesting and beautiful properties, many of which are less famous than they merit. Here we will focus on properties that are unique to parallelograms, that is, those that distinguish parallelograms from other quadrilaterals.

Basic properties and characterizations of parallelograms are studied in most high school geometry textbooks. In such books, and also online, it's easy to find proofs of the following necessary and sufficient conditions for when a quadrilateral is a parallelogram:

- Both pairs of opposite sides have equal length
- Both pairs of opposite angles have equal measure
- One pair of opposite sides are parallel and have equal length
- The diagonals bisect each other

In addition to the definition, that both pairs of opposite sides are parallel, how many other characterizations of parallelograms do you know? There are a few that sometimes are mentioned in textbooks, but it's quite difficult to find sources that have compiled more than ten characterizations as theorems and exercises (the top contender we found is the English Wikipedia page [38], listing eleven characterizations at the time of writing this paper).

Many authors of articles, books, and Olympiad problems have contributed to the knowledge on necessary and sufficient condition for a quadrilateral to be a parallelogram, but they are scattered in many places and it's common for new ones to appear one at a time.

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This is the first of a two part paper on characterizations of parallelograms. Together they contain 103 such characterizations. A few months before the writing process started, we knew of only one third of them. The second third was found online as problems in various mathematics competitions or in less known articles and books. The last third was discovered while we were doing the research and thought about what other properties that might also be sufficient conditions. We have found no references for those as characterizations, but several of them are known properties of parallelograms.

There are no concave or crossed quadrilaterals that can be a parallelogram, so when we write quadrilateral, it means a *convex* quadrilateral.

The disposition of the two papers are such that all characterizations are grouped regarding what they mainly are about, but at the end of the second part we will give a historical overview in chronological order according to the oldest date of publication that we know of for each of them together with references.

2. Equal or parallel sides

While several classes of quadrilaterals are defined in various ways in different textbooks, a parallelogram is almost always defined in the following way. This also makes our first characterization:

Definition 2.1. *A quadrilateral is a parallelogram if and only if both pairs of opposite sides are parallel.*

Euclid did not directly define a parallelogram in his *Elements* [22], which is surprising considering how thorough he was defining most other concepts, but it's nonetheless clear from the context what he meant when using the word parallelogram. According to Proclus [12, p. 325], it was Euclid who invented this word ($\pi\alpha\rho\alpha\lambda\lambda\eta\lambda\acute{o}\gamma\rho\alpha\mu\mu o$ in Greek). The English word parallelogram was used for the first time in 1570 in a translation of the *Elements* according to [25]. An old synonym is rhomboid [41, p. 4], but it usually refers to parallelograms that are neither rhombi nor rectangles.

In our first theorem, we prove five characterizations of parallelograms that deal with equal or parallel sides of a quadrilateral. Condition *(c)* is from [41, p. 26], where it was not stated separately as a sufficient condition, but part of a proof of another sufficient condition, so from this proof it's clear that it implies a quadrilateral to be a parallelogram. We found *(d)* and *(e)* in [10, p. 47], which are related to a congruence criterion of triangles rarely included in textbooks. Condition *(e)* was further investigated in [8].

Theorem 2.1. *A quadrilateral $ABCD$ satisfies any one of:*

- *(a) it has two pairs of opposite sides with equal length*
- *(b) it has one pair of opposite parallel sides that have equal length*
- *(c) it has two pairs of equal alternate angles between a diagonal and the sides*
- *(d) $AB = DC > AC$ and $\angle ACB = \angle CAD$*
- *(e) $AB = DC < AC$ and $\angle B = \angle D$*

if and only if it's a parallelogram.

Proof. *(a)* In a parallelogram $ABCD$, triangles ABC and CDA are congruent (ASA) due to equal alternate angles and the common diagonal, so $AB = CD$ and $BC = DA$ (see Figure 1). Conversely, in a quadrilateral $ABCD$ with $AB = CD$ and $BC = DA$, triangles ABC and CDA are congruent (SSS), so opposite sides are parallel due to equal alternate angles.

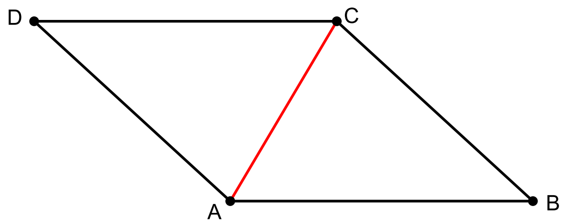

Figure 1. A parallelogram with diagonal AC

(b) In a parallelogram, opposite sides are parallel (by definition) and equal according to *(a)*. Conversely, if in a quadrilateral $ABCD$ it holds that $AB = CD$ and $AB \parallel CD$, then triangles ABC and CDA are congruent (SAS), so $BC = DA$ and $ABCD$ is a parallelogram by *(a)*.

(c) That $ABCD$ is a parallelogram if and only if $\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$ is a direct consequence of a necessary and sufficient condition for parallel lines (equal alternate angles) and Definition 2.1.

(d) In a parallelogram, $AB = DC$ and $\angle ACB = \angle CAD$. Conversely, when $AB = DC > AC$ and $\angle ACB = \angle CAD$, then triangles ABC and CDA are congruent (SSA), see Figure 1, so $ABCD$ is a parallelogram according to *(a)*. Many readers will recognize (SSA) as the ambiguous case, which in general is not a true congruence case, but it *is true* when the given angle is opposite the *longer* of the two considered sides, see [13].

(e) This proof is left to the reader, as it's very similar to *(d)*. □

3. Diagonals

Here we prove nine different characterizations of parallelograms that are about the diagonals. Condition *(b)* is taken from [41, p. 28], where it was not stated separately as a sufficient condition, but part of a proof of another sufficient condition, so from this proof it's clear that it implies a quadrilateral to be a parallelogram. *(c)* and *(e)* are from the old books [41, p. 26] and [32] (exercise on page 381), while *(d)* was found in the paper [1, p. 235], and *(g)* is from the recent book [35, p. 188]. To prove that *(h)* is a property of parallelograms was Problem 155 in [26]. The last characterization is from the Mexican Mathematical Olympiad in 1996 according to [40], but the proof we cite is from [39, p. 34].

Theorem 3.1. *A quadrilateral $ABCD$ with diagonal intersection P satisfies any one of:*

(a) it has bisecting diagonals
(b) the two pairs ABP, CDP and BCP, DAP are congruent triangles

(c) *each diagonal divide it into two congruent triangles with the same orientation*
(d) *each diagonal bisects the perimeter of the quadrilateral*
(e) *each diagonal bisects the area of the quadrilateral*
(f) *the sum of any two adjacent diagonal parts is equal to the sum of the other two diagonal parts*
(g) *it has equal distances from opposite vertices to the corresponding diagonals*
(h) *the line segments from a random point on any diagonal to the opposite vertices together with this diagonal and the sides divide it into two pairs of triangles with equal area*
(i) *E and F are the midpoints of AB and BC respectively, where $E = DM \cap AB$ and $F = DN \cap BC$, and where M and N trisect AC such that $AM = MN = NC$*

if and only if it's a parallelogram.

Proof. *(a)* In a parallelogram $ABCD$, $AB = DC$, $\angle BAC = \angle DCA$ and $\angle ABD = \angle CDB$, so triangles ABP and CDP are congruent (ASA), see Figure 2. Then $AP = CP$ and $BP = DP$.

Conversely, when $AP = CP$ and $BP = DP$ in a quadrilateral $ABCD$, then triangles ABP and CDP are congruent (SAS) due to vertically opposite angles at P, so $AB = DC$. In the same way we get $BC = DA$, so $ABCD$ is a parallelogram according to Theorem 2.1 *(a)*.

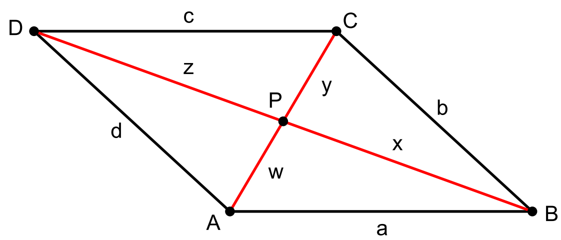

FIGURE 2. The diagonal intersection P

(b) In a parallelogram $ABCD$, the pairs of triangles ABP, CDP and BCP, DAP are congruent due to bisecting diagonals and vertically opposite angles at P (see Figure 2). Conversely, if these opposite triangles are congruent in pairs, then the diagonals bisect each other and $ABCD$ is a parallelogram according to *(a)*.

(c) Any diagonal divide a parallelogram into two congruent triangles (SSS). Conversely, such congruent triangles directly yield that opposite sides are equal, so the quadrilateral is a parallelogram according to Theorem 2.1 *(a)*.

(d) Each diagonal bisects the perimeter of a quadrilateral if and only if

$$\begin{cases} a+b=c+d \\ a+d=b+c \end{cases} \Leftrightarrow \begin{cases} 2a+b+d=2c+b+d \\ b-d=d-b \end{cases} \Leftrightarrow \begin{cases} a=c \\ b=d \end{cases}$$

(see Figure 2), where the last two equalities are equivalent to the quadrilateral being a parallelogram according to Theorem 2.1 *(a)*.

(e) Since any diagonal in a parallelogram divide it into two congruent triangles by *(c)*, these pairs of triangles clearly have the same area.

Conversely, if both triangles ABC and ABD have an area equal to one-half the area of quadrilateral $ABCD$, then they have the same height due to equal bases. Let H_1 and H_2 be the foot points of the two heights (see Figure 3). Then H_1CDH_2 is a parallelogram according to Theorem 2.1 *(a)* since CH_1 and DH_2 are parallel and have the same length, so AB and CD are parallel. In the same way BC and DA are parallel, making $ABCD$ a parallelogram by definition.

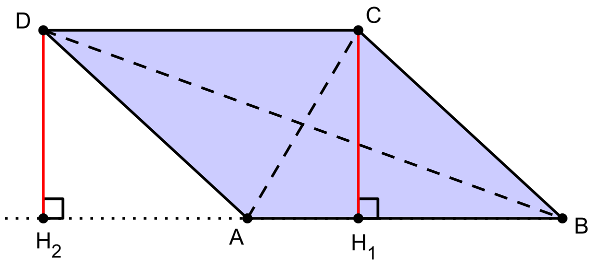

FIGURE 3. Two parallel heights with equal length

(f) Using the notations $w = AP$, $x = BP$, $y = CP$, $z = DP$ for the diagonal parts (see Figure 2), we have that

$$\begin{cases} w + x = y + z \\ z + w = x + y \end{cases} \Leftrightarrow \begin{cases} 2w + x + z = 2y + x + z \\ x - z = z - x \end{cases} \Leftrightarrow \begin{cases} w = y \\ x = z \end{cases}$$

where the last two equalities are equivalent to that the quadrilateral is a parallelogram according to *(a)*.

(g) In a parallelogram $ABCD$, since triangles ABC and CDA have equal area by *(e)* and the same base AC, the distances from B and D to AC are equal, and similarly for triangles ABD and CDB.

Conversely, if the distances from B and D to AC are equal, then triangles ABC and CDA have equal area, and similarly for triangles ABD and CDB. Then $ABCD$ is a parallelogram according to *(e)*.

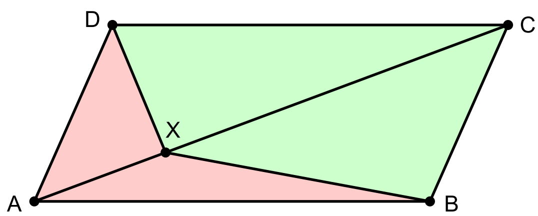

FIGURE 4. $T_{DAX} = T_{ABX}$ and $T_{BCX} = T_{CDX}$

(h) In these pairs of triangles, the bases (on the diagonal in question) are always equal (see Figure 4, where T_{XYZ} denote the area of triangle XYZ). The quadrilateral is a parallelogram if and only if the heights to this diagonal

in all four triangles are equal according to *(g)*, so the pairs of triangles have equal area if and only if the quadrilateral is a parallelogram.

(i) First we join BM and BN (see Figure 5). When $ABCD$ is a parallelogram, AC and BD bisect each other at P, so $BP = PD$ and $AP = PC$. Since $AM = NC$, we get $MP = PN$, so $BNDM$ is a parallelogram. Then $BN \parallel EM$ where M is the midpoint of AN, so E is the midpoint of AB according to the converse of the midpoint theorem. In the same way, F is the midpoint of BC.

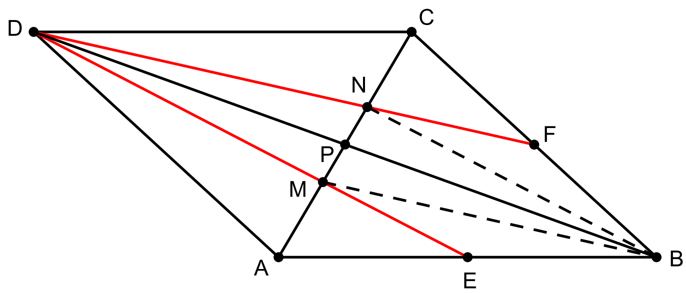

FIGURE 5. The interior parallelogram $BNDM$

Conversely, when E and F are the midpoints of sides AB and BC in a quadrilateral $ABCD$, and M and N are the midpoints of AN and CM, then $EM \parallel BN$ and $FN \parallel BM$ according to the midpoint theorem. Hence $BNDM$ is a parallelogram, so $MP = PN$ and $BP = PD$. But $AM = NC$, so $AP = PC$. Then $ABCD$ is a parallelogram according to *(a)*. □

As a comment to *(d)*, we note that quadrilaterals where one diagonal bisects the perimeter are called extangential quadrilaterals, see [17].

4. Angles and trigonometry

Next we prove seven characterizations of parallelograms that are about angles or trigonometric relations. Condition *(b)* is from [41, p. 26], where it was not stated separately as a sufficient condition, but part of a proof of another sufficient condition, so from this proof it's clear that it implies a quadrilateral to be a parallelogram. The basic *(d)* is for some reason not included in textbooks as a characterization of parallelograms. Conditions *(e)* and *(g)* together with their proofs are from [27] and [31] respectively. To prove *(f)* was a problem in [14], where the published proof applied vectors. The proof we give was found online several years ago and now we cannot find a reference for it.

Theorem 4.1. *A quadrilateral $ABCD$ with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ and diagonal intersection P satisfies any one of:*

- *(a) it has two pairs of equal opposite angles*
- *(b) any pair of adjacent angles are supplementary*
- *(c) the sum of any two adjacent angles is a constant*
- *(d) the two angles between the extensions of opposite sides are zero*

(e) $\frac{\angle CAD}{\angle CAB} = \frac{\angle ACB}{\angle ACD}$ *and* $\frac{\angle DBA}{\angle DBC} = \frac{\angle BDC}{\angle BDA}$
(f) $da\cos A + ab\cos B + bc\cos C + cd\cos D = 0$
(g) $PA\sin A = PC\sin C$ *and* $PB\sin B = PD\sin D$

if and only if it's a parallelogram.

Proof. *(a)* In a parallelogram $ABCD$, triangles ABC and CDA are congruent (ASA), so $\angle B = \angle D$. In the same way, by drawing diagonal BD, we have $\angle A = \angle C$. Conversely, if $\angle A = \angle C$ and $\angle B = \angle D$ in a quadrilateral, then $\angle A + \angle D = \pi$ by the angle sum of a quadrilateral. Thus $\angle A = \delta$, where δ is the exterior angle at D (see Figure 6), proving that $AB \parallel DC$. In the same way $BC \parallel AD$, so $ABCD$ is a parallelogram by definition.

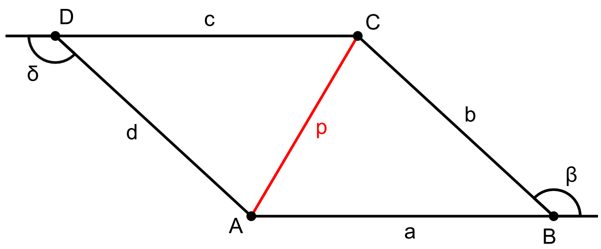

Figure 6. The exterior angles at B and D

(b) In a quadrilateral $ABCD$, let β and δ be the exterior angles at B and D respectively (see Figure 6). Then $\beta + \angle B = \pi$ and $\delta + \angle D = \pi$. We have that $ABCD$ is a parallelogram if and only if $\angle A = \delta$ and $\angle A = \beta$, which is equivalent to $\angle A + \angle B = \pi$ and $\angle A + \angle D = \pi$. Similar expressions hold at each vertex.

(c) In a parallelogram, that constant is π by *(b)*. Conversely, if

$$\angle A + \angle B = \angle B + \angle C = \angle C + \angle D = \angle D + \angle A$$

in a quadrilateral, then it directly follows that $\angle A = \angle C$ and $\angle B = \angle D$. Hence it's a parallelogram according to *(a)*.

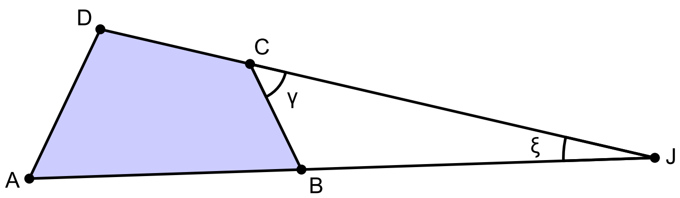

Figure 7. Angle ξ between extensions of AB and DC

(d) In quadrilateral $ABCD$, let the extensions of AB and DC intersect at J, which we assume without loss of generality to be outside of BC (see Figure 7). Then we have

$$AB \parallel DC \quad \Leftrightarrow \quad \angle ABC = \gamma \quad \Leftrightarrow \quad \gamma + \xi = \gamma \quad \Leftrightarrow \quad \xi = 0$$

where γ is the exterior angle at C and we applied the exterior angle theorem in triangle BCJ. In the same way we have that the other pair of opposite sides are parallel if and only if the angle between their extensions is zero. Both conditions must hold for $ABCD$ to be a parallelogram (by definition).

(e) It's obvious that these equalities are satisfied in a parallelogram due to equal alternate angles.

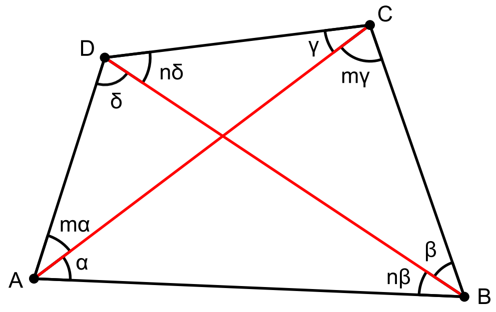

FIGURE 8. The angles α, β, γ, δ

Conversely, suppose we have $\frac{\angle CAD}{\angle CAB} = \frac{\angle ACB}{\angle ACD}$ and $\frac{\angle DBA}{\angle DBC} = \frac{\angle BDC}{\angle BDA}$ in a quadrilateral. We denote these respective quotients by $\bar{A}$, $\bar{C}$, $\bar{B}$, $\bar{D}$ so that $\bar{A} = \bar{C} \equiv m$ and $\bar{B} = \bar{D} \equiv n$. Let α, β, γ, δ denote the angles CAB, DBC, ACD, BDA respectively (see Figure 8). Then we have $\angle CAD = m\alpha$, $\angle ACB = m\gamma$, $\angle DBA = n\beta$, and $\angle BDC = n\delta$. Applying the angle sum in the four subtriangles created by the diagonals yields

$$\delta + m\alpha = \beta + m\gamma \qquad \text{and} \qquad \alpha + n\beta = \gamma + n\delta,$$

so we get $\beta - \delta = m(\alpha - \gamma)$ and $n(\beta - \delta) = \gamma - \alpha$. Hence by substitution, $nm(\alpha - \gamma) = \gamma - \alpha$, that is,

$$(nm + 1)(\alpha - \gamma) = 0.$$

It follows that $\alpha = \gamma$ and in turn $\beta = \delta$, so both pairs of opposite sides are parallel. Then $ABCD$ is a parallelogram by definition.

(f) In a parallelogram $ABCD$ with sides $a = AB = CD = c$ and $b = BC = DA = d$, angles $\angle A = \angle C$ and $\angle B = \angle D$, we get

$$da \cos A + ab \cos B + bc \cos C + cd \cos D = 2ab(\cos A + \cos B) = 0$$

since $\angle A + \angle B = \pi$ according to *(b)*.

Conversely, applying the law of cosines in a quadrilateral $ABCD$ with diagonals $p = AC$ and $q = BD$ (see Figure 6) yields

$$\begin{aligned} p^2 &= a^2 + b^2 - 2ab \cos B, \qquad & q^2 &= b^2 + c^2 - 2bc \cos C, \\ p^2 &= c^2 + d^2 - 2cd \cos D, \qquad & q^2 &= d^2 + a^2 - 2da \cos A. \end{aligned}$$

Adding these four equalities, we get

$$2(p^2 + q^2) = 2(a^2 + b^2 + c^2 + d^2) - 2(da \cos A + ab \cos B + bc \cos C + cd \cos D),$$

and since we now assume the expression in the last parenthesis is zero, this simplifies into

$$p^2 + q^2 = a^2 + b^2 + c^2 + d^2.$$

In any quadrilateral there is the following similar expression, called Euler's quadrilateral theorem:

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$

where v is the distance between the diagonal midpoints. Comparing the last two equalities, we see that we get $4v^2 = 0$, that is $v = 0$. This means that the diagonals bisect each other, so the quadrilateral is a parallelogram according to Theorem 3.1 *(a)*. A proof of Euler's quadrilateral theorem can be found for instance in [3, pp. 9–10], a marvelous book that any enthusiast of quadrilaterals ought to own.

(g) In a parallelogram $ABCD$, it holds that $PA = PC$, $PB = PD$ and $\sin A = \sin B = \sin C = \sin D$ since adjacent angles are supplementary, so $PA \sin A = PC \sin C$ and $PB \sin B = PD \sin D$.

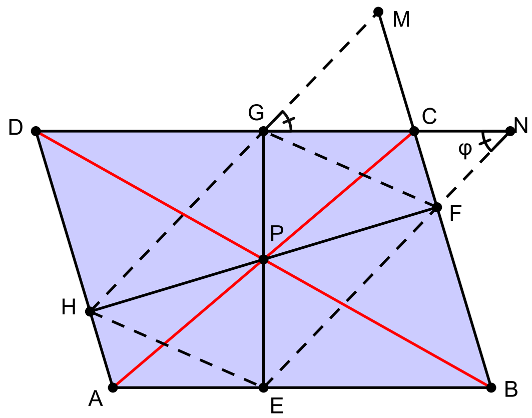

Figure 9. Projections of P onto the sides

Conversely, when these two equalities hold in a quadrilateral, let E, F, G, H be the projections of P onto the sides AB, BC, CD, DA respectively (see Figure 9). Then AEP and AHP are right angles, so $AEPH$ is a cyclic quadrilateral. Applying the law of sines yields

$$\frac{HE}{\sin A} = \frac{AE}{\sin \angle AHE} = \frac{AE}{\sin \angle APE} = \frac{PA}{\sin \frac{\pi}{2}}$$

so $HE = PA \sin A$. In the same way we get $FG = PC \sin C$; hence $HE = FG$. By symmetry, $EF = GH$ and we can thus conclude that $EFGH$ is a parallelogram. Then $EF \parallel GH$ and $HE \parallel FG$. Now let the extensions of HG and EF intersect the extensions of BC and DC at M and N respectively. Next we denote $\angle FNC = \angle CGM = \angle HGD \equiv \varphi$. In cyclic quadrilateral $HPGD$ we have

$$\angle DPH = \angle DGH = \varphi \tag{1}$$

and by the exterior angle theorem in triangle FNC, it holds that $\angle CFN = \angle C - \varphi$. Then

$$\angle EPB = \angle BFE = \angle C - \varphi. \tag{2}$$

Since DB is a straight line segment, we have

$$\angle DPH + \angle HPE + \angle EPB = \pi \tag{3}$$

and in cyclic quadrilateral $AHPE$, it holds

$$\angle HPE = \pi - \angle A. \tag{4}$$

By substituting (1), (4) and (2) into (3), we get

$$\varphi + \pi - \angle A + \angle C - \varphi = \pi.$$

Hence $\angle A = \angle C$. In the same way it can be proved that $\angle B = \angle D$, so $ABCD$ is a parallelogram according to *(a)*. □

5. Angle bisectors and bimedians

In this section we prove eleven necessary and sufficient conditions for when a quadrilateral is a parallelogram concerning angle bisectors or bimedians. A *bimedian* is a line segment between the midpoints of opposite sides (see Figure 11). Conditions *(a)* and *(b)* were stated in the book [35, p. 188], while *(c)* and *(d)* were found at [37] and [5] respectively (the latter concerned a problem cited from an unspecified book that was asked at a math forum, where no solution was given). The characterizations *(f)* and *(g)* were proved in the recent paper [9], while *(i)* was stated as a defining property of parallelograms in connection with a hierarchical classification in [34] and is one of our favorite characterizations of parallelograms. To prove *(j)* was a problem on the Leningrad High School Olympiad in 1980. We give our proof of it, but other proofs can be found in [3, p. 258] and [23]. The last condition was studied in [6, pp. 140–141], where the given proof of the converse is attributed to John Webb. Our simpler proof only applies formulas used to prove *(j)*.

Theorem 5.1. *A quadrilateral $ABCD$ satisfies any one of:*

- *(a) it has two pairs of opposite parallel angle bisectors*
- *(b) one angle bisector is perpendicular to two adjacent angle bisectors*
- *(c) all four angle bisectors form a rectangle*
- *(d) the bimedians quadrisect the area of the quadrilateral*
- *(e) the bimedians quadrisect the perimeter of the quadrilateral*
- *(f) each bimedian bisects the area of the quadrilateral*
- *(g) each bimedian bisects the perimeter of the quadrilateral*
- *(h) the bimedians intersect at a diagonal midpoint*
- *(i) the bimedians and the diagonals are concurrent*
- *(j) the sum of the length of the bimedians equals the semiperimeter*
- *(k) the bimedians have constant length for all quadrilaterals with sides of fixed length*

if and only if it's a parallelogram.

Proof. *(a)* Let $\angle A = 2\alpha$, $\angle C = 2\gamma$, and the angle bisectors at A and C be AE and CF respectively, with $E \in DC$ and $F \in AB$ (see Figure 10). In a parallelogram, $\angle BAE = \angle DEA$, and since $\alpha = \gamma$, then $AE \parallel FC$. In the same way the angle bisectors at B and D are parallel.

Conversely, if $AE \parallel FC$, then $\angle FCD = \angle AED = \angle EAB$, so $\angle C = \angle A$. In the same way $\angle D = \angle B$, so $ABCD$ is a parallelogram according to Theorem 4.1 *(a)*.

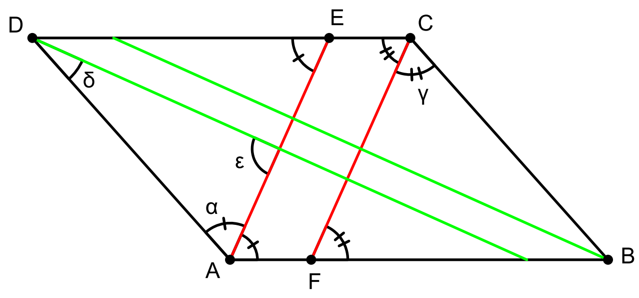

FIGURE 10. The four angle bisectors

(b) Let $\angle A = 2\alpha$, $\angle B = 2\beta$ and $\angle D = 2\delta$. In a parallelogram, $2\alpha+2\delta = \pi$, so the angle ε between the angle bisectors at A and D is

$$\varepsilon = \pi - (\alpha + \delta) = \pi - \tfrac{\pi}{2} = \tfrac{\pi}{2}$$

using the angle sum of a triangle (see Figure 10). In the same way, the angle between the angle bisectors at A and B is a right angle.

Conversely, if in quadrilateral $ABCD$ we have $\varepsilon = \frac{\pi}{2}$, then $\alpha+\delta = \pi-\frac{\pi}{2} = \frac{\pi}{2}$, so $\angle A+\angle D = 2\alpha+2\delta = \pi$. In the same way we prove that $\angle A+\angle B = \pi$, so $ABCD$ is a parallelogram by Theorem 4.1 *(b)*.

(c) This is a weaker condition than the one in *(b)*, but the proof is more or less the same except that here we have more angles to consider. We let the reader write the full proof.

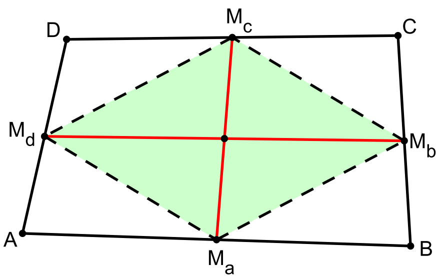

FIGURE 11. Bimedians M_aM_c and M_bM_d

(d) In a parallelogram, each bimedian is parallel to two opposite sides, so the two bimedians divide the parallelogram into four congruent smaller parallelograms with equal area.

Conversely, if the bimedians M_aM_c and M_bM_d divide a quadrilateral $ABCD$ into four regions of equal area, we first consider the Varignon parallelogram $M_aM_bM_cM_d$. Its diagonals are the bimedians in $ABCD$, and they divide $M_aM_bM_cM_d$ into four triangles with equal area since each pair of adjacent triangles have base and height of equal length (see Figure 11). Hence we only need to prove that if the areas of triangles AM_aM_d, BM_bM_a, CM_cM_b, DM_dM_c are equal, then $ABCD$ is a parallelogram. Since $AM_d = DM_d$, we get that M_c and M_a are equidistant from AD, so AD and M_aM_c are parallel. By a similar argument, BC and M_aM_c are parallel, so all three of AD, M_aM_c, BC are parallel. In the same way AB, M_dM_b, DC are parallel, so $ABCD$ is a parallelogram.

(e) The bimedians quadrisect the perimeter of a quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ if and only if

$$\frac{d}{2}+\frac{a}{2}=\frac{a}{2}+\frac{b}{2}=\frac{b}{2}+\frac{c}{2}=\frac{c}{2}+\frac{d}{2}$$

which is equivalent to $a = c$ and $b = d$. This characterizes a parallelogram according to Theorem 2.1 *(a)*.

(f) It has been proved for instance as Proposition 4 in [18] and Theorem 1 (a) in [9] that a bimedian bisects the area of a quadrilateral if and only if the two sides it connects are parallel. Hence it holds that each bimedian bisects the area of a quadrilateral if and only if both pairs of opposite sides are parallel, which is equivalent to the quadrilateral being a parallelogram by definition.

(g) Since a bimedian connects the midpoints of a pair of opposite sides, it's more or less trivial that a bimedian bisects the perimeter of a quadrilateral if and only if a pair of opposite sides have equal lengths (see Figure 12). Hence each bimedian bisects the perimeter if and only if both pairs of opposite sides have equal lengths, which is equivalent to a parallelogram according to Theorem 2.1 *(a)*.

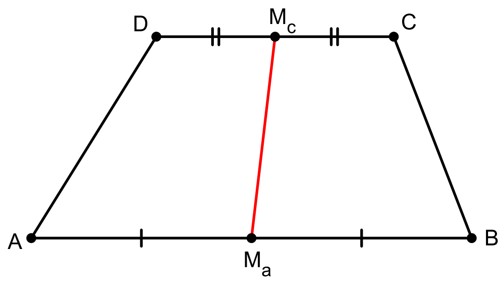

FIGURE 12. M_aM_c bisects the perimeter $\Leftrightarrow$ $BC = DA$

(h) Let M and N be the diagonal midpoints. It's quite well-known that the bimedians intersect at the midpoint of MN in all quadrilaterals (for a proof, see [2, pp. 108–109]). Let Q be the point where the bimedians intersect. Then we have that $Q = M$ is equivalent to $MN = 0$, which according to Theorem 3.1 *(a)* is equivalent to the quadrilateral being a parallelogram.

(i) With the same notations as in *(h)*, we have that $Q = M = N$ is equivalent to that the quadrilateral is a parallelogram ($MN = 0$).

(j) Let $n = M_aM_c$ and $m = M_bM_d$ be the bimedians that connect sides a, c and b, d respectively in a quadrilateral $ABCD$ with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ and diagonals $AC = p$, $BD = q$ (see Figure 11).

In [3, p. 11] it was proved that

$$4n^2 = a^2 + c^2 - b^2 - d^2 + p^2 + q^2$$

and according to Theorem 10 in [18], we have the equality

$$p^2 + q^2 = b^2 + d^2 + 2ac\cos\xi$$

where ξ is the angle between the extensions of opposite sides a and c. Via substitution, we get after applying a well-known trigonometric half-angle formula that

$$n = \sqrt{\left(\frac{a+c}{2}\right)^2 - ac\sin^2\left(\frac{\xi}{2}\right)} \le \frac{a+c}{2} \tag{5}$$

where equality holds if and only if $\xi = 0$. By symmetry there is the similar formula for the other bimedian

$$m = \sqrt{\left(\frac{b+d}{2}\right)^2 - bd\sin^2\left(\frac{\psi}{2}\right)} \le \frac{b+d}{2} \tag{6}$$

where ψ is the angle between the extensions of sides b and d, and equality holds if and only if $\psi = 0$. Then we get

$$m + n \le \tfrac{1}{2}(b+d) + \tfrac{1}{2}(a+c) = s \tag{7}$$

where s is the semiperimeter, and equality holds if and only if $\xi = \psi = 0$, which is equivalent to the quadrilateral being a parallelogram according to Theorem 4.1 *(d)*.

(k) In a parallelogram, it's trivial that each bimedian has the same length as the two opposite sides it's parallel to, and when the side lengths are fixed, the length of the bimedians does not change if we tilt the parallelogram.

Conversely, we have a quadrilateral with sides of fixed length and shall prove that the only quadrilateral where the length of the bimedians does not change as we vary the vertex angles is a parallelogram. Rewriting (5), we get

$$\sin^2\left(\frac{\xi}{2}\right) = \frac{1}{ac}\left(\left(\frac{a+c}{2}\right)^2 - n^2\right)$$

where a, c, n are constant, so the right hand side is constant. Then the left hand side must also be constant since equality holds, and the only possibility for this to happen for all ξ is when $\xi = 0$. In the same way, using (6), we get that $\psi = 0$ is the only possibility. There is just one type of quadrilateral where $\xi = \psi = 0$, the parallelogram according to Theorem 4.1 *(d)*. □

6. Trapezoids

Next we consider ten properties that make a trapezoid a parallelogram. Note that *a parallelogram is a special case of a trapezoid* when using inclusive definitions. This is the preferred way of making definitions nowadays in mathematics, but unfortunately many high school textbooks still use exclusive definitions for trapezoids (as discussed in [19, pp. 75–78]). A perfectly acceptable way of defining a parallelogram, which has very rarely been used in textbooks (according to [36, p. 21]), is that a parallelogram is a trapezoid where both pairs of opposite sides are parallel. This is condition *(a)* in the following theorem, where *(b)* was found in [33]. Condition *(c)* was formulated slightly differently and proved in [28, pp. 283–284], while the direct part of *(g)* was Exercise 132 in [30, p. 140]. *(i)* and its proof are cited from [3, pp. 96–97] and *(j)* was Problem 6 in the final round for Grade 7 on the Moldovan Mathematical Olympiad in 2001 [15].

Theorem 6.1. *In a trapezoid $ABCD$, let M_a, M_b, M_c be the midpoints of AB, BC, CD respectively, and T_a, T_b, T_c, T_d be the areas of triangles ABP, BCP, CDP, DAP respectively, where P is the diagonal intersection. Then $ABCD$ satisfies any one of:*

- *(a) it has two pairs of opposite parallel sides*
- *(b) it has a pair of opposite equal angles*
- *(c) one diagonal bisects the perimeter*
- *(d) one diagonal bisects the area*
- *(e) all transversals between the lateral sides are bisected by the bimedian between the bases*
- *(f) $AB \parallel DC$ and $BJ = \frac{1}{3}BD$, where CM_a intersect BD at J*
- *(g) $AB \parallel DC$ and $AE = \sqrt{EF \cdot EG}$, where E, F, G are the points where a line through A intersects diagonal BD and the sides BC, CD respectively, or their extensions*
- *(h) $AB \parallel DC$ and $T_a = T_c$*
- *(i) $AB \parallel DC$ and $T_a + T_c = T_b + T_d$*
- *(j) $AD \parallel BC$ and $AO = 4OM_c$, where AM_c intersect DM_b at O*

if and only if it's a parallelogram.

Proof. *(a)* All parallelograms are trapezoids when using inclusive definitions of quadrilaterals. Conversely, if a trapezoid has two pairs of opposite parallel sides, then it's a parallelogram by definition.

(b) In a parallelogram, opposite sides are parallel and opposite angles are equal. Conversely, if $AB \parallel CD$ and $\angle A = \angle C$ in a quadrilateral $ABCD$, then, since $\angle ABD = \angle CDB$, triangles ABD and CDB are congruent (AAS), so $AB = CD$. Then $ABCD$ is a parallelogram according to Theorem 2.1 *(b)*.

(c) Any diagonal in a parallelogram bisects the perimeter due to equal opposite sides. Conversely, suppose we have a trapezoid $ABCD$ with $AB \parallel CD$ and that diagonal AC bisects the perimeter so $AB + BC = CD + DA$. Also suppose without loss of generality that $AB \geq CD$. Then there is a point J on AB such that $AJ = CD$, making $AJCD$ a parallelogram (see Figure 13). Thus we also have $AD = JC$ and $JB = AB - CD$. Now we apply the triangle inequality in triangle JBC to get $JB + BC \geq CJ$, which we by substitution can rewrite as $AB - CD + BC \geq AD$. Hence we have

$$\begin{cases} AB + BC \geq CD + DA \\ AB + BC = CD + DA. \end{cases}$$

For both to be true it requires triangle JBC to be degenerate, which can only happen if $JB = 0$. Then trapezoid $ABCD$ is a parallelogram.

(d) Any diagonal in a parallelogram bisects the area since it divides it into two congruent triangles. Conversely, suppose we have a trapezoid $ABCD$ with $AB \parallel CD$ and without loss of generality that $AB \geq CD$. Construct J

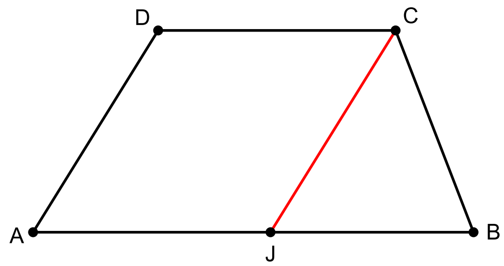

FIGURE 13. Trapezoid with $AJ = DC$

as in Figure 13. Then triangles ADC and CJA are congruent, so they have the same area. Denoting the area of triangle XYZ by T_{XYZ}, we have

$$\begin{cases} T_{ADC} = T_{CJA} + T_{CBJ} \\ T_{ADC} = T_{CJA}. \end{cases}$$

Then $T_{CBJ} = 0$ must hold, and since the height cannot be zero, we get $JB = 0$. Hence $B = J$, which proves that $ABCD$ is a parallelogram.

(e) In parallelogram $ABCD$, a transversal QR with $Q \in AD$ and $R \in BC$ intersects M_aM_c at a point I (see Figure 14). We draw $ST \parallel AB$ through I with $S \in AD$ and $T \in BC$. Then $\angle QIS = \angle RIT$, $\angle QSI = \angle RTI$ and $SI = AM_a = BM_a = TI$ since AM_aIS and BM_aIT are parallelograms. Hence triangles QIS and RIT are congruent (ASA), so $QI = RI$.

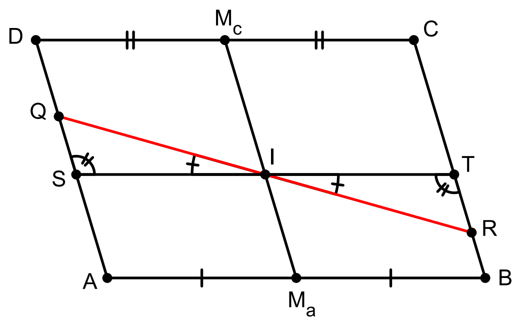

FIGURE 14. An arbitrary transversal QR

Conversely, in a trapezoid where $QI = RI$ and $AB \parallel DC$, we have $SI = TI$ since any line segment parallel to the bases is bisected by the bimedian between these bases. Also, $\angle QIS = \angle RIT$ due to vertically opposite angles, so triangles QIS and RIT are congruent (SAS). Hence $\angle QSI = \angle RTI$, implying that $AD \parallel BC$, and this proves that $ABCD$ is a parallelogram due to both pairs of opposite sides being parallel.

(f) In a parallelogram we have $AB = DC$ and $AB \parallel DC$. Then $M_aB = \frac{1}{2}DC$ and triangles M_aBJ and CDJ are similar (AA), so $2BJ = JD$. It follows that $BJ = \frac{1}{3}BD$.

We prove the converse with a contrapositive proof. Suppose without loss of generality that $AB < DC$, so $M_aB < \frac{1}{2}DC$ (see Figure 15). Triangles

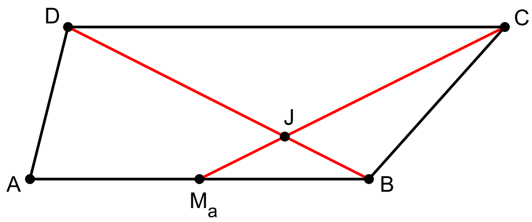

FIGURE 15. Trapezoid with $AB \parallel DC$

M_aBJ and CDJ are still similar (AA) since we assume that $AB \parallel DC$, so $2BJ < JD$. It follows that $BJ < \frac{1}{3}BD$.

(g) In a parallelogram, triangles BEF and DEA are similar (AA), as are AEB and GED (AA), so we get

$$\frac{EF}{AE} = \frac{BE}{DE}, \qquad \frac{AE}{EG} = \frac{BE}{DE}$$

and it directly follows that $(AE)^2 = EF \cdot EG$ (see Figure 16).

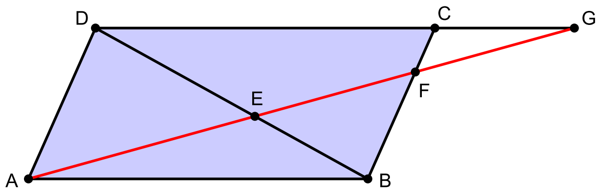

FIGURE 16. The transversal AG

Conversely, if $(AE)^2 = EF \cdot EG$ holds in a trapezoid with $AB \parallel DC$, so that triangles ABE and GDE are similar (AA), then we have

$$\frac{EF}{AE} = \frac{AE}{EG}, \qquad \frac{AE}{EG} = \frac{BE}{DE}.$$

Hence $\frac{BE}{DE} = \frac{EF}{AE}$, which together with equal vertical angles at E implies that triangles BFE and DAE are similar. Then $AD \parallel BC$ and together with the assumption $AB \parallel DC$, we have that $ABCD$ is a parallelogram.

(h) The areas of triangles ABP and CDP are in a trapezoid $ABCD$ with $a = AB$ and $c = CD$ given by

$$T_a = \frac{a^2h}{2(a+c)} \qquad \text{and} \qquad T_c = \frac{c^2h}{2(a+c)}$$

when $a \parallel c$ according to [3, p. 96] (note that opposite sides are denoted a and b in that book), where h is the height of the trapezoid (the distance between a and c). Hence $T_a = T_c$ if and only if $a = c$. Together with $a \parallel c$, this characterizes parallelograms according to Theorem 2.1 *(b)*.

(i) In addition to the two formulas for T_a and T_c in the proof of *(h)*, it's also proved in [3, p. 96] that

$$T_b = T_d = \frac{ach}{2(a+c)}$$

so we have that $T_a + T_c \geq T_b + T_d$ holds in all trapezoids where $a \parallel c$ due to the AM-GM-inequality $a^2 + c^2 \geq 2ac$, with equality if and only if $a = c$. This case is equivalent to the trapezoid being a parallelogram according to Theorem 2.1 *(b)*.

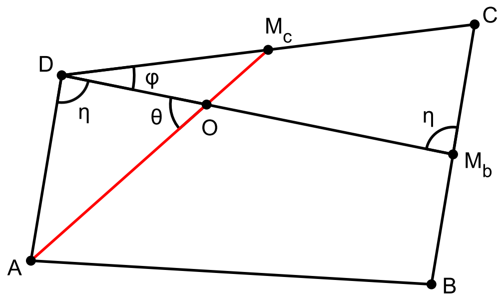

Figure 17. Trapezoid with $AD \parallel BC$

(j) We use notations as in Figure 17. Applying the law of sines three times yields

$$\frac{AO}{\sin\eta} = \frac{AD}{\sin\theta}, \qquad \frac{OM_c}{\sin\varphi} = \frac{\frac{1}{2}CD}{\sin(\pi-\theta)}, \qquad \frac{\sin\eta}{CD} = \frac{\sin\varphi}{\frac{1}{2}BC}$$

so we get

$$\frac{AO}{OM_c} = \frac{AD\sin\eta}{\sin\theta}\cdot\frac{2\sin\theta}{CD\sin\varphi} = \frac{2AD\sin\eta}{\frac{1}{2}BC\sin\eta} = \frac{4AD}{BC}.$$

Then

$$\frac{AO}{OM_c} = 4 \quad \Leftrightarrow \quad BC = AD$$

which together with $BC \parallel AD$ characterizes a parallelogram according to Theorem 2.1 *(b)*. $\square$

7. Bisect-diagonal quadrilaterals

In this section we will prove seven necessary and sufficient conditions for when a bisect-diagonal quadrilateral is a parallelogram. A *bisect-diagonal quadrilateral* is a quadrilateral where at least one diagonal is bisected by the other diagonal. This not so well-known type of quadrilateral has been studied in [21] and [7]. Characterizations *(a)*, *(f)* and *(g)* in the following theorem were found in [33], where the author discusses student attempts to find new sufficient conditions for parallelograms, but the proof we give for *(f)* is cited from [11, p. 36] (Example 1.4.3). *(g)* was only stated in [33, p. 210] but no proof was given (it was left as a challenge for the reader), and we have not been able to find it neither stated nor proved anywhere else. Condition *(e)* was proved in another way in [21, p. 217].

Theorem 7.1. *A bisect-diagonal quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ and diagonal intersection P satisfies any one of:*

(a) it has one pair of opposite parallel sides

(b) *P is at equal distance from a pair of opposite sides*
(c) *two opposite vertices are at equal distance from the bisected diagonal*
(d) *angles MAP and NCP are equal, where M and N are the midpoints of DP and BP respectively and $AP = PC$*
(e) *$a^2 + b^2 = c^2 + d^2$ and $AP = CP$*
(f) *it has one pair of opposite equal angles and the diagonal that joins the vertices of those angles bisects the other diagonal*
(g) *it has one pair of opposite equal angles, the diagonal that joins the vertices of those angles is bisected by the other diagonal, and the diagonals may not be perpendicular unless all four sides have equal length*

if and only if it's a parallelogram.

Proof. *(a)* We know that one diagonal is bisected and one pair of opposite sides are parallel in a parallelogram, so let's prove the converse. Suppose that $AP = PC$ and $AD \parallel BC$ in a quadrilateral $ABCD$. Then angles DAP and BCP are equal, and together with equal vertical angles at P we have that triangles APD and CPB are congruent (ASA), so $BP = PD$. Hence $ABCD$ is a parallelogram according to Theorem 3.1 *(a)*.

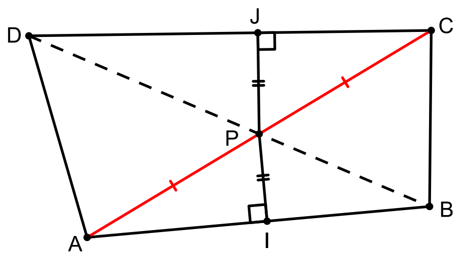

FIGURE 18. $AP = PC$ and $IP = PJ$

(b) In a parallelogram, P is at equal distance from any pair of opposite sides since opposite subtriangles created by the diagonals are congruent. Conversely, if $AP = PC$ and $IP = PJ$ in a quadrilateral where I and J are the projections of P on AB and DC respectively, then triangles AIP and CJP are congruent (RHS), see Figure 18. This implies that angles IAP and JCP are equal, so $AB \parallel DC$ making $ABCD$ a parallelogram according to *(a)*.

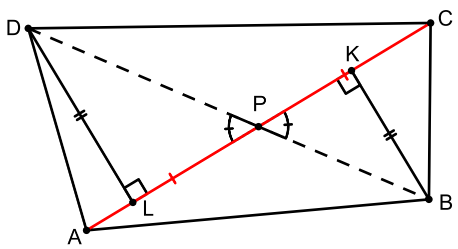

FIGURE 19. $AP = PC$ and $BK = DL$

(c) In a parallelogram, any pair of opposite vertices are at equal distance to the diagonal not containing those two vertices due to congruent triangles. Conversely, suppose $AP = PC$ and $BK = DL$ in a quadrilateral (see Figure 19). Then triangles APD and CPB have equal area, so we get

$$\tfrac{1}{2}AP \cdot DP \sin \angle APD = \tfrac{1}{2}CP \cdot BP \sin \angle CPB$$

implying that $DP = BP$ due to equal vertical angles at P. Hence $ABCD$ is a parallelogram according to Theorem 3.1 *(a)*.

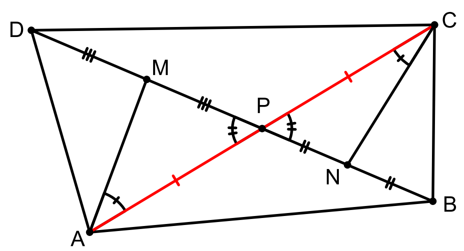

FIGURE 20. $AP = PC$ and $\angle MAP = \angle NCP$

(d) In a parallelogram, triangles APM and CPN are congruent (SAS) since $MP = NP$, so $\angle MAP = \angle NCP$. Conversely, suppose angles MAP and NCP are equal in a quadrilateral where $AP = PC$ (see Figure 20). Then triangles APM and CPN are congruent (ASA), so $MP = NP$. This implies that $DP = BP$ since M and N are midpoints on DP and BP, making $ABCD$ a parallelogram according to Theorem 3.1 *(a)*.

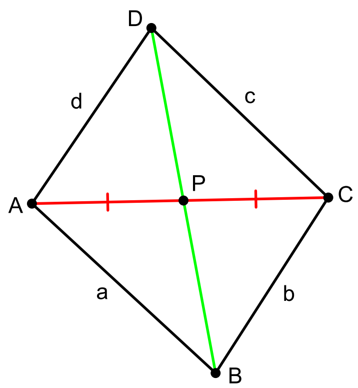

FIGURE 21. Two triangle medians must be equal

(e) A quadrilateral where $AP = CP$ is a parallelogram if and only if also $BP = DP$ according to Theorem 3.1 *(a)*. Applying the median formula in triangles ABC and ACD (see Figure 21) where $AC = p$, we get that the quadrilateral is a parallelogram if and only if

$$\tfrac{1}{2}\sqrt{2(a^2+b^2)-p^2} = \tfrac{1}{2}\sqrt{2(c^2+d^2)-p^2}$$

which is equivalent to $a^2 + b^2 = c^2 + d^2$.

(f) The direct theorem is trivially true, so let's prove the converse. Suppose we have a quadrilateral where $\angle B = \angle D$ and $AP = CP$, but let for the sake of contradiction the other diagonal not be bisected (see Figure 22). First we assume that $BP < DP$. Then we can extend BD to a point H such that $DP = HP$, making $AHCD$ a parallelogram by Theorem 3.1 *(a)*, so $\angle H = \angle D$ according to Theorem 4.1 *(a)*. But we also have that $\angle B = \angle D$, so $\angle B = \angle H$ must hold. Next we apply the exterior angle theorem twice to get

$$\begin{aligned}\angle B &= \angle ABD + \angle CBD \\ &= \angle BAH + \angle BHA + \angle BHC + \angle BCH \\ &= \angle BAH + \angle H + \angle BCH > \angle H.\end{aligned}$$

This is a contradiction to $\angle B = \angle H$, so the assumption that $BP < DP$ was wrong. In the same way it can be proved that $BP > DP$ is wrong, so we must have $BP = DP$. This means that the diagonals bisect each other, so $ABCD$ is a parallelogram according to Theorem 3.1 *(a)*.

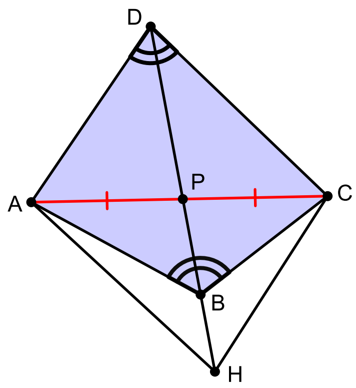

Figure 22. $\angle B = \angle D$ and $AP = CP$

(g) The direct theorem is obviously satisfied in all parallelograms. The exception that the diagonals may not be perpendicular unless all four sides have equal lengths is necessary to exclude general kites but to include rhombi and squares, which are simultaneously parallelograms and kites.

For the converse, suppose without loss of generality that $\angle ABC = \angle ADC$, $BP = DP$, and $BD \not\perp AC$, and for the sake of contradiction, also that $AP < CP$. Extend diagonal AC beyond A to E such that $EP = CP$. Then $EBCD$ is a parallelogram according to Theorem 3.1 *(a)*, so $\angle EBC = \angle EDC$. By subtracting equal angles, we get $\angle EBA = \angle EDA$. Now let D' be the reflection of D in the line EC. Next we draw the circumcircle to triangle AED'. Then, with notations as in Figure 23, we get

$$\angle EDA = \angle ED'A = \angle EFA = \angle FEB + \angle EBA > \angle EBA$$

where we used that angles subtended by the same chord on the same segment of a circle are equal, and the exterior angle theorem. We have reached a contradiction: $\angle EBA = \angle EDA$ and $\angle EBA < \angle EDA$. Hence the assumption that $AP < CP$ was wrong, and in the same way we can prove that

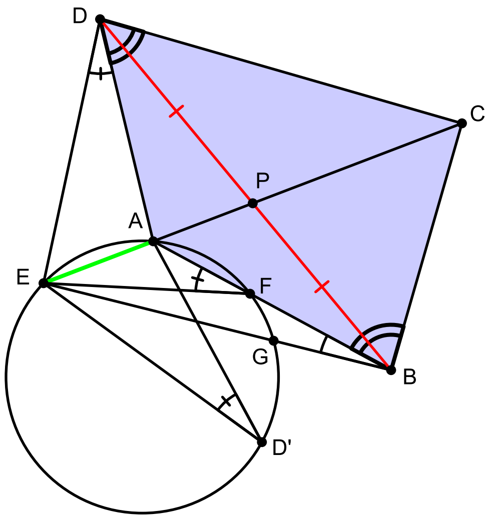

FIGURE 23. $\angle B = \angle D$, $BP = DP$ and $BD \not\perp AC$

$AP > CP$ is also wrong. Thus $AP = CP$, which together with $BP = DP$ implies that $ABCD$ is a parallelogram according to Theorem 3.1 *(a)*. $\square$

8. Two-dimensional metric relations

In the following theorem we prove twelve characterizations of parallelograms that are two-dimensional metric relations. Several of these were discovered while conducting the research before and during the writing of this paper. In a way, *(b)* is the most fundamental of all necessary and sufficient conditions of parallelograms since it's expressed as only *one* equality and in terms of the sides alone. *(d)* is quite famous and one of the oldest characterizations, due to Euler according to [29, p. 35], and to prove the inequality case in *(f)* was a problem on the Mediterranean Mathematics Olympiad in the year 2000 [4]. Condition *(g)* is from the book [35, p. 188] and *(h)* was proved in [24] as a part of the proof of the converse of our Theorem 3.1 *(e)*. The authors of [9] remember seeing *(i)* as a problem in some journal, but they cannot recall a reference, and we have not been able to track it down either. Characterization *(j)* is from the Croatian Mathematical Olympiad in 1996 [16], while *(l)* and its proof are cited from [28, p. 187].

Theorem 8.1. *In a quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$; diagonals p, q; bimedians m, n; semiperimeter s, and area K, let M_a, M_b, M_c, M_d be the midpoints of a, b, c, d respectively, $m_A = AM_b$, $m_B = BM_c$, $m_C = CM_d$, $m_D = DM_a$ be the medians in triangles ABC, BCD, CDA, DAB respectively, and T_a, T_b, T_c, T_d be the areas of triangles ABP, BCP, CDP, DAP respectively, where P is the diagonal intersection. Then $ABCD$ satisfies any one of:*

(a) $ab = cd$ *and* $da = bc$
(b) $(a - c)^2 + (b - d)^2 = 0$
(c) $(a + c)^2 + (b + d)^2 = 2(p^2 + q^2)$

(d) $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$
(e) $a^2 + b^2 + c^2 + d^2 = 2(m^2 + n^2)$
(f) $a^2 + b^2 + c^2 + d^2 = \frac{4}{5}(m_A^2 + m_B^2 + m_C^2 + m_D^2)$
(g) $T_a = T_b = T_c = T_d$
(h) $T_a = T_c$ *and* $T_b = T_d$
(i) $T_b = T_d = \frac{1}{4}K$
(j) $T_b^2 = T_aT_c$ *and* $2T_d = T_a + T_c$
(k) $K = \frac{1}{2}\sqrt{(s^2 - p^2)(s^2 - q^2)}$
(l) $T_{ABC} \leq T_{CDA} \leq T_{BCD} \leq T_{DAB}$

if and only if it's a parallelogram.

Proof. *(a)* We solve the system of equations

$$\begin{cases} ab = cd \\ da = bc \end{cases} \Leftrightarrow \begin{cases} a^2bd = bc^2d \\ \frac{ab}{da} = \frac{cd}{bc} \end{cases} \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases}$$

which proves that the quadrilateral is a parallelogram if and only if these two equations are satisfied according to Theorem 2.1 *(a)*.

(b) It's trivial that $(a - c)^2 + (b - d)^2 = 0$ holds in a parallelogram. Conversely, since algebraic squares are non-negative for real numbers, there is just one solution to the equation $(a - c)^2 + (b - d)^2 = 0$, and that is for both terms to be zero, yielding $a = c$ and $b = d$. This implies that the quadrilateral is a parallelogram according to Theorem 2.1 *(a)*.

(c) Theorem 10 in [18] states that in a convex quadrilateral with sides a, b, c, d and diagonals p, q, it holds that

$$p^2 + q^2 = b^2 + d^2 + 2ac\cos\xi$$

where ξ is the angle between the extensions of sides a and c. By symmetry we have

$$p^2 + q^2 = a^2 + c^2 + 2bd\cos\psi$$

where ψ is the angle between the extensions of sides b and d. Adding these two equations yields

$$2(p^2 + q^2) = a^2 + c^2 + b^2 + d^2 + 2ac\cos\xi + 2bd\cos\psi$$

and using the assumption $(a + c)^2 + (b + d)^2 = 2(p^2 + q^2)$, we get after expansion and simplification

$$2ac + 2bd = 2ac\cos\xi + 2bd\cos\psi$$

which is equivalent to

$$2ac(\cos\xi - 1) = 2bd(1 - \cos\psi)$$

and, in turn, holds if and only if

$$4ac\sin^2\left(\frac{\xi}{2}\right) + 4bd\sin^2\left(\frac{\psi}{2}\right) = 0.$$

This equation only has one possible solution, that both terms on the left hand side are zero, which is equivalent to $\xi = \psi = 0$. Hence the only type of quadrilateral where the equality $(a + c)^2 + (b + d)^2 = 2(p^2 + q^2)$ holds is a parallelogram according to Theorem 4.1 *(d)*.

(d) This characterization is a direct consequence of Euler's quadrilateral theorem (for a proof, see [3, pp. 9–10])

$$a^2+b^2+c^2+d^2=p^2+q^2+4v^2\ge p^2+q^2 \tag{8}$$

with equality if and only if the distance v between the diagonal midpoints is zero, which is equivalent to a parallelogram according to Theorem 3.1 *(a)*.

(e) It's well known that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram (discovered by Varignon), whose diagonals are the bimedians of the quadrilateral and that its sides are half the length of the diagonals of the quadrilateral. Applying the equality from *(d)* in this Varignon parallelogram, we get

$$\left(\frac{p}{2}\right)^2+\left(\frac{q}{2}\right)^2+\left(\frac{p}{2}\right)^2+\left(\frac{q}{2}\right)^2=m^2+n^2,$$

that is,

$$p^2+q^2=2(m^2+n^2). \tag{9}$$

Inserting this into (8) yields

$$a^2+b^2+c^2+d^2=2(m^2+n^2)+4v^2\ge 2(m^2+n^2)$$

with equality if and only if the quadrilateral is a parallelogram ($v=0$).

(f) Applying the triangle median formula (see Figure 24), we get that

$$\begin{aligned}
&4(m_A^2+m_B^2+m_C^2+m_D^2)\\
&=2(a^2+p^2)-b^2+2(b^2+q^2)-c^2+2(c^2+p^2)-d^2+2(d^2+q^2)-a^2\\
&=a^2+b^2+c^2+d^2+4(p^2+q^2)\\
&=a^2+b^2+c^2+d^2+4(a^2+b^2+c^2+d^2-4v^2)\\
&=5(a^2+b^2+c^2+d^2)-16v^2\le 5(a^2+b^2+c^2+d^2)
\end{aligned}$$

where we inserted (8) in the third step. We have equality if and only if the quadrilateral is a parallelogram ($v=0$, according to Theorem 3.1 *(a)*).

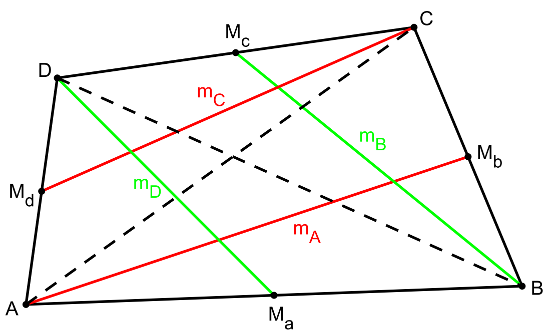

FIGURE 24. The medians m_A, m_B, m_C, m_D

(g) Since the diagonals bisect each other in a parallelogram, adjacent subtriangles have equal area due to equal heights, so $T_a=T_b=T_c=T_d$.

Conversely, we have

$$\begin{cases} T_a = T_b \\ T_a = T_d \end{cases} \quad \Rightarrow \quad \begin{cases} w = y \\ x = z \end{cases}$$

where $w = AP$, $x = BP$, $y = CP$, $z = DP$ are the diagonal parts and we used that adjacent subtriangles have equal heights (see Figure 25). This proves that in fact only three of the four subtriangles need to have equal area in order for us to be able to conclude that the quadrilateral is a parallelogram according to Theorem 3.1 *(a)*.

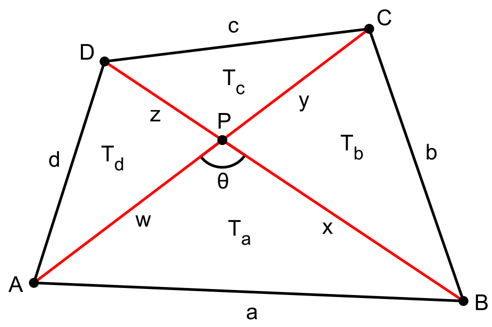

Figure 25. The diagonal parts and the four subtriangle areas

(h) Applying a well-known formula for the area of a quadrilateral (see Corollary 1.5.2 in [3, p. 15]), we have that

$$\begin{cases} T_a = T_c \\ T_b = T_d \end{cases} \quad \Leftrightarrow \quad \begin{cases} \frac{1}{2}wx\sin\theta = \frac{1}{2}yz\sin\theta \\ \frac{1}{2}xy\sin(\pi-\theta) = \frac{1}{2}zw\sin(\pi-\theta) \end{cases}$$

where w, x, y, z are the diagonal parts and θ is one of the angles between the diagonals (see Figure 25). This is equivalent to

$$\begin{cases} wx = yz \\ xy = zw \end{cases} \quad \Leftrightarrow \quad \begin{cases} wx^2y = wyz^2 \\ \frac{wx}{xy} = \frac{yz}{zw} \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = z \\ y = w \end{cases}$$

which holds if and only if the quadrilateral is a parallelogram according to Theorem 3.1 *(a)*.

(i) It's trivial that $T_b = T_d = \frac{1}{4}K$ are satisfied in a parallelogram according to *(g)*. For the converse, we rewrite these equalities as

$$\tfrac{1}{2}wz\sin\theta = \tfrac{1}{2}xy\sin\theta = \tfrac{1}{4}\cdot\tfrac{1}{2}(w+y)(x+z)\sin\theta$$

(see Figure 25), which is equivalent to

$$4wz = 4xy = wx + wz + yx + yz.$$

Hence by simplification, we have the system of equations

$$\begin{cases} wz = xy \\ 2wz = wx + yz \end{cases} \tag{10}$$

which we solve by writing $2wz$ as $wz + wz$ and use $wz = xy$ for one of the terms. Then the second equation yields $wz + xy = wx + yz$, which is equivalent to

$$(z - x)(w - y) = 0$$

with solutions $z = x$ and then $y = w$, or $w = y$ which implies that $z = x$. These identical cases imply the quadrilateral is a parallelogram according to Theorem 3.1 *(a)*.

(j) In a parallelogram, $T_a = T_b = T_c = T_d$ according to *(h)*, so both $T_b^2 = T_aT_c$ and $2T_d = T_a + T_c$ are satisfied.

Conversely, rewriting the equality $T_b^2 = T_aT_c$, we get (see Figure 25)

$$\left(\tfrac{1}{2}xy\sin(\pi - \theta)\right)^2 = \tfrac{1}{2}wx\sin\theta \cdot \tfrac{1}{2}yz\sin\theta$$

which is equivalent to $xy(xy - wz) = 0$. Hence $xy = wz$ since $xy \neq 0$. The second equality $2T_d = T_a + T_c$ can in the same way be rewritten as $2wz = wx + yz$, so again we have the system of equations (10). In the proof of *(j)* we showed that it implies a parallelogram.

(k) In [20] it was proved that the area of a convex quadrilateral can be calculated with the formula

$$K = \tfrac{1}{2}\sqrt{[(m+n)^2 - p^2][p^2 - (m-n)^2]}.$$

This can, with the help of (9), be rewritten as

$$K = \tfrac{1}{2}\sqrt{[(m+n)^2 - p^2][(m+n)^2 - q^2]} \leq \tfrac{1}{2}\sqrt{(s^2 - p^2)(s^2 - q^2)}$$

where equality holds if and only if the quadrilateral is a parallelogram according to (7).

(l) In a parallelogram we obviously have equality throughout. Conversely, from $T_{ABC} \leq T_{BCD}$ we get $T_{ABP} \leq T_{CDP}$ by subtracting the common area of triangle BCP (see Figure 25). In the same way $T_{CDA} \leq T_{DAB}$ yields $T_{CDP} \leq T_{ABP}$, so $T_{ABP} \leq T_{CDP} \leq T_{ABP}$ and thus we must have $T_{CDP} = T_{ABP}$. It follows that $T_{CDA} = T_{DAB}$, so C and B have the same distance to DA, that is, $CB \parallel DA$. In the same way it's proved that $AB \parallel DC$, so $ABCD$ is a parallelogram by definition. □

9. Part 2

In the second part of this compilation of characterizations of parallelograms we will study characterizations that are about one-dimensional metric relations, symmetry, vectors, and coincidences.

References

[1] Al-Sharif, A. I.; Hajja, M.; Krasopoulos, P. T., Coincidences of Centers of Plane Quadrilaterals, *Results Math.*, **55(3)** (Nov. **2009**) 231–247.

[2] Alsina, C. and Nelsen, R. B., *Charming Proofs. A Journey into Elegant Mathematics*, MAA Press, **2010**.

[3] Alsina, C. and Nelsen, R. B., *A Cornucopia of Quadrilaterals*, MAA Press, **2020**.

[4] Andreescu, T.; Feng, Z.; Lee Jr., G., *Mathematical Olympiads 2000–2001. Problems and Solutions From Around the World*, MAA, **2003**.

[5] Cola, C., Given a quadrilateral with 4 equal areas, prove that it is a parallelogram, *Mathematics Stack Exchange*, **2021**,
`https://math.stackexchange.com/questions/4057138/`

[6] De Villiers, M., *Some Adventures in Euclidean Geometry*, Lulu Press, **2009**.

[7] De Villiers, M., Some more properties of the bisect-diagonal quadrilateral, *Math. Gaz.*, **105** (Nov. **2021**) 474–480.

[8] Gant, P., When is a Parallelogram Not a Parallelogram?, *Math. Gaz.*, **39** (Sep. **1955**) 191–195.

[9] Hajja, M. and Krasopoulos, P. T., More characterisations of parallelograms, *Math. Gaz.*, **107** (March **2023**) 76–83.

[10] Halsted, G. B., *Elementary Synthetic Geometry*, John Wiley & Sons, New York, **1896**.

[11] Hang, K. H. and Wang, H., *Solving Problems in Geometry. Insights and Strategies for Mathematical Olympiad*, World Scientific, Singapore, **2018**.

[12] Heath, T. L., *The Thirteen Books of The Elements*, Vol. 1 (Books I and II), Dover Publications, New York, **1956**.

[13] Hirschhorn, D. B., Why Is the SsA Triangle-Congruence Theorem Not Included in Textbooks?, *Math. Teacher*, **83**(**5**) (May **1990**) 358–361.

[14] Hoehn, L., Problem 515, *The College Mathematics Journal*, **24**(**5**) (Nov. **1993**) 473; Solution ibid, **25**(**5**) (Nov. **1994**) 467–468.

[15] IMOmath, 45-th Moldova Mathematical Olympiad 2001, n.d., available at:
`https://imomath.com/othercomp/Mda/MdaMO01.pdf`

[16] jasperE3 (username), area chasing iff parallelogram, *AoPS Online*, **2021**,
`https://artofproblemsolving.com/community/c6h2728683p23760535`

[17] Josefsson, M., Similar metric characterizations of tangential and extangential quadrilaterals, *Forum Geom.*, **12** (**2012**) 63–77.

[18] Josefsson, M., Characterizations of trapezoids, *Forum Geom.*, **13** (**2013**) 23–35.

[19] Josefsson, M., On the classification of convex quadrilaterals, *Math. Gaz.*, **100** (March **2016**) 68–85.

[20] Josefsson, M., Heron-like formulas for quadrilaterals, *Math. Gaz.*, **100** (Nov. **2016**) 505–508.

[21] Josefsson, M., Properties of bisect-diagonal quadrilaterals, *Math. Gaz.*, **101** (July **2017**) 214–226.

[22] Joyce, D. E., *Euclid's Elements.* Book I, **1996**,
`https://mathcs.clarku.edu/~djoyce/java/elements/bookI/bookI.html`

[23] Klamkin, M. S., The Olympiad Corner, *Crux Math.*, **10**(**9**) (Nov. **1984**) 289.

[24] Klamkin, M. S., The Olympiad Corner, *Crux Math.*, **11**(**1**) (Jan. **1985**) 12–13.

[25] *MacTutor*, Earliest Known Uses of Some of the Words of Mathematics (P), University of St Andrews, Scotland, n.d.,
`https://mathshistory.st-andrews.ac.uk/Miller/mathword/p/`

[26] Mead, G. F., *The Mathematical Visitor*, **1**(**4**) (Jan. **1880**) 165.

[27] Okumura, H., A Characterisation of Parallelograms, *Math. Gaz.*, **72** (June **1988**) 117–118.

[28] Pop, O. T.; Minculete, N.; Bencze, M., *An introduction to quadrilateral geometry*, Editura Didactică şi Pedagogică, Bucharest, Romania, **2013**.

[29] Sandifer, C. E., *How Euler Did It*, MAA, **2007**.

[30] Saul, M., *Hadamard's Plane Geometry. A Reader's Companion*, AMS, **2010**.

[31] Seimiya, T., Problem 1841, *Crux Math.*, **20**(**4**) (April **1994**) 111–112.

[32] Todhunter, I., *The elements of Euclid for the use of schools and colleges*, Macmillan & Co., London, **1871**.

[33] Toumasis, C., When is a Quadrilateral a Parallelogram, *Math. Teacher*, **87**(**3**) (March **1994**) 208–211.

[34] Tydd, M., Flying Two Kites: Part 2, *AMESA KZN Mathematics Journal*, **9** (**2005**) 51–54, available at:
`http://dynamicmathematicslearning.com/matthew-tydd-two-kites.pdf`

[35] Ufnarovski, V.; Wikström, F.; Madjarova, J., *Mathematical Buffet. Problem solving at the Olympiad level*, Studentlitteratur, Lund, Sweden, **2016**.

[36] Usiskin, Z. and Griffin, J., *The Classification of Quadrilaterals. A Study of Definition*, Information Age Publishing, **2008**.

[37] vinod19 (username), If the bisectors of angles of a quadrilateral enclose a rectangle, then show it is a parallelogram, *brainly*, **2016**, https://brainly.in/question/1003966

[38] *Wikipedia*, accessed July **2023**, https://en.wikipedia.org/wiki/Parallelogram

[39] Wong Y. L., *MA2219 An Introduction to Geometry*, **2009**, available at: https://bpb-us-w2.wpmucdn.com/blog.nus.edu.sg/dist/2/12659/files/2021/08/Notes_MA2219-1.pdf

[40] Woodrow, R. E., 10^{th} Mexican Mathematics Olympiad National Contest November 1996, *Crux Math.*, **26**(**2**) (**2000**) 71.

[41] Young, J. R., *Elements of Geometry, with Notes*, Carey, Lea & Blanchard, **1833**.

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