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ON THE PLATO'S POLYHEDRA

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Abstract. The aim of this study is to propose formulas making it possible to calculate the number of faces, edges and vertices of 3−dimensional Platonic polyhedra by assigning specific numerical values to each polyhedron without using the L. Schläfli symbol.

1. Introduction

In his Timaeus [1], Plato followed the [2] the Pythagorean metaphysical method when describing the formulation of the world by the half-urge Cf.[4] . He used this method which is a mathematical philosophy to find mathematical foundations that guarantee the legitimacy of his principles. It is from this foundation that he undertakes to infer the structures of the "physical universe" which are mathematical Cf.[1]. For Plato, there are two categories of bodies: the bodies of the sensible world and those of the celestial world. The bodies of the sensible world are composed of five elements or particles. They are: earth, water, air, fire and ether. And these five elements are geometrically and respectively represented by the regular convex polyhedra, the cube, the icosahedron, the octahedron, the tetrahedron and the dodecahedron, the shape closest to the sphere Cf.[4]. The classification and the numerical constraints of these regular polyhedra give rise to Euler's formula: $f - a + s = 2$ where f, a, s denote respectively the numbers of faces, edges and vertices of a spherical polyhedron Cf.[2]. By using Euler's formula, we can deduce the triplets (f, a, s) from the symbol of L. Schläfli, denoted by the couple $[x, y)$. The numbers x and y are both at least equal to three.

Keywords and phrases: Integer convex polyhdra, Plato's polyhedra, Characteristic polynomial, Polyhedron characteristic, Schläfli symbol

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[1] In old greeck, that of last Plato's dialogues, is considered since the antiquity as Plato's essential work.

[2] The half-urge of the Plato's Timaeus or the mythical representation. In the Timaeus, Plato explains the origine of the world expresses a likely mythical doing intervene a divine half-urge.

The Schläfli symbols are the pairs determined by:

$$(x, y) \in \{(3,3); (3,4); (4,3); (3,5); (5,3)\}.$$

These couples vary according to the regular polyhedra (tetrahedron, cube, icosahedron, octahedron and dodecahedron). In our work, we have built a polynomial model that we call characteristic polynomial denoted by p_c Cf.[5] and a mathematical formulation in the form of a recurrent relation which allows us to calculate the number of faces of a polyhedron of dimension d, with $1 \leq d \leq 3$ such that $f = 4 + 2^{n'}$, for $1 \leq n' \leq 4$. The characteristic polynomial p_c and the recurrence formula make it possible to find the triple (f, a, s) without using L. Schläfli's symbol via Euler's formula.
We also obtained the Schläfli symbol from the characteristic polynomial and the formula which give the number of faces.

2. Preliminaries

2.1. Characteristic polynomial of a convex polytope.

2.1.1. *Definition and construction.* Cf.[5] *We call characteristic polynomial of an integer convex polytope P of dimension d, with $(d \geq 1)$, the polynomial noted p_c making it possible to calculate the number of vertices of this convex polytope defined by:*

- *for $1 \leq d \leq 2$, p_c is of degree k, where k is an integer such that:*
  - *if d is even, $deg(p_c) = k = \dfrac{d}{2}$ and $p_c(n) = kn^k + k + 1$ as well as k divides $2k + 1$.*
  - *if d is odd, p_c is of degree $k = \dfrac{d+1}{2}$, $p_c(n) = kn^k + k$ and k divide $2k$.*
- *for $d = 3$, $deg(p_c) = k - 1$, where k is an integer such that $k = \dfrac{d+1}{2}$, $p_c(n) = k + \sum_{i=0}^{k-1}(k-i)n^{k-i-1}$, $\forall n,\ k \geq 1$ and k divides $k + \sum_{i=0}^{k-1}(k-i)$.*

Example 2.1. *Cf.*[5]

2.1.2. *Calculation of the number of vertices of five Platonic polyhedra Cf.*[5].
We know that the five polyhedra of Plato are convex polytopes of dimension 3. Therefore, we use the characteristic polynomial of a convex polytope of dimension 3.
For $d = 3,\ k = 2$, then $p_c(n) = 2n + 2$ and 2 divides 4.
Evaluation of p_c according to the values of n.

- **•:** $n = 1,\ p_c(1) = 4 =$ vertices, coefficients of p_c so $s = 4$, P has 4 vertices, it is a tetrahedron.
- **•:** $n = 2,\ p_c(2) = 6 =$, P has 6 vertices, it is an octahedron.
- **•:** $n = 3,\ p_c(3) = 8 =$, P has 8 vertices, it is an hexahedron.
- **•:** $n = 5,\ p_c(5) = 12$, it is an icosahedron.
- **•:** $n = 9,\ p_c(9) = 20$, P is a dodecahedron.

2.1.3. *Calculation of the number of vertices of certain regular polytopes P.* For an regular polygon of dimension 2, $p_c(n) = n + 2$

- $n = 1$, $p_c(1) = 3 = s$, P has 3 vertices, P is a triangle.
- $n = 2$, $p_c(2) = 4$, P has 4 vertices, P is a parallelogram (rectangle, square or diamond or rhombus).

Remark 2.1. *Cf.*[5]

- *The convex polytopes of dimension* 4 *and* 5 *do not have the characteristic polynomials.*
- *From the dimension* $d \geq 6$*, only the convex polytopes whose dimensions obey the recurrence relation* $d_n = 4n + 6$ *have characteristic polynomials of degree equal to* $k_n = 2n + 3$*, and of constant term equals to* $2n + 4$ *and, which are written as follows:*

$$p_c(n) = k_n + \sum_{i=0}^{k_n - 1} (k_n - i) n^{k_n - i - 1}, \quad \forall n \in \mathbb{N}.$$

Example 2.2. *Cf.*[5]

Remark 2.2.

- *The number of vertices of a convex polytope is equal to its characteristic polynomial.*
- *Convex polytopes of different dimensions can have the same number of vertices.*

Example 2.3.
- *For* $d = 3$*, we get* $p_c(n) = 2n + 2$ *so* $s = p_c(9) = 20$*.*
- *For* $d = 6$*, we have* $p_c(n) = 3n^2 + 2n + 4$ *then* $s = p_c(2) = 20$*.*
- *For* $d = 10$*, we get* $p_c(n) = 5n^4 + 4n^3 + 3n^2 + 2n + 6$ *then if* $n = 1$ *we obtain* $s = p_c(1) = 20$*, number of vertices of the dodecahedron.*

3. On the number of faces of Plato's five polyhedra

3.1. **Characteristic of a polyhedron.**

Definition 3.1. *We call characteristic of a convex polyhedron of dimension* d*, with* $(d \geq 1)$*, the positive constant noted* c*,* $(c \geq 2)$ *equal to the product of the degree of the characteristic polynomial of the convex polyhedron* P *by the number of its vertex.*

Notation

Let P be a convex polyhedron of dimension d, (≥ 1) and p_c its characteristic polynomial of degree k and s its number of vertices. We have

$$c = k.s = k.p_c(n).$$

We also denote $c(n) = k.p_c(n)$.

Proposition 3.1. *Any polyhedron or more generally any convex polytope of dimension* d*,* $(d \geq 1)$ *which has a characteristic polynomial admits a constant* c *which characterizes it.*

Proof. *Let* P *be a convex polytope of dimension* d*.*

- *If* $d = 1$, *then* P *has a characteristic polynomial of degree* $k = 1$, $p_c(n) = n + 2 = s$, *so* P *admits a constant* $c = k.s = k(n + 2)$.
- *If* $d = 2$, $k = 1$ *and* $p_c(n) = 2n + 2$ *then* $c(n) = k(2n + 2)$.
- *If* $d \geq 6$ *and* $d_n = 4n + 6$, $k_n = 2n + 3$; *so* $p_c(n) = k_n + \sum_{i=0}^{k_n-1}(k_n - i)n^{k_n-i-1}, \quad \forall n \in \mathbb{N}$. *The*

$$\begin{aligned} c(n) &= k_n.s = k_n p_c(n) \\ &= (2n+3)s \\ &= (2n+3)[(2n+3) + \sum_{i=0}^{2n+2}(2n+3-i)n^{2n+2-i}] \end{aligned}$$

Which ends the proof.

Example 3.1. P *is a convex polytope of dimension* d, $(d \geq 1)$*:*

- $d = 1$, $k = 1$ *so* $p_c(n) = n + 2$, *we then have* $c(n) = n + 2$.
- $d = 1 = n$, $k = 1$ *so we get* $c(1) = 3$.
- $d = 6$, $k = 3$ *so* $p_c(n) = 3n^2 + 2n + 4$ *we then have* $c(n) = 3(3n^2 + 2n + 4)$ *and* $c(1) = 3(9) = 27$.

3.2. **Formulation of the number of faces of platonic polyhedra.** Let f be the number of faces of the polyhedron P of dimension d, $(d = 3)$, c its characteristic and p_c its characteristic polynomial of degree k and s the corresponding number of vertices.
As $c = ks = kp_c(n)$. For $d = 3$, $k = 2$, then we obtain $p_c(n) = 2n + 2$. Therefore, $c = 2(2n + 2) = 4n + 4 = 4 + 4n = 4 + 2^2 n$. But we know that for $1 \leq n \leq 4$, we have $n \geq 2^{n-2}$. So $c = 4 + 2^2 n \geq 4 + 2^n$ for $1 \leq n \leq 4$. Let $f = 4 + 2^{n'}$, for $1 \leq n' \leq 4$. The number of faces is then $f = 4 + 2^{n'}$, $1 \leq n' \leq 4$.

Theorem 3.1. *Any polyhedron of Plato of dimension* 3 *admits the recurrence relation* $f = 2^{n'} + 4$ *for the number of faces, where* $1 \leq n' \leq 4$.

Proof. *See above.*

Remark 3.1.

- *To avoid any confusion on the variable* n, *we set* $f = 4 + 2^{n'}$ *with* $1 \leq n' \leq 4$ *for the number of faces.*
- *In the case of tetrahedron,* $f = 4 + 4n'$ *where* $n' = 0$.

Example 3.2. *Consider a regular convex polyhedron* P *of dimension* 3.

- $n' = 1$, $f = 4 + 2^1 = 6$, P *is therefore a hexahedron or cube.*
- $n' = 2$, $f = 4 + 2^2 = 8$, *then* P *is a octahedron.*
- $n' = 3$, $f = 12$, *so we have a dodecahedron* P.
- $n' = 4$, $f = 20$, *then* P *is a icosahedron.*

4. Number of edges of the five polyhedra of Plato

4.1. **Determination of thr number of edges.** According to Euler's formula, we have $s - a + f = 2$, that is $a = f + s - 2$ where $f = 4 + 4n'$ for the tetrahedron, $n' = 0$.
$f = 4 + 2^{n'}$ for $1 \leq n' \leq 4$ where f is the number of faces of the four convex polyhedra Platonic regulars and s is the number of vertices, a is the one of edges, with $s = p_c(n) = 2n + 2$ where $d = 3$, $n \geq 1$.

- $a = 4 + 4n' + 2n + 2 - 2$, $a = 2n + 4n' + 4$, case of the tetrahedron with $n' = 0$ and $n = 1$.
- $a = 4 + 2^{n'} + 2n + 2 - 2$, $a = 2n + 2^{n'} + 4$, $n \geq 1$ and $1 \leq n' \leq 4$ for the four regular polyhedra of Plato. So, for $n \geq 1$ and $1 \leq n' \leq 4$, we have $a = 2n + 2^{n'} + 4$.

Example 4.1.

- *Tetrahedron:* $a = 4 + 4n' + 2n$. *We have* $n' = 0$ *and* $n = 1$, *so* $a = 4 + 2 = 6$.
- *Hexahedron:* $a = 4 + 2n + 2^{n'}$. *We obtain* $n = 3$ *and* $n' = 1$, *then* $a = 4 + 6 + 2 = 12$.
- *Octahedron:* $a = 4 + 2n + 2^{n'}$. *With* $n = 2$ *and* $n' = 2$, *we have* $a = 12$.
- *Dodecahedron:* $a = 4 + 2n + 2^{n'}$. *We have* $n = 9$ *and* $n' = 3$, *then* $a = 4 + 18 + 2^3 = 4 + 18 + 8 = 30$.
- *Icosahedron:* $a = 4 + 2n + 2^{n'}$. *We obtain* $n = 5$ *and* $n' = 4$, *so* $a = 4 + 10 + 16 = 30$.

Remark 4.1. *For the regulars convex polyhedra other than the tetrahedron, we have:*

$$\begin{aligned} a &= 4 + 2n + 2^{n'} \\ &= 2 + 2n + 2 + 2^{n'} \\ &= 2n + 2 + 2(1 + 2^{n'-1}) \\ &= p_c(n) + 2(1 + 2^{n'-1}),\ 1 \leq n' \leq 4. \end{aligned}$$

We obtain another expression of the formula which gives the number of edges of regulars convex polyhedra other than the tetrahedron.

$$a = p_c(n) + 2(1 + 2^{n'-1}), \quad 1 \leq n' \leq 4,$$

where $p_c(n)$ *is the characteristic polynomial of an integer convex polytope of dimension* 3.

Example 4.2. *As* $a = p_c(n) + 2(1 + 2^{n'-1})$, *with* $1 \leq n' \leq 4$, *then for* $n = 5$ *and* $n' = 4$, *therefore* $a = p_c(5) + 2(1 + 2^{4-1}) = 12 + 2(1 + 2^3) = 30$, *this is the case of an icosahedron.*

4.2. **Calculation of the Schläfli symbol** (x, y)**.** The Schläfli symbol is defined by the couple

$$(x, y) \in \{(3, 3),\ (3, 4),\ (4, 3),\ (3, 5),\ (5, 3)\}.$$

Each pair has its specific regular polyhedron.

- For the tetrahedron: $(x, y) = (3, 3)$
- Hexahedron: $(x, y) = (4, 3)$
- Octahedron: $(x, y) = (3, 4)$
- Dodecahedron: $(x, y) = (5, 3)$
- Icosahedron: $(x, y) = (3, 5)$

We can see that every polyhedron has its Schläfli symbol.

4.2.1. *Determination of the Schläfli symbol.* We have $a = p_c(n) + 2(1 + 2^{n'-1})$, $1 \leq n' \leq 4$; $n \geq 1$ or $a = 2n + 4 + 4n'$ for the tetrahedron, with $n \geq 1$ and $1 \leq n' \leq 4$ as well as $f = 4 + 4n'$ for the tetrahedron, with $n' = 0$. For the other polyhedra other than the tetrahedron, $f = 4 + 2^{n'}$, with $1 \leq n' \leq 4$.

- Tetrahedron: $x = \frac{2a}{f} = \frac{2(2n+4+4n')}{4+4n'} = \frac{n+2+2n'}{n'+1}$. For $n' = 0$, so $x = n + 2$ where $n \geq 1$.
- for the other polyhedra: $x = \frac{2(p_c(n) + 2(1+2^{n'-1}))}{4+2^{n'}} = \frac{p_c(n) + 2(1+2^{n'-1})}{2+2^{n'-1}} = \frac{2n+2+2(1+2^{n'-1})}{2+2^{n'-1}}$. For $1 \leq n' \leq 4$ and $n \geq 1$.
- As $y = \frac{xf}{s}$, with $s = p_c(n)$; then $y = \frac{n+2}{p_c(n)}(4+4n') = \frac{4(n+2)}{2n+2}$, where $n' = 0$ and $n \geq 1$. Therefore, for the tetrahedron, $y = 2\frac{(n+2)}{n+1}$, with $n \geq 1$.
- For the other polyhedra of Plato

$$\begin{aligned} y &= \frac{[2n+2+2(1+2^{n'-1})](4+2^{n'})}{(2+2^{n'-1})(2n+2)} \\ &= \frac{2(2+2^{n'-1})[2n+2+2(1+2^{n'-1})]}{2(2+2^{n'-1})(n+1)} \\ &= \frac{2n+2+2(1+2^{n'-1})}{n+1} \\ &= \frac{2(n+2+2^{n'-1})}{n+1}, \quad n \geq 1 \quad \text{and } 1 \leq n' \leq 4. \end{aligned}$$

So the Schläfli symbol is obtained by the following formula:

- For the tetrahedron:

$$(x, y) = (n+2, 2\frac{n+2}{n+1}), \quad n \geq 1. \tag{1}$$

- For the other Platonic polyhedra:

$$(x, y) = (\frac{2n+2+2(1+2^{n'-1})}{2+2^{n'-1}}, \frac{2(n+2+2^{n'-1})}{n+1}), \quad \text{for} n \geq 1 \text{ and } 1 \leq n' \leq 4. \tag{2}$$

Remark 4.2. *In any formula where the variables n and n' appear, numerical values are specific for each regular polyhedron of Plato. For:*

- *tetrahedron:* $n = 1$ *and* $n' = 0$.
- *hexahedron:* $n = 3$ *and* $n' = 1$.
- *octahedron:* $n = 2$ *and* $n' = 2$.
- *icosahedron:* $n = 5$ *and* $n' = 4$.
- *dodecahedron:* $n = 9$ *and* $n' = 3$.

Example 4.3. *By using the above equations , we respectively get:*

- *tetrahedron:* $(x, y) = (3, 3)$.
- *hexahedron:* $(x, y) = (4, 3)$.
- *octahedron:* $(x, y) = (3, 4)$.

- *icosahedron:* $(x, y) = (3, 5)$.
- *dodecahedron:* $(x, y) = (5, 3)$.

5. Formulation of the number of faces, of vertices and of edges of the five Plato's regular convex polyhedra

5.1. Characteristic symbol.

Definition 5.1. *We call characteristic symbol of each Platonic regular convex polyhedron, denoted* (n, n') *the pair of integers which gives the numerical value of the number of faces, vertices and edges of the polyhedron.*

Example 5.1.

- *tetrahedron:* $(n, n') = (1, 0)$.
- *hexahedron:* $(n, n') = (3, 1)$.
- *octahedron:* $(n, n') = (2, 2)$.
- *dodecahedron:* $(n, n') = (9, 3)$.
- *icosahedron:* $(n, n') = (5, 4)$.

Lemma 5.1. *Let* $\mathfrak{N}$ *be the set of the characteristic symbols of every Plato's regular convex polyhedron. Then* $\mathfrak{N} = \{(1, 0), (3, 1), (2, 2), (5, 4), (9, 3)\}$.

Proof. *Let us consider the triplets* (f, a, s) *where* f *is the number of faces,* a *the number of edges and* s *that of the vertices of the regular convex polyhedron* P *of dimension* 3. *We know that these triplets belong to the* $\{(4, 6, 4), (6, 12, 8), (8, 12, 6), (12, 30, 20), (20, 30, 12)\}$ *Cf.*[2]. *Therefore, the only possible cases for the pairs* (n, n') *are* $\{(1, 0), (3, 1), (2, 2), (5, 4), (9, 3)\}$ *through the formulation of the number of faces, edges and vertices of the regular convex polyhedron* P. *Hence* $\mathfrak{N}$.

Proposition 5.1. *Any regular convex polyhedron* P *of dimension* d, $(1 \leq d \leq 3)$ *admits a characteristic symbol.*

Proof. *This is obtained from the above Definition5.1 and Lemma 5.1.*

Proposition 5.2. *The number of vertices, faces and edges of all regular convex polyhedron of dimension* 3 *depends on the value of the characteristic symbol* (n, n') *of the polyhedron.*

Proof. *Immediate.*

5.2. Formulation of the number of vertices, faces and edges of Platonic regular convex polyhedron .

5.2.1. *For tetrahedron.* Let P be a regular convex polyhedron of dimension 3. We have $p_c(n) = 2n + 2$.

- **Vertices:**
  $p_c(n) = 2n + 2 = s$ Cf.[5], for $n = 1$.
- **Faces:**
  The number of faces of the tetrahedron is given by the recurrence formula: $f = 4 + 4n'$ with $n' = 0$.
- **Edges:**
  The number of edges is $a = 4 + 4n' + 2n$, with $n = 1$ and $n' = 0$.

5.2.2. *For the Platonic regular convex polyhedra other than the tetrahedron.*

- **Vertices:**
  $p_c(n) = 2n + 2 = s$ Cf.[5], with $n \in \{2, 3, 5, 9\}$.
- **Faces:**
  $f = 4 + 2^{n'}$, with $1 \leq n' \leq 4$.
- **Edges:**
  $a = 4 + 2n + 2^{n'}$, with $n \in \{2, 3, 5, 9\}$ and $n' \in \{1, 2, 3, 4\}$.

5.3. **Verification of Euler's formula from recurrence formulas.**

Proposition 5.3. *The recurrence formulas giving the number of vertices, faces and edges of any regular convex polyhedron of dimension* 3 *verify the Euler's formula.*

Proof. *The Euler's formula is written:* $f - a + s = 2$ *where* f *is the number of faces,* a *is the one of edges and* s *the one of vertices of regular convex polyhedron. Then we have:*

- *For the tetrahedron,*

$$\begin{aligned} f - a + s &= 4 + 4n' - (4 + 4n' + 2n) + 2n + 2 \\ &= 4 + 4n' - 4 - 4n' - 2n + 2n + 2 \\ &= 2 \end{aligned}$$

- *For the others (hexahedron, octahedron, icosahedron, dodecahedron)*

$$\begin{aligned} f - a + s &= 4 + 2^{n'} - (4 + 2n + 2^{n'}) + 2n + 2 \\ &= 4 + 2^{n'} - 4 - 2^{n'} - 2n + 2 + 2n \\ &= 2 \end{aligned}$$

Which ends the proof.

Proposition 5.4. *The number of vertices, faces and edges of a regular convex polyhedron is obtained from the recurrence formula knowing the characteristic symbol of each polyhedron.*

Proof. *See recurrence formulas and characteristic symbol.*

Remark 5.1.

- *We can calculate the triplet* (f, a, s) *of an regular convex polyhedron with recurrence formulas without going through the Schläfli symbol.*
- *The Schläfli symbol and the characteristic symbol are two pairs of different integers,* $(x, y) \neq (n, n')$. *They have about the same role in determining the number of faces, edges and vertices of regular convex polyhedra.*

6. Roles of Schläfli symbol and characteristic symbol

6.1. **For the Schläfli symbol** (x, y)**.** It allows to calculate the number of faces, edges and vertices of regular convex polyhedra.

$f = \dfrac{4y}{2x + 2y - xy}$, $(x, y) \neq (0, 0)$ with $(x, y) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$

$a = \dfrac{xf}{2}$
$s = \dfrac{xf}{y}$, where $y \neq 0$.

6.2. **For the characteristic symbol** (n, n')**.** It allows to calculate the number of faces, edges and vertices of Platonic regular convex polyhedra by using the recurrence formulas with

$$(n, n') \in \{(1,0), (2,2), (3,1), (5,4), (9,3)\}, \text{Cf. } 5.2.$$

7. Conclusion

Plato, in his Pythagorean metaphysical philosophy, when he describes the formation of the world Cf.[4], affirms the existence and uniqueness of the five regular convex polyhedra which are the tetrahedron, the hexahedron, the icosahedron, the octahedron and dodecahedron.
Euclid in Book XIII of his Elements, takes up Plato's idea of the existence and uniqueness of the five regular convex poyhedra and demonstrated using two approaches. In the first, he uses analysis and trigonometry to demonstrate the uniqueness of the five polyhedra (an empirical approach to the problem of uniqueness). In the other, an abstract approach to the problem of the existence of Plato's polyhedra.

After studying the existence and uniqueness of Plato's five polyhedra, Euler made a geometric study of these Platonic solids and established a famous formula that relates the number of faces, edges and vertices of regular convex polyhedra. In other words, $f - a + s = 2$. L. Schläfli uses Euler's formula and has introduced a symbol, couple (x, y) specific to each polyhedron which allows him to calculate the number of faces, edges and of vertices of Plato's five regular convex polyhedra using Euler's formula Cf. subsection 6.1. And this we have just worked on.

We found some recurrence relations making it possible to find the same results by the characteristic symbol (n, n') without using that of Schĺalfi. Thus the recurrence relations and the characteristic symbol (n, n') that we have found allow us to calculate the three (03) parameters f, a, s of five Platonic polyhedra. But the extension of these recurrence formulas for regular convex polyhedra of dimension greater than three still remains to be desired.

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