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THE BEST MAJORATION FOR THE SUMS $\sum \sin\frac{A}{2}$, $\sum \cos\frac{A}{2}$, $\sum \frac{1}{\sin\frac{A}{2}}$ AND $\sum \frac{1}{\cos\frac{A}{2}}$ IN A TRIANGLE

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Abstract. The purpose of this article is to find the best homogenous functions $f(R,r)$, $g(R,r)$ which minimize and maximize the sums $\sum \sin\dfrac{A}{2}$, $\sum \cos\dfrac{A}{2}$, $\sum \dfrac{1}{\sin\frac{A}{2}}$ and $\sum \dfrac{1}{\cos\frac{A}{2}}$ in a triangle.

1. Introduction

In this section we will recall some known results, which we will use in the following.
In a given triangle ABC, we denote the lengths of the sides with $AB = c$, $BC = a$, $CA = b$, F the area, r, R the radius of the inscribed circle with the center I, respectively of the circumscribed circle with the center O of the triangle, $s = \dfrac{a+b+c}{2}$ the semiperimeter, the distance between O and I by $d = \sqrt{R^2 - 2Rr}$ and with A, B, C the measures of the angles of the triangle ABC.

W.J. Blundon in [7] has proved in 1965 the following inequalities

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr} \le s^2 \le$$
$$\le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}. \tag{1}$$

The inequalities from (1) represent necessary and sufficient conditions for the existence of a triangle with given elements R, r and s.

In several papers, the cases of equality from (1) were studied. For example, the cases of equality were obtained in [12] by A. Lupaş with the algebraic way, in [6] T. Bîrsan geometrically, in [15] Shan-He Wu and Yu-Ming Chu trigonometrcally and in [3] D. Andrica, C. Barbu and L.I. Pişcoran by extremal way.

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Also, in the paper [10] by M. Drăgan and N. Stanciu, equality condition are given from (1).

In this paper, we will use the above methods and results from the paper [10]. In the following, we mention some results from these papers.

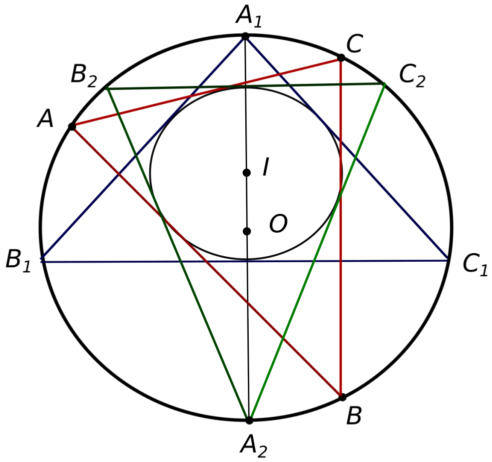

Figure 1

In the following we consider given the triangle ABC, $\mathcal{C}(O,R)$ the circumscribed circle and $\mathcal{C}(I,r)$ the inscribed circle. The half-lines $(OI$, $(IO$ intersect $\mathcal{C}(O,R)$ in A_1, respectively A_2.

According to Poncelet's Theorem, are obtained the triangles $A_1B_1C_1$ and $A_2B_2C_2$, tangent to the circle $\mathcal{C}(O,I)$ (see Figure 1).

Lemma 1.1. (i) *The lengths of the sides of the triangle $A_1B_1C_1$ are given by*

$$a_1 = 2\sqrt{R^2-(r-d)^2}, \quad b_1 = c_1 = \sqrt{2R(R+r-d)}, \tag{2}$$

while those of the sides of the triangle $A_2B_2C_2$ are given by

$$a_2 = 2\sqrt{R^2-(r+d)^2}, \quad b_2 = c_2 = \sqrt{2R(R+r+d)}, \tag{3}$$

(ii) *The semiperimeter of the triangle $A_1B_1C_1$ is*

$$s_1 = \sqrt{\frac{(R+r-d)^3}{R-r-d}} = \sqrt{2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3}}, \tag{4}$$

while that of the triangle $A_2B_2C_2$ is

$$s_2 = \sqrt{\frac{(R+r+d)^3}{R-r+d}} = \sqrt{2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}}. \tag{5}$$

Remark 1.1. It is immediately verified that $a_1 > b_1$ and $a_2 < b_2$.

Theorem 1.1 (Fundamental triangle inequalities of Blundon)**.** *The following inequalities*

$$s_1 \le s \le s_2 \tag{6}$$

hold. The equality occurs on the left-side and right-side of inequality if and only if the triangle ABC becomes triangle $A_1B_1C_1$, respectively triangle $A_2B_2C_2$, with the sides from the Lemma 1.1.

In the following we will study the case $d \ge r$, equivalently $R \ge \left(\sqrt{2}+1\right) r$, that is the case when the point O is exterior to the circle $\mathcal{C}(I,r)$. The tangents through O to the circle $\mathcal{C}(I,r)$ intersect the circle $\mathcal{C}(O,R)$ in two pairs of points B_3, C_3 and B_4, C_4. Next we construct the right triangles $A_3B_3C_3$ and $A_4B_4C_4$ inscribed in $\mathcal{C}(O,R)$ and circumscribed to $\mathcal{C}(I,r)$ (see the Figure 2).

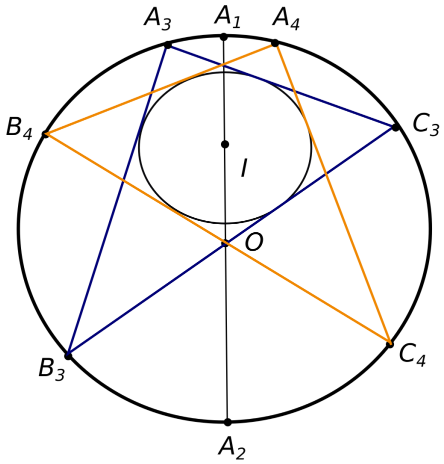

Figure 2

Remark 1.2. Immediately check that triangles $A_3B_3C_3$ and $A_4B_4C_4$ are congruent.

Lemma 1.2. *The right triangle $A_3B_3C_3$ has the sides*

(7) $a_3 = 2R$, $b_3 = R + r - \sqrt{R^2 - 2Rr - r^2}$, $c_3 = R + r + \sqrt{R^2 - 2Rr - r^2}$

and semiperimeter

$$s_3 = 2R + r. \tag{8}$$

Lemma 1.3 (C. Ciamberlini, see [8])**.** *A triangle is an acute or right triangle if and only if*

$$s \geq 2R + r. \tag{9}$$

Equality occurs if and only if the triangle is right triangle.

Lemma 1.4. *If $d \geq r$, then the inequality*

$$s_1 \leq s_3 \tag{10}$$

holds. The equality occurs if and only if $\dfrac{R}{r} = 1 + \sqrt{2}$.

Proof. The inequality (10) is equivalent with
$2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3} \leq 4R^2 + 4Rr + r^2$, equivalent with $-R^2 + 3Rr - r^2 \leq \sqrt{R(R-2r)^3}$. If $\dfrac{R}{r} > \dfrac{3+\sqrt{5}}{2}$, then $-R^2 + 3Rr - r^2 < 0$, so inequality holds. If $2 \leq \dfrac{R}{r} \leq \dfrac{3+\sqrt{5}}{2}$, then $-R^2 + 3Rr - r^2 \geq 0$ and by squaring we have $(-R^2 + 3Rr - r^2)^2 \leq R(R-2r)^3$, equivalent after performing some calculation with $0 \leq R^2 - 2Rr - r^2$, or $\dfrac{R}{r} \geq 1 + \sqrt{2}$.
From the above it follows that inequality from (10) and the case of equality occurs.

Theorem 1.2 (Blundons inequalities in non-obtuse triangle)**.** *In any accute or right triangle, the following inequalities*

$$s_1 \leq s \leq s_2 \quad \textit{if} \ \ 2 \leq \frac{R}{r} \leq \sqrt{2} + 1 \tag{11}$$

and

$$s_3 \le s \le s_2, \text{ if } \frac{R}{r} \ge \sqrt{2}+1 \tag{12}$$

hold. The equality occurs in the left-side, right-side of inequalities (11) *if and only if the triangle ABC becomes triangle $A_1B_1C_1$, respectively $A_2B_2C_2$, with the sides from the Lemma* 1.1*. The equality occurs in the left-side, right-side of inequalities* (12) *if and only if the triangle ABC becomes triangle $A_3B_3C_3$ with the sides from the Lemma* 1.2*, respectively triangle $A_2B_2C_2$ from Lemma 1.1.*

2. Main results

In this section, first we will prove some identities and after that using them we will prove several inequalities.

Lemma 2.1. *In $A_1B_1C_1$ triangle are true the equalities*

$$\sum \sin\frac{A_1}{2} = \sqrt{\frac{R-r+d}{2R}} + \sqrt{2-\sqrt{\frac{2(R-r+d)}{R}}} = f_1(R,r), \tag{13}$$

$$\sum \cos\frac{A_1}{2} = \sqrt{\frac{R+r-d}{2R}} + \sqrt{2+\sqrt{\frac{2(R-r+d)}{R}}} = g_1(R,r), \tag{14}$$

$$\sum \frac{1}{\sin\frac{A_1}{2}} = \sqrt{\frac{2R}{R-r+d}} + \sqrt{\frac{8\sqrt{2R}}{\sqrt{2R}-\sqrt{R-r-d}}} = u_1(R,r), \tag{15}$$

and

$$\sum \frac{1}{\cos\frac{A_1}{2}} = \sqrt{\frac{2R}{R+r-d}} + \sqrt{\frac{8\sqrt{2R}}{\sqrt{R-r+d}+\sqrt{2R}}} = v_1(R,r). \tag{16}$$

Proof. In triangle $A_1B_1C_1$, with the sides from Lemma 1.1 we have

$$\sin\frac{A_1}{2} = \sqrt{\frac{(s_1-b_1)(s_1-c_1)}{b_1c_1}} = \frac{s_1-b_1}{b_1} = \frac{a_1+c_1-b_1}{2b_1} = \sqrt{\frac{R-r+d}{2R}}$$

and

$$\sin\frac{B_1}{2} = \sin\frac{C_1}{2} = \sqrt{\frac{(s_1-b_1)(s_1-a_1)}{a_1b_1}} = \frac{1}{2}\sqrt{\frac{2b_1-a_1}{b_1}} =$$
$$= \frac{\sqrt{2}}{2}\sqrt{\frac{\sqrt{2R}-\sqrt{R-r+d}}{\sqrt{2R}}}$$

from where it follows (13). We have

$$\cos\frac{A_1}{2} = \sqrt{\frac{s_1(s_1-a_1)}{4c_1}} = \frac{1}{2}\sqrt{\frac{(b_1+c_1)^2-a_1^2}{b_1^2}} = \frac{1}{2}\sqrt{\frac{4b_1^2-a_1^2}{b_1^2}} =$$
$$= \sqrt{\frac{R+r-d}{2R}}$$

and

$$\cos\frac{B_1}{2} = \cos\frac{C_1}{2} = \sqrt{\frac{s_1(s_1-b_1)}{a_1b_1}} = \frac{1}{2}\sqrt{\frac{a_1+2b_1}{b_1}} = \frac{1}{2}\sqrt{2+2\sqrt{\frac{R-r+d}{2R}}}$$

from which it follows (14).
From the above relations it follows that the identities (15) and (16) it's true.

Lemma 2.2. *In $A_2B_2C_2$ triangle are true the equalities*

$$\sum \sin\frac{A_2}{2} = \sqrt{\frac{R-r-d}{2R}} + \sqrt{2-\sqrt{\frac{2(R-r-d)}{R}}} = f_2(R,r), \tag{17}$$

$$\sum \cos\frac{A_2}{2} = \sqrt{\frac{R+r+d}{2R}} + \sqrt{2+\sqrt{\frac{2(R-r-d)}{R}}} = g_2(R,r), \tag{18}$$

$$\sum \frac{1}{\sin\frac{A_2}{2}} = \sqrt{\frac{2R}{R-r-d}} + \sqrt{\frac{8\sqrt{2R}}{\sqrt{2R}-\sqrt{R-r-d}}} = u_2(R,r) \tag{19}$$

and

$$\sum \frac{1}{\cos\frac{A_2}{2}} = \sqrt{\frac{2R}{R+r+d}} + \sqrt{\frac{8\sqrt{2R}}{\sqrt{R-r-d}+\sqrt{2R}}} = v_2(R,r). \tag{20}$$

Proof. In triangle $A_2B_2C_2$, with the sides from Lemma 1.1, we have

$$\sin\frac{A_2}{2} = \sqrt{\frac{(s_2-b_2)(s_2-c_2)}{b_2c_2}} = \frac{a_2+c_2-b_2}{2b_2} = \frac{a_2}{2b_2} = \sqrt{\frac{R-r-d}{2R}}$$

and

$$\sin\frac{B_2}{2} = \sin\frac{C_2}{2} = \sqrt{\frac{(s_2-a_2)(s_2-b_2)}{a_2b_2}} = \frac{1}{2}\sqrt{\frac{2b_2-a_2}{b_2}} =$$
$$= \frac{1}{2}\sqrt{2-\sqrt{\frac{2(R-r-d)}{R}}}$$

from where (17) follows.
We have

$$\cos\frac{A_2}{2} = \sqrt{\frac{s_2(s_2-a_2)}{b_2c_2}} = \frac{1}{2}\sqrt{\frac{4b_2^2-a_2^2}{b_2^2}} = \sqrt{\frac{R+r+d}{2R}}$$

and

$$\cos\frac{B_2}{2} = \cos\frac{C_2}{2} = \sqrt{\frac{s_2(s_2-b_2)}{a_2b_2}} = \frac{1}{2}\sqrt{\frac{2b_2+a_2}{b_2}} =$$
$$= \frac{1}{2}\sqrt{2+\sqrt{\frac{2(R-r-d)}{R}}},$$

from where is it obtained (18).
Taking the relations above into account, the identities (19) and (20) immediately result.

Lemma 2.3. *In $A_3B_3C_3$ triangle are true the equalities*

$$\sum \sin\frac{A_3}{2} = \frac{1}{\sqrt{2}} + \sqrt{\frac{r\left(R-\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}} + \sqrt{\frac{r\left(R-\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}} = f_3(R,r), \tag{21}$$

$$\sum \cos\frac{A_3}{2} = \frac{1}{\sqrt{2}} + \sqrt{\frac{(2R+r)\left(R+\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}} + \sqrt{\frac{(2R+r)\left(R-\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r-\sqrt{R^2-2Rr-r^2}\right)}} = g_3(R,r), \tag{22}$$

$$\sum \frac{1}{\sin\frac{A_3}{2}} = \sqrt{2} + \sqrt{\frac{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}{r\left(R-\sqrt{R^2-2Rr-r^2}\right)}} + \sqrt{\frac{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}{r\left(R-\sqrt{R^2-2Rr-r^2}\right)}} = u_3(R,r) \tag{23}$$

and

$$\sum \frac{1}{\cos\frac{A_3}{2}} = \sqrt{2} + \sqrt{\frac{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}{(2R+r)\left(R+\sqrt{R^2-2Rr-r^2}\right)}} + \sqrt{\frac{2R\left(R+r-\sqrt{R^2-2Rr-r^2}\right)}{(2R+r)\left(R-\sqrt{R^2-2Rr-r^2}\right)}} = v_3(R,r). \tag{24}$$

Proof. In triangle $A_3B_3C_3$, with the sides from Lemma 1.2, we have
$s_3-a_3=r,\ \ s_3-b_3=R+\sqrt{R^2-2Rr-r^2}$,
$s_3-c_3=R-\sqrt{R^2-2Rr-r^2}$. So

$$\sin\frac{B_3}{2} = \sqrt{\frac{(s_3-a_3)(s_3-c_3)}{a_3c_3}} = \sqrt{\frac{r\left(R-\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}}$$

and

$$\sin \frac{C_3}{2} = \sqrt{\frac{(s_3-a_3)(s_3-b_3)}{a_3b_3}} = \sqrt{\frac{r\left(R+\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r-\sqrt{R^2-2Rr-r^2}\right)}},$$

from where (21) is obtained.
We have

$$\cos \frac{B_3}{2} = \sqrt{\frac{s_3(s_3-b_3)}{a_3c_3}} = \sqrt{\frac{(2R+r)\left(R+\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r+\sqrt{R^2-2Rr-r^2}\right)}}$$

and

$$\cos \frac{C_3}{2} = \sqrt{\frac{s_3(s_3-c_3)}{a_3b_3}} = \sqrt{\frac{(2R+r)\left(R-\sqrt{R^2-2Rr-r^2}\right)}{2R\left(R+r-\sqrt{R^2-2Rr-r^2}\right)}},$$

from which it follows (22).
Identities (23) and (24) it result immediately from above.

Lemma 2.4. *In every ABC triangle are true the equalities*

$$\left(\sum \cos \frac{A}{2}\right)^2 - \left(\sum \sin \frac{A}{2} + 1\right)^2 = \frac{r}{R} \tag{25}$$

and

$$2R\sum \cos \frac{A}{2}\left(\sum \sin \frac{A}{2} - 1\right) = s. \tag{26}$$

Proof. By calculation in the left member of (25) we have

$$\left(\sum \cos \frac{A}{2}\right)^2 - \left(\sum \sin \frac{A}{2} + 1\right)^2 = \sum \cos^2 \frac{A}{2} + 2\sum \cos \frac{A}{2} \cos \frac{B}{2} -$$
$$-\sum \sin^2 \frac{A}{2} - 2\sum \sin \frac{A}{2} \sin \frac{B}{2} - 2\sum \sin \frac{A}{2} - 1 =$$
$$= \sum \left(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}\right) + 2\sum \left(\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}\right) -$$
$$-2\sum \sin \frac{A}{2} - 1 = \sum \cos A + 2\sum \cos \frac{A+B}{2} - 2\sum \sin \frac{A}{2} - 1.$$

From identities $\sum \cos A = \dfrac{R+r}{R}$ and $\sum \cos \dfrac{A+B}{2} = \sum \sin \dfrac{A}{2}$ it follows (25).
Also we have

$$\sum \cos \frac{A}{2}\left(\sum \sin \frac{A}{2} - 1\right) = \sum \cos \frac{A}{2} \sin \frac{A}{2} + \sum \cos \frac{A}{2} \sin \frac{B}{2} - \sum \cos \frac{A}{2} =$$
$$= \frac{1}{2}\sum \sin A + \sum \sin \frac{B+C}{2} - \sum \cos \frac{A}{2}$$

and since $\sin \dfrac{B+C}{2} = \cos \dfrac{A}{2}$ with analogs and $\sum \sin A = \dfrac{s}{R}$, it follows (26).
If we denote with $f = \sum \sin \dfrac{A}{2}$ and $g = \sum \cos \dfrac{A}{2}$, from (25) and (26) we

obtain the system $\begin{cases} g^2-(f+1)^2 &= \dfrac{r}{R} \\ g(f-1) &= \dfrac{s}{2R}. \end{cases}$

Substituting g from the second equation, after perform some calculations we get

$$(27)\qquad \begin{cases} f^4+\left(\dfrac{r}{R}-2\right)f^2-\dfrac{2r}{R}f+1+\dfrac{r}{R}-\dfrac{s^2}{4R^2}=0 \\ g=\dfrac{s}{2R}\cdot\dfrac{1}{f-1}. \end{cases}$$

From the relationship $g=\dfrac{s}{2R}\cdot\dfrac{1}{f-1}$ it follows that $f>1$ since $g>0$.
Let be the function $F:(1,\infty)\to\mathbb{R}$ defined by

$$F(t)=t^4+\left(\frac{r}{R}-2\right)t^2-\frac{2r}{R}t+1+\frac{r}{R}-\frac{s^2}{4R^2}\,,\ t\in(1,\infty).$$

We have $F'(t)=4t^3+2\left(\dfrac{r}{R}-2\right)t-\dfrac{2r}{R}=(t-1)\left(4t^2+4t+\dfrac{2r}{R}\right)>0$, since $t>1$, so F is increasing on $(1,\infty)$. We have $\lim\limits_{\substack{t\to1\\t>1}}F(t)=-\dfrac{s^2}{4R^2}<0$ and $\lim\limits_{t\to\infty}F(t)=+\infty$, it follows that the equation $F(t)=0$ has a unique solution on $(1,\infty)$. From (27) we have that $F(f)=0$, so f represent the unique root of the equation $F(t)=0$.

In the following we keep R and r constant.

Remark 2.1. Taking into account the formulas for solving the 4^{th} degree equation, it turns out that the 4^{th} equation with unknown f from (27) depends on the variable s. The second relationship from (27) shows that g also depends on s.
Taking Remark 2.1 into account, we define the functions $f,g:[s_1,s_2]\to\mathbb{R}$, $f(s)=\sum\sin\dfrac{A}{2}$, $g(s)=\sum\cos\dfrac{A}{2}$, which are the unique solutions of system from Lemma 2.4.

Theorem 2.1. *The functions from above are strictly increasing on* $[s_1,s_2]$.

Proof. From Lemma 2.4 we have $g^2-(f+1)^2=\dfrac{r}{R}$ and $2Rg(f-1)=s$.
After differentiate we obtain $gg'=f'(f+1)$ and $g'(f-1)+gf'=\dfrac{1}{2R}$, or

$$(28)\qquad f'\left(\frac{f^2-1}{g}+g\right)=\frac{1}{2R}\,.$$

From (27) we have that $g(s)>0$ and $f(s)>1$, $s\in[s_1,s_2]$.
According with (28), it follows that $f'(s)>0$, $s\in[s_1,s_2]$, so f is strictly increasing function. But since $g\cdot g'=f'(f+1)$ and $g(s),f(s),\ f'(s)>0$, $s\in[s_1,s_2]$, it follows that $g'(s)>0$, $s\in[s_1,s_2]$, so g is strictly increasing function.

In the following we consider the functions

$$u, v : [s_1, s_2] \to \mathbb{R},\ \ u(s) = \sum \frac{1}{\sin\frac{A}{2}},\ \ v(s) = \sum \frac{1}{\cos\frac{A}{2}}.$$

Theorem 2.2. *The function u is strictly increasing on $[s_1, s_2]$ and the function v is strictly decreasing on $[s_1, s_2]$.*

Proof. For $s \in [s_1, s_2]$ we have

$$f^2(s) = \left(\sum \sin\frac{A}{2}\right)^2 = \sum \sin^2\frac{A}{2} + 2\sum \sin\frac{A}{2}\sin\frac{B}{2} = $$
$$= \sum \frac{1-\cos A}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum\frac{1}{\sin\frac{A}{2}},$$

so $f^2(s) = \dfrac{2R-r}{2R} + \dfrac{r}{2R}u(s)$. Since f is strictly increasing on $[s_1, s_2]$, it follows that u is also strictly increasing on $[s_1, s_2]$.

In the same way $g^2(s) = \sum \cos^2\dfrac{A}{2} + 2\sum \cos\dfrac{A}{2}\cos\dfrac{B}{2} = \sum\dfrac{1+\cos A}{2} + 2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}v(s)$, so $g^2(s) = \dfrac{4R+r}{2R} + \dfrac{s}{2R}v(s)$.

From (25) we have $g^2(s) = (f(s)+1)^2 + \dfrac{r}{R}$ and then, the equality from above becomes $(f(s)+1)^2 + \dfrac{r}{R} = \dfrac{4R+r}{2R} + \dfrac{s}{2R}v(s)$, from where

$s \cdot v(s) = 2R\big(f(s)+1\big)^2 + r - 4R,\ s \in [s_1, s_2]$.

After differentiate we obtain $sv'(s) + v(s) = 4R(f(s)+1)f'(s)$, or

$sv'(s) = 4R\big(f(s)+1\big)f'(s) - v(s),\ s \in [s_1, s_2]$.

Substituting $v(s)$ in the above relation, we get

$$(29)\quad s \cdot v'(s) = 4R\big(f(s)+1\big)f'(s) - \frac{2R\big(f(s)+1\big)^2}{s} + \frac{4R-r}{s},\ s \in [s_1, s_2].$$

Since the sine function is concave, we have

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le 3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3},$$

from where $f(s) \le 3\sin\dfrac{\pi}{6} = \dfrac{3}{2} < 2$, so $f(s) < 2,\ s \in [s_1, s_2]$.

On the other hand, from relations (26) we have that $f(s) > 1,\ s \in [s_1, s_2]$.

Taking (28) and that $f(s) - 1 > 0,\ s \in [s_1, s_2]$ into account, we have

$\dfrac{1}{2R}g(s) = f'(s)\big(f^2(s) - 1 + g^2(s)\big) > f'(s) \cdot g^2(s),\ s \in [s_1, s_2]$, or

$f'(s) < \dfrac{1}{2R}\cdot\dfrac{1}{g(s)},\ s \in [s_1, s_2]$. Also using the relationship (26), we have $f'(s) < \dfrac{1}{2R}\cdot\dfrac{1}{g(s)} = \dfrac{1}{2R}\cdot\dfrac{2R}{s}\big(f(s)-1\big)$, so $f'(s) < \dfrac{1}{s}\big(f(s)-1\big),\ s \in [s_1, s_2]$.

Taking the above inequalities into account, we obtain

$$s\cdot v'(s)=4R\big(f(s)+1\big)f'(s)-\frac{2R\big(f(s)+1\big)^2}{s}+\frac{4R-r}{s}<$$
$$<4R\big(f(s)+1\big)\frac{1}{s}\big(f(s)-1\big)-\frac{2R\big(f(s)+1\big)^2}{s}+\frac{4R-r}{s}=$$
$$=\frac{2Rf(s)\big(f(s)-2\big)-2R-r}{s}<0,\ s\in[s_1,s_2].$$

Then $v'(s)<0$, $s\in[s_1,s_2]$, so v is strictly decreasing function on $[s_1,s_2]$.

Remark 2.2. It is immediately seen that

$$f(s_1)=\sin\frac{A_1}{2}+\sin\frac{B_1}{2}+\sin\frac{C_1}{2}=f_1(R,r),$$
$$g(s_1)=\cos\frac{A_1}{2}+\cos\frac{B_1}{2}+\cos\frac{C_1}{2}=g_1(R,r)$$

and analogue.

Taking into account Lemma 2.1–Lemma 2.3, Theorem 1.2, Theorem 2.1 and Theorem 2.2, we give the following inequalities verified by $\sum\sin\frac{A}{2}$, $\sum\cos\frac{A}{2}$, $\sum\frac{1}{\sin\frac{A}{2}}$, $\sum\frac{1}{\cos\frac{A}{2}}$. For the equalities see Theorem 1.2.

Theorem 2.3. *In every ABC triangle are true the inequalities*

$$f_1(R,r)\le\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le f_2(R,r)\tag{30}$$

and

$$g_1(R,r)\le\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le g_2(R,r).\tag{31}$$

Theorem 2.4. *In every ABC acute or right triangle are true the inequalities*

$$f_1(R,r)\le\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le f_2(R,r)\ \textit{if}\ 2\le\frac{R}{r}\le\sqrt{2}+1,\tag{32}$$

$$f_3(R,r)\le\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le f_2(R,r)\ \textit{if}\ \frac{R}{r}\ge\sqrt{2}+1,\tag{33}$$

$$g_1(R,r)\le\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le g_2(R,r)\ \textit{if}\ 2\le\frac{R}{r}\le\sqrt{2}+1\tag{34}$$

and

$$g_3(R,r)\le\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le g_2(R,r)\ \textit{if}\ \frac{R}{r}\ge\sqrt{2}+1.\tag{35}$$

Theorem 2.5. *In every ABC triangle are true the inequalities*

$$u_1(R,r)\le\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\le u_2(R,r)\tag{36}$$

and

$$v_2(R,r)\le\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\le v_1(R,r).\tag{37}$$

Theorem 2.6. *In every ABC acute or right triangle are true the inequalities*

$$u_1(R,r) \leq \frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} \leq u_2(R,r) \quad \text{if} \quad 2 \leq \frac{R}{r} \leq \sqrt{2}+1,$$

$$u_3(R,r) \leq \frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} \leq u_2(R,r) \quad \text{if} \quad \frac{R}{r} \geq \sqrt{2}+1,$$

$$v_2(R,r) \leq \frac{1}{\cos \frac{A}{2}} + \frac{1}{\cos \frac{B}{2}} + \frac{1}{\cos \frac{C}{2}} \leq v_1(R,r) \quad \text{if} \quad 2 \leq \frac{R}{r} \leq \sqrt{2}+1$$

and

$$v_2(R,r) \leq \frac{1}{\cos \frac{A}{2}} + \frac{1}{\cos \frac{B}{2}} + \frac{1}{\cos \frac{C}{2}} \leq v_3(R,r) \quad \text{if} \quad \frac{R}{r} \geq \sqrt{2}+1.$$

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