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SYMMEDIAN LOCUS CONFIGURATION PART II

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Abstract. This paper is a continuation of our paper entitled "*Symmedian locus Configuration, Part I*". Assume that $\mathcal{C}$ is a circle with center O and radius r. Fix a point H with $r < OH < 3r$. For the $(\mathcal{C}, H)$-locus, there exists a unique $(\overline{\mathcal{C}}, \overline{H})$-locus satisfying the properties: (1) $\overrightarrow{OH}$ and $\overrightarrow{O\overline{H}}$ are in opposite direction, (2) the $(\mathcal{C}, H)$- and $(\overline{\mathcal{C}}, \overline{H})$-loci have the same symmedian circle, i.e. $\mathcal{C}_H = \overline{\mathcal{C}}_{\overline{H}}$, and (3) $\mathcal{C} \cap \mathcal{C}_H \cap \overline{\mathcal{C}} = \{P, Q\}$. We also define the concept of a degenerate triangle inscribed in a circle and show that its orthocenter and symmedian point are well-defined. The above intersection points P and Q turn out to be the symmedian points of two pairs of degenerate triangles, one pair from the $(\mathcal{C}, H)$-locus and the other from the $(\overline{\mathcal{C}}, \overline{H})$-locus.

1. Introduction

The current paper is a continuation of [7]. Part I established the following results: let $\mathcal{C}$ be a circle with center O and radius r. Fix a point H such that the distance $0 < OH < 3r$. Set $h = OH$. Theorem 2.3 proves that for each $A \in \mathcal{C}$ such that $AH < 2r$ and $2AH^2 \neq h^2 - r^2$, there is a unique $\triangle ABC$ inscribed in $\mathcal{C}$ with orthocenter H. This set $\mathcal{T}_H$ of triangles is called the $(\mathcal{C}, H)$-locus. Theorem 3.1 proves that the locus of symmedian points of the $(\mathcal{C}, H)$-locus lies on the circle $\mathcal{C}_H$ with radius

$$r_H = \tfrac{2r^2h^2}{9r^2-h^2}$$

and center the point O_H defined by the vector equation

$$\overrightarrow{OO_H} = \tfrac{6r^2}{9r^2-h^2}\overrightarrow{OH}.$$

Included in Theorem 3.1 is the following vector description of a symmedian point K of a triangle $\triangle ABC$ in the $(\mathcal{C}, H)$-locus for $A \neq H$:

$$\overrightarrow{OK} = \tfrac{8r^2-2AH^2}{9r^2-h^2}\overrightarrow{OA} + \tfrac{2r^2(r^2-h^2+2AH^2)}{(9r^2-h^2)AH^2}\overrightarrow{AH}.$$

As A varies, the vectors $\overrightarrow{OA}$ and $\overrightarrow{AH}$ form a moving frame of the canonical vector space $\mathcal{V}$ of geometric vectors. Part of the strategy in proving Theorem 3.1 uses this moving frame.

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Next, Theorem 4.1 addresses the inverse problem. Explicitly, given $K \in \mathcal{C}_H$ such that $OK < r$, we prove that there is a unique cubic polynomial $X^3 + pX + q$ with three real roots that pick out a triangle $\triangle ABC$ in the $(\mathcal{C}, H)$-locus with symmedian point K. Theorem 4.2 proves that the triangle is unique.

The final result of Part I, Theorem 5.1, addresses the symmedian arc $\mathcal{A}_H$ of a $(\mathcal{C}, H)$-locus. Theorem 5.1 proves that $\mathcal{A}_H$ is the entire symmedian circle $\mathcal{C}_H$ when $0 < OH < r$ and is the open arc of $\mathcal{C}_H$ lying inside $\mathcal{C}$ when $r \le OH < 3r$.

Part II focusses on the case where $r < h < 3r$. In this range, the symmedian locus $\mathcal{A}_H$ is not a circle. The *Complementary Circle Theorem*, Theorem 2.1, proves that the complementary arc of $\mathcal{A}_H$ has a natural realization as the symmedian locus of a complementary circle with respect to a fixed orthocenter. More explicitly, let Σ denote the set of all circumcircle-orthocenter pairs $(\mathcal{C}, H)$ where $\mathcal{C}$ is a circle with center O, radius r, and H is a point such that $r < OH < 3r$. Theorem 2.1 shows that for each $(\mathcal{C}, H) \in \Sigma$, there is a unique $(\overline{\mathcal{C}}, \overline{H}) \in \Sigma$ such that the two symmedian circles guaranteed by Theorem 3.1 coincide; the vectors $\overrightarrow{OH}$ and $\overrightarrow{\overline{O}\,\overline{H}}$ are opposite in direction; and the three circles $\mathcal{C}, \overline{\mathcal{C}},$ and $\mathcal{C}_H = \overline{\mathcal{C}}_{\overline{H}}$ intersect in two points $\{P, Q\}$ and are coaxal. Corollary 2.1 goes on to prove that the common symmedian circle is the disjoint union of the two symmedian arcs along with $\{P, Q\}$:

$$\mathcal{C}_H = \mathcal{A}_H \cup \mathcal{A}_{\overline{H}} \cup \{P, Q\}.$$

Section §3 studies the geometric meaning of the endpoints $\{P, Q\}$ of a symmedian arc $\mathcal{A}_H$.

2. Complementary Circle Theorem

2.1. **Circumcircle-Orthocenter Pairs.** Let $\mathcal{C}$ be a circle with center O and radius r; let H be a point such that $r < OH < 3r$. We call $(\mathcal{C}, H)$ a *circumcircle-orthocenter pair.* Let Σ denote the set of all such pairs.

2.2. **Main Theorem.**

Theorem 2.1. *For each $(\mathcal{C}, H) \in \Sigma$, there exists a unique $(\overline{\mathcal{C}}, \overline{H}) \in \Sigma$ with the following properties:*

(1) *The vectors $\overrightarrow{OH}$ and $\overrightarrow{\overline{O}\,\overline{H}}$ are opposite in direction.*
(2) *The two pairs $(\mathcal{C}, H)$ and $(\overline{\mathcal{C}}, \overline{H})$ have the same symmedian circle, i.e. $\mathcal{C}_H = \overline{\mathcal{C}}_{\overline{H}}$.*
(3) *The three circles $\mathcal{C}, \overline{\mathcal{C}},$ and $\mathcal{C}_H$ intersect at two points. Hence the three circles are coaxal.*

Proof. The objective is to produce the points $\overline{O}$ and $\overline{H}$. Once accomplished, everything is determined. See (2.6) and Lemma 2.1 ahead. The unique complementary pair $(\overline{\mathcal{C}}, \overline{H}) \in \Sigma$ is then described by (2.16).

Let $h = OH$ and $\overline{h} = \overline{O}\,\overline{H}$; let r and $\overline{r}$ be the radii of $\mathcal{C}$ and $\overline{\mathcal{C}}$, resp. If the two circumcircle-orthocenter pairs are to have the same symmedian circle, then $O_{\overline{H}} = O_H$

so that by Theorem 3.1 in [7]

$$\overrightarrow{OO_H} = \frac{6r^2}{9r^2-h^2}\overrightarrow{OH}, \qquad \overrightarrow{\overline{O}O_H} = \frac{6\overline{r}^2}{9\overline{r}^2-\overline{h}^2}\overrightarrow{\overline{O}\,\overline{H}} \tag{2.1}$$

$$\frac{2rh^2}{9r^2-h^2} = \frac{2\overline{r}\overline{h}^2}{9\overline{r}^2-\overline{h}^2} \qquad \text{—the radius of } \overline{\mathcal{C}}_H = \mathcal{C}_{\overline{H}}. \tag{2.2}$$

By (2.1),

$$OO_H = \frac{6r^2h}{9r^2-h^2}, \qquad \overline{O}O_H = \frac{6\overline{r}^2\overline{h}}{9\overline{r}^2-\overline{h}^2}.$$

The vectors $\overrightarrow{OO_H}$ and $\overrightarrow{\overline{O}O_H}$ are opposite in direction. Since $\overrightarrow{OO_H}$ and $\overrightarrow{OH}$ are in the same direction and since the vectors $\overrightarrow{\overline{O}O_H}$ and $\overrightarrow{\overline{O}\,\overline{H}}$ are also in the same direction, the vectors $\overrightarrow{OO_H}$ and $\overrightarrow{\overline{O}O_H}$ are opposite in direction. Hence,

$$\overrightarrow{\overline{O}\,\overline{H}} = -\frac{\overline{h}}{h}\overrightarrow{OH}, \tag{2.3}$$

$$\overrightarrow{\overline{O}O_H} = -\frac{\overline{r}^2\overline{h}(9r^2-h^2)}{r^2h(9\overline{r}^2-\overline{h}^2)}\,\overrightarrow{OO_H}. \tag{2.4}$$

Finally, for use ahead in the proof, let

$$\mathbf{u} = \frac{\overrightarrow{OO_H}}{OO_H}, \qquad \overline{\mathbf{u}} = \frac{\overrightarrow{\overline{O}O_H}}{\overline{O}O_H}.$$

Of course, $\overline{\mathbf{u}} = -\mathbf{u}$. Then for any point P

$$\mathrm{Proj}_{\overline{\mathbf{u}}}(\overrightarrow{O_HP}) = -\mathrm{Proj}_{\mathbf{u}}(\overrightarrow{O_HP}).$$

On the other hand, by definition the vector projections

$$\mathrm{Proj}_{\mathbf{u}}(\overrightarrow{O_HP}) = \left(\frac{\overrightarrow{O_HP}\cdot\overrightarrow{OO_H}}{OO_H}\right)\mathbf{u},$$

$$\mathrm{Proj}_{\overline{\mathbf{u}}}(\overrightarrow{O_HP}) = \left(\frac{\overrightarrow{O_HP}\cdot\overrightarrow{\overline{O}O_H}}{\overline{O}O_H}\right)\overline{\mathbf{u}}.$$

Hence

$$\frac{\overrightarrow{O_HP}\cdot\overrightarrow{\overline{O}O_H}}{\overline{O}O_H} = -\frac{\overrightarrow{O_HP}\cdot\overrightarrow{OO_H}}{OO_H}. \tag{2.5}$$

By Lemma 3.1 ahead in section §3.3, the circles $\mathcal{C}$ and $\mathcal{C}_H$ intersect at two points P and Q that lie on the nine point circle of the $(\mathcal{C}, H)$-locus. Also, the points P and Q are symmetric about the line l_{OH}. So in order for the circles $\mathcal{C}, \mathcal{C}_H$, and $\overline{\mathcal{C}}$ to intersect at two points, $\overline{O}P = \overline{r}$ is a necessary condition. To analyze this condition, we compare OP^2 and $\overline{O}P^2$, to wit,

$$\overrightarrow{OP} = \overrightarrow{OO_H} + \overrightarrow{O_HP}, \qquad \overrightarrow{\overline{O}P} = \overrightarrow{\overline{O}O_H} + \overrightarrow{O_HP}.$$

Using the algebraic properties of the dot product together with the distances $OP = r$, $OO_H = \frac{6r^2h}{9r^2-h^2}$, $\overline{O}O_H = \frac{6\overline{r}^2\overline{h}}{9\overline{r}^2-\overline{h}^2}$, and $O_HP = \frac{2rh^2}{9r^2-h^2}$, we get

$$\begin{aligned}\overrightarrow{O_HP}\cdot\overrightarrow{OO_H} &= \frac{OP^2-O_HP^2-OO_H^2}{2}\\ &= \frac{3r^2(27r^4-18r^2h^2-h^4)}{2(9r^2-h^2)^2}.\end{aligned}$$

Similarly,

$$\overrightarrow{O_H P}\cdot\overrightarrow{\overline{O}O_H}=\tfrac{3\overline{r}^2(27\overline{r}^4-18\overline{r}^2\overline{h}^2-\overline{h}^4)}{2(9\overline{r}^2-\overline{h}^2)^2}.$$

Hence, equation (2.5) is equivalent to

$$\tfrac{27r^4-18r^2h^2-h^4}{h(9r^2-h^2)}=-\tfrac{27\overline{r}^4-18\overline{r}^2\overline{h}^2-\overline{h}^4}{\overline{h}(9\overline{r}^2-\overline{h}^2)}.$$

Conclusion: The two pairs $(\mathcal{C},H)$ and $(\overline{\mathcal{C}},\overline{H})$ satisfy the statement of the theorem iff the system

$$\begin{aligned}\tfrac{27r^4-18r^2h^2-h^4}{h(9r^2-h^2)}&=-\tfrac{27\overline{r}^4-18\overline{r}^2\overline{h}^2-\overline{h}^4}{\overline{h}(9\overline{r}^2-\overline{h}^2)}\\ \tfrac{rh^2}{9r^2-h^2}&=\tfrac{\overline{r}\overline{h}^2}{9\overline{r}^2-\overline{h}^2}\end{aligned}\tag{2.6}$$

has a unique solution for $(\overline{r},\overline{h})$. Lemma 2.1 proves this. $\square$

Lemma 2.1. *To simplify notation, let* $u=\overline{r}$ *and* $v=\overline{h}$. *The system*

$$\tfrac{27r^4-18r^2h^2-h^4}{h(9r^2-h^2)}=-\tfrac{27u^4-18u^2v^2-v^4}{v(9u^2-v^2)}\tag{2.7}$$

$$\tfrac{rh^2}{9r^2-h^2}=\tfrac{uv^2}{9u^2-v^2}\tag{2.8}$$

is equivalent to the cubic equation

$$X^3+pX+q=0\tag{2.9}$$

where

$$p=\tfrac{2(r^2-h^2)}{3h^2},\qquad q=\tfrac{(7r^2+h^2)(r^2-h^2)}{27rh^3},\tag{2.10}$$

and

$$X=\tfrac{u}{v}-\tfrac{r}{3h}\qquad\Longleftrightarrow\qquad\tfrac{u}{v}=X+\tfrac{r}{3h}.$$

The cubic equation has the unique real solution

$$\tfrac{u}{v}=X+\tfrac{r}{3h}=\tfrac{r\kappa}{3h},$$

where

$$\kappa=\left(1+\sqrt[3]{\frac{h^2-r^2}{r^2}}\right)^2.\tag{2.11}$$

Proof. Rewrite (2.8) by $\frac{9u^2-v^2}{uv^2}=\frac{9r^2-h^2}{rh^2}$ to get

$$\tfrac{v^4+18u^2v^2-27u^4}{uv^3}=\tfrac{27r^4-18r^2h^2-h^4}{rh^3}.\tag{2.12}$$

Rearranging (2.12) leads to

$$27\Big[\big(\tfrac{u}{v}\big)^3+\big(\tfrac{r}{h}\big)^3\Big]-18\big(\tfrac{u}{v}+\tfrac{r}{h}\big)-\big(\tfrac{v}{u}+\tfrac{h}{r}\big)=0.$$

Factor out $\frac{u}{v}+\frac{r}{h}$ to get

$$27\Big[\big(\tfrac{u}{v}\big)^2-\tfrac{r}{h}\big(\tfrac{u}{v}\big)+\big(\tfrac{r}{h}\big)^2\Big]-18-\tfrac{h}{r}\cdot\tfrac{v}{u}=0.$$

Multiply by $\frac{u}{v}$ and divide by 27 to get the cubic equation

$$\big(\tfrac{u}{v}\big)^3-\tfrac{r}{h}\big(\tfrac{u}{v}\big)^2+\Big[\big(\tfrac{r}{h}\big)^2-\tfrac{2}{3}\Big]\big(\tfrac{u}{v}\big)-\tfrac{h}{27r}=0.\tag{2.13}$$

Using the substitution $X = \frac{u}{v} - \frac{r}{3h}$, the cubic equation (2.13) becomes

$$X^3 + \frac{2(r^2-h^2)}{3h^2}X + \frac{(7r^2+h^2)(r^2-h^2)}{27rh^3} = 0. \tag{2.14}$$

Let p and q be given by (2.10). Then the discriminant of (2.14) is

$$\begin{aligned}\Delta &= -(4p^3 + 27q^2) \\ &= -\frac{(r^2-h^2)^2(9r^2-h^2)^2}{27r^2h^6} \\ &< 0\,.\end{aligned}$$

Since the discriminant is negative, (2.14) has exactly one real root. This root is

$$\begin{aligned}X &= \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \\ &= \frac{2}{3h}\sqrt[3]{r(h^2-r^2)} + \frac{1}{3h}\sqrt[3]{\frac{(h^2-r^2)^2}{r}}.\end{aligned}$$

In turn,

$$\begin{aligned}\frac{u}{v} &= X + \frac{r}{3h} \\ &= \frac{1}{3h}\left(r + 2\sqrt[3]{r(h^2-r^2)} + \frac{1}{3h}\sqrt[3]{\frac{(h^2-r^2)^2}{r}}\right) \\ &= \frac{r}{3h}\left(1 + \sqrt[3]{\frac{h^2-r^2}{r^2}}\right)^2.\end{aligned}$$

Define

$$\kappa = \left(1 + \sqrt[3]{\frac{h^2-r^2}{r^2}}\right)^2. \tag{2.15}$$

Then

$$\frac{u}{v} = \frac{\kappa r}{3h} \qquad \Longleftrightarrow \qquad u = \frac{\kappa r}{3h}v. \qquad \square$$

Remark 2.1. Using Lemma 2.1, since $\frac{uv^2}{9u^2-v^2} = \frac{rh^2}{9r^2-h^2}$, see (2.8),

$$u = \frac{r(\kappa^2r^2-h^2)}{9r^2-h^2}, \qquad v = \frac{3h(\kappa^2r^2-h^2)}{\kappa(9r^2-h^2)}.$$

In summary, given $(\mathcal{C}, H) \in \Sigma$, the complementary pair $(\overline{\mathcal{C}}, \overline{H})$ is described as follows:

$$\begin{aligned}\kappa &= \left(1 + \sqrt[3]{\frac{h^2-r^2}{r^2}}\right)^2, \\ \overline{r} &= \frac{r(\kappa^2r^2-h^2)}{9r^2-h^2}, \\ \overline{h} &= \frac{3h(\kappa^2r^2-h^2)}{\kappa(9r^2-h^2)} \\ \overrightarrow{\overline{O}O_H} &= -\frac{\kappa}{3}\overrightarrow{OO_H}, \\ \overrightarrow{\overline{O}\,\overline{H}} &= -\frac{3(\kappa^2r^2-h^2)}{\kappa(9r^2-h^2)}\overrightarrow{OH}.\end{aligned} \tag{2.16}$$

Corollary 2.1. *Given $(\mathcal{C}, H) \in \Sigma$, let $(\overline{\mathcal{C}}, \overline{H})$ be the complementary pair. Let*

$$\{P, Q\} = \mathcal{C} \cap \overline{\mathcal{C}} \cap \mathcal{C}_H.$$

Let $\mathcal{A}_H$ and $\mathcal{A}_{\overline{H}}$ be the symmedian arcs of $\mathcal{C}$ and $\overline{\mathcal{C}}$, resp. The endpoints of both arcs are the points P and Q. Finally,

$$\mathcal{C}_H = \mathcal{A}_H \cup \mathcal{A}_{\overline{H}} \cup \{P, Q\}$$

is a disjoint union.

Proof. By Corollary 3.1 in section §3.3 ahead, the points P and Q are the endpoints of $\mathcal{A}_H$ and $\mathcal{A}_{\overline{H}}$. On the other hand, by Theorem 3.1 in [7] the arc $\mathcal{A}_H$ lies inside the circle $\mathcal{C}$, while $\mathcal{A}_{\overline{H}}$ lies inside $\overline{\mathcal{C}}$. By Theorem 4.1 in [7], $\mathcal{C} \backslash (\mathcal{A}_H \cup \{P, Q\}) = \mathcal{A}_{\overline{H}}$. □

Corollary 2.2. *Given $(\mathcal{C}, H) \in \Sigma$ and the corresponding pair $(\overline{\mathcal{C}}, \overline{H})$, let $\mathcal{D}, \overline{\mathcal{D}}$, and $\mathcal{N}, \overline{\mathcal{N}}$ denote the D-loci circles and nine point circles of $(\mathcal{C}, H)$ and $(\overline{\mathcal{C}}, \overline{H})$, resp. The seven circles*

$$\mathcal{C},\ \mathcal{N},\ \mathcal{D},\ \overline{\mathcal{C}},\ \overline{\mathcal{N}},\ \overline{\mathcal{D}},\ \mathcal{C}_H$$

are coaxal of intersecting type. The radical axis of the family is the line through the points P and Q where $\mathcal{C}$ and $\overline{\mathcal{C}}$ intesect.

Proof. Bt Theorem 3.4 of [7], the circle $\mathcal{D}$ intersects $\mathcal{C}$ at P and Q. Likewise, $\overline{\mathcal{D}}$ intersects $\overline{\mathcal{C}}$ at P and Q. Hence the result. See Chapter 8, pages 194 and 202 (Discussion 445) of [2]. □

3. The Endpoints of the Symmedian Arc

Assume that $\mathcal{C}$ is a circle with center O and radius r and that H is a point satisfying $r \leq OH < 3r$. The symmedian arc $\mathcal{A}_H$ associated to the $(\mathcal{C}, H)$-locus $\mathcal{T}_H$ has two endpoints P and Q; namely, $\mathcal{C}_H \cap \mathcal{C} = \{P, Q\}$. Notice that $P = Q$ when H lies on $\mathcal{C}$. The points $E, F \in \mathcal{C}$ with $HE = HF = 2r$ completely determine P and Q. In this section we study the geometric meaning of the endpoints of the symmedian arc $\mathcal{A}_H$.

3.1. **Degenerate Triangles.** Given a nondegenerate chord $\overline{AB}$ of a circle $\mathcal{C}$ with center O and radius r, denote the tangent line to $\mathcal{C}$ at B by l_{BB}. We will call the pair $(\overline{AB}, l_{BB})$ a **degenerate triangle** inscribed in $\mathcal{C}$, written $\triangle ABB$. The point B is called the **double point** of $\triangle ABB$ and the tangent line l_{BB} is regarded as the side opposite to vertex A.

Note: The notation l_{BB} for the tangent line is deliberate. Its use is intended to emphasize that tangency at B determines a unique line.

The centroid G of a degenerate triangle $\triangle ABB$ with double point B inscribed in a circle $\mathcal{C}$ with center O is defined following (2.3) by the vector equation

$$3\overrightarrow{OG} = \overrightarrow{OA} + 2\overrightarrow{OB}.$$

Let H be the point defined by the vector equation

$$\overrightarrow{OH} = 3\overrightarrow{OG} = \overrightarrow{OA} + 2\overrightarrow{OB}.$$

Note that

$$\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = (\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OA} = 2\overrightarrow{OB}.$$

Then $\overline{AH}$ and $\overline{OB}$ are parallel to each other and $AH = 2OB = 2r$. Since $\overline{OB}$ is perpendicular to l_{BB}, $\overline{AH}$ is also perpendicular to l_{BB}. On the other hand, we have

$$\begin{aligned}\overrightarrow{BH} \cdot \overrightarrow{AB} &= (\overrightarrow{OH} - \overrightarrow{OB}) \cdot (\overrightarrow{OB} - \overrightarrow{OA}) \\ &= [(\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OB}] \cdot (\overrightarrow{OB} - \overrightarrow{OA}) \\ &= (\overrightarrow{OB} + \overrightarrow{OA}) \cdot (\overrightarrow{OB} - \overrightarrow{OA}) \\ &= OB^2 - OA^2 \\ &= 0.\end{aligned}$$

Then $\overrightarrow{BH}$ is perpendicular to $\overline{AB}$. So H is the orthocenter of $\triangle ABB$.

The orthocenter H satisfies

$$\begin{aligned}OH^2 &= \overrightarrow{OH} \cdot \overrightarrow{OH} \\ &= (\overrightarrow{OA} + 2\overrightarrow{OB}) \cdot (\overrightarrow{OA} + 2\overrightarrow{OB}) \\ &= OA^2 + 4\overrightarrow{OA} \cdot \overrightarrow{OB} + 4OB^2 \\ &= 5r^2 + 4\overrightarrow{OA} \cdot \overrightarrow{OB} \\ &\geq 5r^2 - 4OA \cdot OB \\ &= 5r^2 - 4r^2 \\ &= r^2\end{aligned}$$

and the equality holds if and only if $\overrightarrow{OA} \cdot \overrightarrow{OB} = -OA \cdot OB$, if and only if $\overrightarrow{OA} = -\overrightarrow{OB}$ by the Cauchy-Schwartz inequality, if and only if $\overline{AB}$ is a diameter of $\mathcal{C}$.

A degenerate triangle $\triangle ABB$ of a circle $\mathcal{C}$ has a well defined symmedian point, namely, the double point B. To justify this definition, the triangle $\triangle ABB$ has two natural angles at B, namely $\angle ABE$ and $\angle ABE'$ as shown in the figure. The median of $\triangle ABB$ at vertex B is the ray $\overrightarrow{BA}$. The reflection of this ray across the angle bisectors $\overrightarrow{BD}$ of $\angle ABE$ and $\overrightarrow{BD'}$ of $\angle ABE'$ are $\overrightarrow{BE}$ and $\overrightarrow{BE'}$, resp. On the other hand, the angle at vertex A of $\triangle ABB$ is 0 so that the ray $\overrightarrow{AB}$ is the symmedian ray at vertex A. Since the point B is the only point that is common to $\overrightarrow{BE}, \overrightarrow{BE'}$, and $\overrightarrow{AB}$, B is the symmedian point following the isogonality definition of the symmedian point; see Figure 1.

Finally, the vector form of the midpoint characterization of the symmedian point also extends to $\triangle ABB$. Here, $B = M = S = D$. Moreover, since $AH = 2r$, $5r^2 + AH^2 - h^2 = 9r^2 - h^2$. Hence by Theorem 2.4 in [7]

$$\overrightarrow{OK} = \tfrac{1}{2}\overrightarrow{AH} = \overrightarrow{OB}.$$

3.2. **Nine Point Circle of a Degenerate Triangle.** The well known properties of the orthocenter of a nondegenerate triangle also hold for degenerate triangles. These properties are summarized in Theorem 3.1 just ahead. Degenerate triangles arise naturally in the description of the symmedian arc. A degenerate triangle $\triangle ABB$ has a nine point circle. If N is the midpoint of O and H, then the nine point circle of $\triangle ABB$ is the circle $\mathcal{N}$ with center N and radius $\frac{r}{2}$.

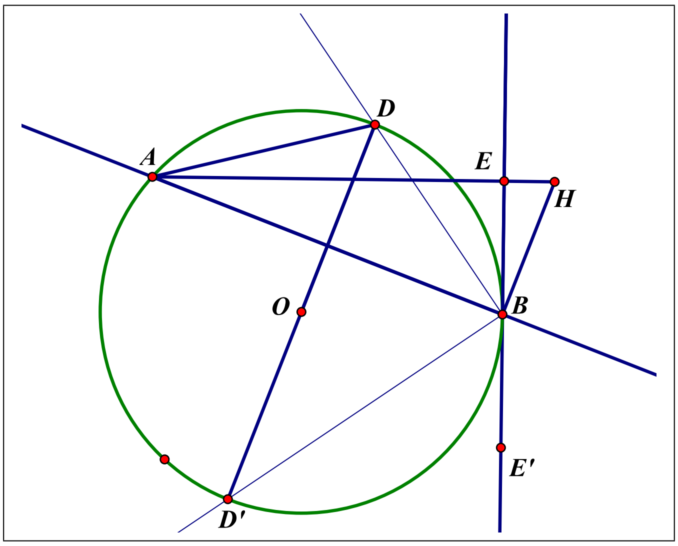

FIGURE 1

Theorem 3.1. *Given a degenerate triangle $\triangle ABB$ inscribed in a circle $\mathcal{C}$ of radius r with center O and orthocenter H, let A' be the second point where the altitude at vertex A intersects $\mathcal{C}$; let A_h be the intersection of l_{BB} and l_{AH}. Then A' is the reflection of H across the line l_{BB}. Let A'' be the reflection of H across the line l_{AB}. Then $\overline{AA''}$ is a diameter of the circle $\mathcal{C}$.*

Let N be the midpoint of O and H; let $\mathcal{N}$ be the circle with center N and radius $\frac{r}{2}$. Let E_a (resp. E_b) be the midpoint of A and H (resp. the midpoint of B and H); let M_b be the midpoint of A and B. The circle $\mathcal{N}$ contains the six points A_h, B (counted twice), M_b, E_a, and E_b. The segments $\overline{BE_a}$ and $\overline{M_bE_b}$ are diameters of $\mathcal{N}$; see Figure 2.

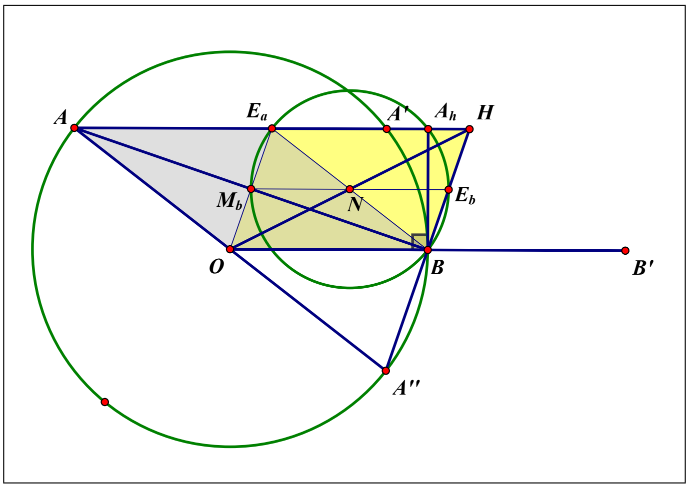

FIGURE 2

Proof. As observed just above, $\overrightarrow{AB} \perp \overrightarrow{BH}$. On the other hand, let A_h be the intersection of l_{BB} and l_{AH}. Then $\overrightarrow{AH} = 2\overrightarrow{OB}$ implies

$$\overrightarrow{BA_h} \cdot \overrightarrow{AH} = 2\overrightarrow{BA_h} \cdot \overrightarrow{OB} = 0$$

so that $l_{BB} \perp l_{AH}$. So H is the point where the altitudes at vertices A and B meet. Next, let $\overrightarrow{OB'} = 2\overrightarrow{OB}$. Then $\overrightarrow{OB'} = \overrightarrow{AH}$ so that $\square OAHB'$ is a parallelogram. Hence $B'H = OA$.

Next, since reflection across a line is an isometry, the reflection of H across l_{BB}, call it A' lies on the circle $\mathcal{C}$, that is, B' reflects to O so that $r = B'H = OA'$.

Let A'' denote the reflection of H across the line l_{AB}. Then $A, B \in \mathcal{C}$ and $\angle A''BA = \angle HBA = 90$ imply $A'' \in \mathcal{C}$ so that $\overline{AA''}$ is a diameter of $\mathcal{C}$.

Finally, the quadrilaterals $\square OE_aHB$ and $\square OAE_bB$ are parallelograms. From this, it follows that $A_h, B, M_b, E_a, E_c \in \mathcal{N}$ and also that $\overline{BE_a}$ and $\overline{M_bE_b}$ are diameters. $\square$

3.3. The Geometry of the Endpoints of $\mathcal{A}_H$.

Lemma 3.1. *Let $\mathcal{C}$ be a circle with center O and radius r and let H be a point with $r \le OH < 3r$. Set $h = OH$ and write $\mathcal{N}$ for the nine-point circle associated to the $(\mathcal{C}, H)$-locus.*

1. *If $h > r$, then $\mathcal{C} \cap \mathcal{N} \cap \mathcal{C}_H = \{P, Q\}$, where P and Q are two distinct points on $\mathcal{C}$.*
2. *If $h = r$, then $\mathcal{C} \cap \mathcal{N} \cap \mathcal{C}_H = \{H\}$.*

Proof. Let N be the center of $\mathcal{N}$, i.e. the midpoint of O and H. Note that $\mathcal{C}_H \cap \mathcal{C} = \{P, Q\}$ and that $\mathcal{N}$ and $\mathcal{C}$ are coaxal; see page 201 of [2]. It suffices to check that $P \in \mathcal{N}$ and $Q \in \mathcal{N}$.

Suppose first that $r < h < 3r$. Then the circles $\mathcal{N}$ and $\mathcal{C}$ intersect at two distinct points by the Two-Circles Theorem, see Theorem 13.4 on page 112 of [3]. Note that $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OH}$, $\overrightarrow{OO_H} = \frac{6r^2}{9r^2-h^2}\overrightarrow{OH}$, and

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \overrightarrow{OP} - \tfrac{9r^2-h^2}{12r^2}\overrightarrow{OO_H}.$$

Also note that $\overrightarrow{OP} \cdot \overrightarrow{OO_H} = \frac{OP^2+OO_H^2-O_HP^2}{2}$ by the Law of Cosines. Then

$$\begin{aligned}
NP^2 &= \left(\overrightarrow{OP} - \tfrac{9r^2-h^2}{12r^2}\overrightarrow{OO_H}\right) \cdot \left(\overrightarrow{OP} - \tfrac{9r^2-h^2}{12r^2}\overrightarrow{OO_H}\right) \\
&= OP^2 - \tfrac{9r^2-h^2}{6r^2}\overrightarrow{OP} \cdot \overrightarrow{OO_H} + \left(\tfrac{9r^2-h^2}{12r^2}\right)^2 OO_H^2 \\
&= OP^2 - \tfrac{9r^2-h^2}{6r^2} \cdot \tfrac{OP^2+OO_H^2-O_HP^2}{2} + \left(\tfrac{9r^2-h^2}{12r^2}\right)^2 OO_H^2 \\
&= r^2 - \tfrac{9r^2-h^2}{6r^2} \cdot \frac{r^2+\left(\frac{6r^2h}{9r^2-h^2}\right)^2-\left(\frac{2rh^2}{9r^2-h^2}\right)^2}{2} + \left(\tfrac{9r^2-h^2}{12r^2}\right)^2\left(\tfrac{6r^2h}{9r^2-h^2}\right)^2 \\
&= r^2 - \tfrac{3r^2+h^2}{4} + \tfrac{h^2}{4} \\
&= \tfrac{r^2}{4}.
\end{aligned}$$

Since $NP = \frac{r}{2}$, the point P lies on the circle $\mathcal{N}$. Similarly, one shows that $Q \in \mathcal{N}$. When $h = r$, the circles $\mathcal{C}$ and $\mathcal{N}$ are tangent at H. So $\mathcal{C} \cap \mathcal{N} \cap \mathcal{C}_H = \{H\}$. $\square$

Theorem 3.2. *Let $\mathcal{C}$ be a circle with center O and radius r and let H be a point with $r \le OH < 3r$. Set $h = OH$.*

(1) *If $h > r$, then there are exactly two degenerate triangles $\triangle FPP$ and $\triangle EQQ$ inscribed in $\mathcal{C}$ with orthocenter H, where E and F are the intersection points of the circles $\mathcal{C}$ and $\mathcal{C}(H;2r)$ such that F,P are on the opposite side of l_{OH} and E,Q are on the opposite side of l_{OH}. In this case, the symmedian points of $\triangle FPP$ and $\triangle EQQ$ are P and Q, respectively; see Figure 3.*

(2) *If $h = r$, then there is a unique degenerate triangle $\triangle EHH$ inscribed in $\mathcal{C}$ with orthocenter H, where $\overline{EH}$ is the diameter of $\mathcal{C}$ with one endpoint H. In this case, the symmedian point of $\triangle EHH$ is H.*

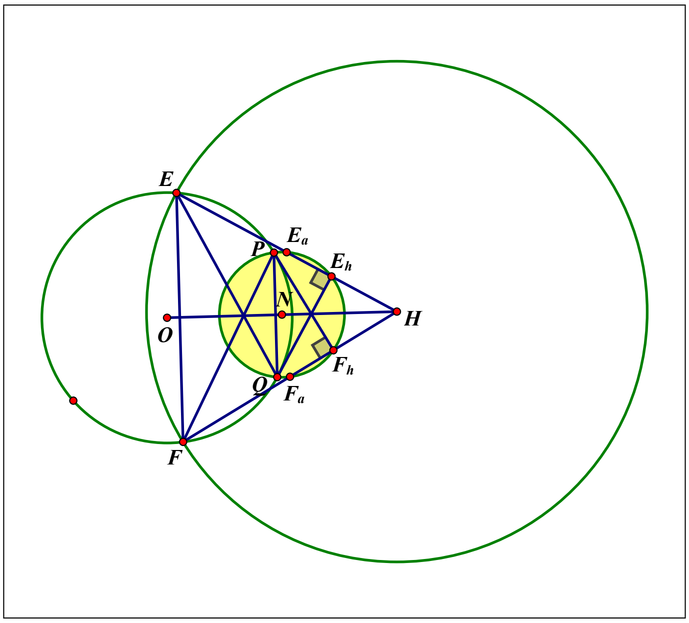

FIGURE 3

Proof. Define the points P' and Q' by the vector equations

$$2\overrightarrow{OP'} = \overrightarrow{FH} \quad \text{and} \quad 2\overrightarrow{OQ'} = \overrightarrow{EH}.$$

Since $OP' = \frac{1}{2}FH = r$ and $OQ' = \frac{1}{2}EH = r$, we get $P', Q' \in \mathcal{C}$. Since $\triangle FP'P'$ and $\triangle EQ'Q'$ are degenerate triangles inscribed in $\mathcal{C}$ with orthocenter H, we get $P', Q' \in \mathcal{N}$, where $\mathcal{N}$ is the nine-point circle associated to the $(\mathcal{C}, H)$-locus. So $P', Q' \in \mathcal{C} \cap \mathcal{N}$. Note that $\mathcal{C} \cap \mathcal{N} = \{P, Q\}$ by Lemma 3.1. The points F, P' are on the opposite side of l_{OH}, while E and Q' are also on the opposite side of l_{OH}. Hence, $P' = P$ and $Q' = Q$. So $\triangle FPP$ and $\triangle EQQ$ are degenerate triangles inscribed in $\mathcal{C}$ with orthocenter H.

Next, if $\triangle ABB$ is a degenerate triangle inscribed in $\mathcal{C}$ with orthocenter H, then either it is $\triangle FPP$ or $\triangle EQQ$. Indeed, since $\overrightarrow{OA} + 2\overrightarrow{OB} = \overrightarrow{OH}$, we get

$$\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = (\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OA} = 2\overrightarrow{OB}$$

which implies $AH = 2OB = 2r$ and that l_{OB} is parallel to l_{AH}. The first conclusion forces $A \in \{E, F\}$ since two circles cannot intersect in more than two points. On the other hand, $A \in \{E, F\}$ forces $B \in \{P, Q\}$. Note that A and B are on the opposite side of l_{OH}. So the degenerate $\triangle ABB$ is either $\triangle FPP$ or $\triangle EQQ$.

The two circles $\mathcal{C}$ and $\mathcal{C}(H; 2r)$ are coaxal as are the circles $\mathcal{N}$ and $\mathcal{C}$; see page 201 of [2]. Moreover, the three centers $O, N,$ and H are collinear. Hence the chords $\overline{EF}$ and $\overline{PQ}$ are parallel so that $\square EFQP$ is an isosceles trapezoid. Consequently, $\triangle EQQ$ is the reflection of $\triangle FPP$ across the line l_{OH}. The above figure illustrates the configuration.

Next, suppose $OH = r$, that is, $H \in \mathcal{C}$. In this case, $\mathcal{C} \cap \mathcal{C}(H; 2r) = \{E\}$. Indeed, the circles are tangent at the point E and the segment $\overline{EH}$ is a diameter of $\mathcal{C}$. We will prove that $\triangle EHH$ is a degenerate triangle inscribed in $\mathcal{C}$ with orthocenter H. Since EH is a diameter of $\mathcal{C}$, we get $\overrightarrow{OE} = -\overrightarrow{OH}$ and then

$$\overrightarrow{OE} + 2\overrightarrow{OH} = -\overrightarrow{OH} + 2\overrightarrow{OH} = \overrightarrow{OH}.$$

So H is the orthocenter of the degenerate triangle $\triangle EHH$.

Next, we prove that if $\triangle ABB$ is a degenerate triangle inscribed in $\mathcal{C}$ with orthocenter H, then it is $\triangle EHH$. Since $\overrightarrow{OA} + 2\overrightarrow{OB} = \overrightarrow{OH}$, we get

$$\overrightarrow{AH} = \overrightarrow{OH} - \overrightarrow{OA} = (\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OA} = 2\overrightarrow{OB}$$

which implies $AH = 2OB = 2r$. Since both A and H lie on the circle $\mathcal{C}$, $\overline{AH}$ is a diameter of $\mathcal{C}$. The equation $2\overrightarrow{OB} = \overrightarrow{AH}$ and the fact $A \neq B$ force $B = H$. So the degenerate triangle $\triangle ABB$ is $\triangle EHH$. ☐

The following corollary is a consequence of the proof of Theorem 3.2.

Corollary 3.1. *The endpoints of the symmedian arc are exactly the double points of the degenerate triangles inscribed in $\mathcal{C}$ with orthocenter H, where $r \leq OH < 3r$.*

Proof. By Theorem 3.2, each endpoint of the symmedian arc $\mathcal{A}_H$ associated to the $(\mathcal{C}, H)$-locus is the double point of some degenerate triangle inscribed in $\mathcal{C}$ with orthocenter H and the double point is its symmedian point by definition. This proves the corollary. ☐

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