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LIGHTLIKE SUBMERSIONS OF INDEFINITE L.C.A COSYMPLECTIC MANIFOLDS

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Abstract. The purpose of this paper, is the study of the generalized Cauchy-Riemannian submersions from the indefinite almost locally conformal cosymplectic manifold onto semi-Riemannian manifolds. A proper smallest-dimensional generalized Cauchy-Riemannian indefinite a.l.c cosymplectic manifold is thirteen dimensional; a relevant example is reported. The tangent bundle's structure has been established using the horizontal lift.

1. Introduction

The non-trivial intersection of the tangent bundle and its normal bundle is what makes studying lightlike submanifolds challenging ([1], [11]). The machinery that Bejancu and Duggal so skillfully described in their book [8] is what makes the transversal bundle exist; without this conception, it would be difficult to demonstrate its existence. Physics, such as general relativity, astrophysics, and other fields, are of relevance to this construction. The screen distribution, the radical distribution, the co-screen distribution, and the lightlike transversal distribution are the four distributions that roughly correspond to a given submanifold M. Consequently, a decomposition of the tangent bundle of $\overline{M}$ results from their respective bundles, that is,

$$T\overline{M}|_M = S(TM) \perp S(TM^{\perp}) \perp (RadTM \oplus ltr(TM)). \quad (1)$$

The useful Gauss and Weingartein equations are the focus of this decomposition, where $S(TM)$, $S(TM^{\perp})$, $RadTM$, $ltr(TM)$ are respectively the screen distribution, Co-screen distribution, radical distribution and lightlike transversal distribution. Immersions and submersions play a crucial role in Riemannian geometry, in particular when the related manifolds have additional structure see [12], [17] and references therein for more details. The Riemannian submersions are specific tools in differential geometry.

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idea of Riemannian submersion dates back to the 1950s, when B. O'Neil [16] and A. Gray [9], separately created the core of this theory, which has since undergone significant development. Several unfinished projects are still ongoing today, see [14] for instance. Let M and B be Riemannian manifolds. A Riemannian submersion $f : M \to B$ is a mapping of M onto B satisfying the following

a. f has maximal rank.
b. f_* preserves the lengths of horizontal vectors.

For each $b \in B$, $f^{-1}(b)$ is a submanifold of M of dimension $\dim M - \dim B$. The submanifolds $f^{-1}(b)$ are called the fibers and a vector field on M is vertical if it is always tangent to the fibers, horizontal if it's always orthogonal to the fibers see [17], [18] and references therein for more details.
We examine the so-called screen lightlike submersions, a lightlike variant of Riemannian submersion, in this paper [7]. The inheritance nature from total space to base space was hightlighted with a focus on indefinite almost locally conformal cosymplectic structures (see [6], [13]). The concepts of vertical, horizontal, and complete lift were taken into consideration in order to emphasize submersions on tangent bundles as one of the applications of submersions (see [5], [10] and references therein for more details). The framework of the article is as follows. In Section 2 the definition of indefinite almost locally conformal cosymplectic manifolds and the equivalent definition in terms of the Levi-Civita connection and the Lee vector field are given. In Section 3, the Gauss and Weingartein formulas are applied to the screen, co-screen, radical and transversal distributions, then we obtain a global decomposition of ambiant manifolds in terms of these distributions. In Section 4, generalized Cauchy-Riemannian (GCR) lightlike submanifolds of an almost l.c. cosymplectic are studied, and an example is given to support the result. Note that the least dimension of a proper GCR is 13. Lightlike submersions are defined and according to the null dimension, types of submersions have been defined, that is, r-lightlike, isotropic and co-isotropic submersion, in Section 5. The last Section revolves around the notion of lifting. Vertical, horizontal and complete lifts are studied in order to use horizontal lifts of an indefinite a.l.c. cosymplectic manifold for obtaining a locally conformal Kählerian structure on the tangent bundle, which is thought of as an even-dimensional manifold.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)-$dimensional manifold endowed with an almost contact structure $(\bar{\phi}, \xi, \eta)$ where $\bar{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field called structure vector field or Reeb vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \overline{\phi} = 0 \text{ and } \overline{\phi}\xi = 0. \tag{2}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector fields $\overline{X}, \overline{Y}$ on $\overline{M}$

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}),\ \overline{g}(\overline{\phi X}, \overline{\phi Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}). \tag{3}$$

The fundamental 2-form $\overline{\Phi}$ of $\overline{M}$ is defined by

$$\overline{\Phi}(\overline{X},\overline{Y}) = \overline{g}(\overline{X},\overline{\phi}\,\overline{Y}), \tag{4}$$

for any $\overline{X},\overline{Y}$ vector fields on $\overline{M}$.
$\overline{M}$ is said to be locally conformal almost cosymplectic if there is an open covering $\{U_i\}$ endowed with smooth functions

$$\sigma_i : U_i \longrightarrow \mathbb{R}$$

such that over each U_i the almost contact metric structure given by

$$\phi_i = \phi,\ \xi_i = e^{\sigma_i}\xi,\ \eta_i = e^{-\sigma_i}\eta,\ \overline{g}_i = e^{-2\sigma_i}\overline{g} \tag{5}$$

is cosymplectic that is $d\Phi_i = 0,\ d\eta_i = 0$. The Lee form is given by $\omega = d\sigma_i$, its dual $\omega^\sharp = B$ is called the Lee vector field. It is globally defined in $\overline{M}$ by $\omega(\overline{X}) = \overline{g}(\overline{X}, B)$ and locally on U_i by $B = grad\ \sigma_i$. The vector field $V = -\phi B$ is called an anti-Lee vector field, its dual denoted by θ is the anti-Lee form meanly, $\theta(\overline{X}) = \overline{g}(\overline{X}, V)$. By straightforward calculations one gets

$$d\overline{\Phi} = 2\omega \wedge \overline{\Phi},\ \ d\eta = \omega \wedge \eta,\ \ d\omega = 0. \tag{6}$$

It is also known that the following relation

$$\overline{\nabla}^i_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} - \omega(\overline{X})\overline{Y} - \omega(\overline{Y})\overline{X} + g(\overline{X},\overline{Y})B \tag{7}$$

where $\overline{\nabla}^i$ is the restriction of $\overline{\nabla}$ in U_i and $\overline{X},\overline{Y}$ vector fields on $\overline{M}$.
As in the Riemannian case, the relation (6) remains valid also in indefinite metric. The contact metric structure $(\overline{\phi},\xi,\eta,\overline{g})$ is said to be normal if the Nijenhuis tensor

$$N^{(1)}_{\overline{\phi}} = [\overline{\phi},\overline{\phi}] + d\eta \otimes \xi$$

of $\overline{\phi}$ vanishes (see [4]), that is, $[\overline{\phi},\overline{\phi}](\overline{X},\overline{Y}) + 2d\eta(\overline{X},\overline{Y})\xi = 0$, where

$$[\overline{\phi},\overline{\phi}](\overline{X},\overline{Y}) = [\overline{\phi}\,\overline{X},\overline{\phi}\,\overline{Y}] - \overline{\phi}[\overline{\phi}\,\overline{X},\overline{Y}] - \overline{\phi}[\overline{X},\overline{\phi}\,\overline{Y}] + \overline{\phi}^2[\overline{X},\overline{Y}]. \tag{8}$$

And in this case, the locally conformal almost cosymplectic manifold is said to be the locally conformal cosymplectic manifold.
In terms of the Levi-Civita connection $\overline{\nabla}$ of the indefinite metric $\overline{g}$, we have:

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \omega(\overline{\phi}\,\overline{Y})\overline{X} - \omega(\overline{Y})\overline{\phi}\,\overline{X} - \overline{g}(\overline{X},\overline{\phi}\,\overline{Y})B + \overline{g}(\overline{X},\overline{Y})\overline{\phi}B, \tag{9}$$

for any vector fields $\overline{X},\overline{Y}$ on $\overline{M}$. Without sacrificing generality, ξ is taken to be unit-spacelike, which means, $\overline{g}(\xi,\xi) = 1$.
Let's say that $b = \overline{g}(B,B) \in \mathcal{C}^\infty(\overline{M})$ and $\text{Sign}(B) = \{x \in \overline{M} : B_x = 0\}$, b and $\text{Sing}(B)$ defines the causal character of B, hence, it is possible that $b = 0$ and $\text{Sing}(B) = \emptyset$ when B is lightlike. The second equation of (3) implies $\overline{g}(V,V) = b - \omega(\xi)^2$. We can therefore establish the causal nature of V based on the values of b and $\omega(\xi)$. Putting $\lambda = \omega(\xi)$ then V is spacelike if $b \geq \lambda^2$, timelike if $b \leq \lambda^2$ and lightlike if $b = \lambda^2$.
The last case yields to

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = f\{\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}\}. \tag{10}$$

This equation can be found in [15], with $\omega = f\eta$ where f is a function such that $\mathrm{d}f \wedge \eta = 0$. In (9) and (10), ξ is simply substituted, yielding the following results

$$\overline{\nabla}_{\overline{X}}\xi = \omega(\xi)\overline{X} - \eta(\overline{X})B \tag{11}$$

and

$$\overline{\nabla}_{\overline{X}}\xi = -f(-\overline{X} + \eta(\overline{X})\xi), \tag{12}$$

which implies $\overline{\nabla}_{\xi}\xi = 0$ for the equation (12). Consequently, if B is colinear to ξ then B or ξ is a Killing vector field. Let's introduce a (1,1)-tensor field h on $\overline{M}$ taking

$$\overline{h}\,\overline{X} = \overline{\nabla}_{\overline{X}}\xi - \omega(\xi)\overline{X} + \eta(\overline{X})B \tag{13}$$

and on each U_i

$$\exp{(-\sigma_i)}\overline{\nabla}^{i}_{\overline{X}}\xi_i = \overline{h}\,\overline{X}. \tag{14}$$

We notice that if $\overline{M}$ is l.c. cosymplectic then $\overline{h} = 0$. The (1,1)-tensor $\overline{h}$ has the following properties

$$\overline{h}\,\overline{\phi} + \overline{\phi}\,\overline{h} = 0,\ \overline{h}\xi = 0,\ tr\overline{h} = 0 \,\text{and}\, \overline{g}(\overline{h}\,\overline{X}, \overline{Y}) = \overline{g}(\overline{h}\,\overline{Y}, \overline{X}). \tag{15}$$

3. Submanifold of indefinite l.c. Cosymplectic manifold

Assume that $(\overline{M}, \overline{g})$ is a real $(m+n)$-dimensional semi-Riemannian manifold, $\overline{g}$ is a semi-Riemannian metric of constant index $q \in \{1, ..., m+n-1\}$ on $\overline{M}$, and M is a m-dimensional submanifold. In [8] it is known that for a lightlike submanifold M there exists smooth distributions, namely, the radical $Rad\,T_pM$, the screen $S(TM)$, the transversal $tr(TM)$, the lightlike transversal $ltr(TM)$ and the co-screen $S(TM^{\perp})$, where

$$Rad\,T_pM = T_pM \cap T_pM^{\perp} \neq \{0\}, \forall p \in M.$$

Hence, the following decompositions hold:

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}) \tag{16}$$

$$\begin{aligned} T\overline{M}|_M &= TM \oplus tr(TM) \\ &= S(TM) \perp S(TM^{\perp}) \perp (RadTM \oplus ltr(TM)). \end{aligned} \tag{17}$$

We establish four distinct types of submanifolds r-lightlike, co-isotropic, isotropic, and totally lightlike manifolds—based on the ranks of each of these distributions. For further information, see [7]. We shall take into account the local quasi-orthonormal frame of $\overline{M}$ along M:

$$\{E_1, ..., E_r,\ N_1, ..., N_r,\ X_{r+1}, ..., X_m,\ W_{r+1}, .., W_n\} \tag{18}$$

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite almost contact manifold $(\overline{M}, \overline{g})$. We put

$$\overline{\phi}X = PX + FX,\ \forall\, X \in \Gamma(TM), \tag{19}$$

$$\overline{\phi}W = BW + CW,\ \forall\, W \in \Gamma(ltr(TM), \tag{20}$$

where $\{PX, BW\}$ and $\{FX, CW\}$ are the tangential and transversal parts, respectively. Moreover, P is skew-symmetric on $S(TM)$. The Gauss and the Weingartein formulas lead us to:

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{21}$$

$$\overline{\nabla}_X V = -A_V X + D^l_X V + D^s_X V, . \tag{22}$$

We recollect certain relationships indicated in [6, ch 5] using the Otsuki connections,

$$\overline{\nabla}_X V = -A_V X + \nabla^l_X(LV) + \nabla^s_X(SV) + D^l_X(LV) + D^s_X(LV). \tag{23}$$

In particular from (23), one has

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{24}$$

$$\overline{\nabla}_X W = -A_W X + D^l(X,W) + \nabla^s_X W, \tag{25}$$

where l and s are projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively, $D^l(X,W)$ and $D^s(X,N)$ are the projection of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively.
By using the inner product, one gets

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y) \tag{26}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{27}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of the tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X PY + h^*(X, \overline{P}Y), \tag{28}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E \tag{29}$$

for $X, Y \in \Gamma(TM)$ and $E \in \Gamma(Rad(TM))$, where h^*, A^* are the second fundamental form and the shape operator of distributions $S(TM)$ and $Rad(TM)$, respectively. Applying the inner product yields the following:

$$\begin{aligned} &\overline{g}(h^l(X, \overline{P}Y), E) = g(A^*_E X, \overline{P}Y), \\ &\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \\ &\overline{g}(h^l(X,E),E) = 0, \;\; A^*_E E = 0. \end{aligned} \tag{30}$$

In general the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (21), we obtain

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^s(X,Z),Y). \tag{31}$$

Lemma 3.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of an indefinite almost locally conformal cosymplectic manifold $(\overline{M}, \overline{g})$, with ξ tangent to M. If h^l and h^s are parallel and the Lee vector field B is tangent to M, then M is totally geodesic if and only if B is colinear to ξ.*

Proof. Indeed, from (11), (21) and (23), we obtain $\nabla_X \xi = \omega(\xi)X - \eta(X)B_T$ and $h^l(X,\xi) + h^s(X,\xi) = 0$ that implies, $h^\alpha(X,\xi) = 0$ for any $\alpha \in \{l, s\}$, therefore $0 = (\nabla_X h^\alpha)(Y,\xi) = -\omega(\xi)h^\alpha(Y,X) + \eta(X)h^\alpha(Y,B_T)$. This completes the proof.

Therefore, using the quasi-orthonormal frame and without assuming that B is tangent to M, $h^{\alpha}(X, xi)$ may be expressed as

$$h^l(X,\xi) = \sum_{i=1}^{r} \eta(X)\omega(E_i)N_i \,\text{and}\, h^s(X,\xi) = \sum_{i=r+1}^{m} \eta(X)\omega(W_i)W_i. \tag{32}$$

Definition 3.1. A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,

$$h(X,Y) = Hg(X,Y). \tag{33}$$

It is obvious that there exists $H^l, H^s \in \Gamma(ltr(TM))$ such that

$$h^l(X,Y) = H^l g(X,Y),\ h^s(X,Y) = H^s g(X,Y),\ D^l(X,W) = 0 \tag{34}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$.

Definition 3.2. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if

(i) $h^s = 0$ on $Rad(TM)$;
(ii) $trace\, h = 0$, where trace is with respect to g restricted to $S(TM)$.

Definition 3.3. If $Rad(TM)$ and $S(TM)$ are respectively invariant and anti-invariant with respect to the tensor ϕ, then the lightlike submanifold M of the indefinite almost l.c. cosymplectic manifold $\overline{M}$ is a screen real submanifold.

4. Generalized Cauchy-Riemannian (GCR) Lightlike submanifolds

Definition 4.1. [6] With ξ tangent to M, let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite almost l.c. cosymplectic manifold $(\overline{M}, \overline{g})$. If the following criteria are met, M is referred to as a generalized Cauchy-Riemannian lightlike submanifold of $\overline{M}$:

(I) There exists two subbundles D_1 and D_2 of $Rad(TM)$ on M such that

$$Rad\,TM = D_1 \oplus D_2,\ \phi D_1 = D_1,\ \phi(D_2) \subset S(TM). \tag{35}$$

(II) There exists two vector subbundles D_0 and D' of $S(TM)$ such that over M

$$S(TM) = \{\phi(D_2) \oplus D'\} \perp D_0 \perp \{\xi\},\ \phi D_0 = D_0,\ \phi(D') = L_1 \perp L_2 \tag{36}$$

where D_0 is the nondegenerate, L_1 and L_2 are vector subbundles of $S(TM^{\perp})$ and $ltr(TM)$, respectively.

The result is the decomposition shown below:

$$TM = D \oplus D' \perp \{\xi\},\ D = Rad\,TM \perp \phi(D_2) \perp D_0 \tag{37}$$

A contact GCR-lightlike is said to be proper if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $L_1 \neq \{0\}$. Thus, from the Definition 4.1, we have:

(i) Condition (I) implies that $\dim(Rad(TM)) \geq 3$.

(ii) Condition (II) implies that $\dim(D) \geq 2s \geq 6$ and $\dim(D_2) = \dim(L_2)$. Thus $\dim(M) \geq 9$ and $\dim\overline{M} \geq 13$.
(iii) Any proper 9-dimensional contact GCR-lightlike submanifold is 3-lightlike.
(iv) (i) and contact distribution ($\eta = 0$) imply that $\mathrm{index}(\overline{M}) \geq 4$.

Example 4.1. Let $\overline{M} = (\mathrm{R}^{13}_4, \overline{g})$ be a semi-euclidian space, where $\overline{g}$ is of signature $(-,-,+,+,+,+,--,+,+,+,+,+)$ with respect to the canonical basis

$$(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z) \tag{38}$$

and $x_1 z \geq 0$.

Consider the case when $\sigma(p) = \ln x_1 z$, where $\omega = \dfrac{1}{x_1}\mathrm{d}x_1 + \dfrac{1}{z}\mathrm{d}z$ implies $\mathrm{d}\eta = \omega \wedge \eta$. Since $\Phi = e^{2\sigma}\mathrm{d}x_1 \wedge \mathrm{d}y_1$ is the only non-zero component of the 2-fundamental form, one can obtain $\mathrm{d}\Phi = 2\omega \wedge \Phi$ by employing differentiation on both sides. Take into account the submanifold M of R^{13}_4, defined by

(39)

$$x^4 = x^1 \cos\theta - y^1 \sin\theta,\ y^4 = x^1 \sin\theta + y^1 \cos\theta,\ x^2 = y^3,\ (x^5)^2 = 1 + (y^5)^2, y^5 \neq 1.$$

Then a local frame of TM is given by

$$Z_1 = e^{-2\sigma}(\partial x^1 + \cos\theta \partial x^4 + \sin\theta \partial y^4),\ Z_2 = e^{-2\sigma}(-\sin\theta \partial x^4 + \partial y^1 + \cos\theta \partial y^4),$$

$$Z_3 = e^{-2\sigma}(\partial x^2 + \partial x^3),\ Z^4 = e^{-2\sigma}(\partial x^3 - \partial y^2),\ Z^5 = e^{-2\sigma}(\partial x^6,\ Z_6 = \partial y_6),$$

(40)

$$Z_7 = e^{-2\sigma}(y^5 \partial x^5 + x^5 \partial y^5),\ Z_8 = e^{-2\sigma}(\partial x^3 + \partial y^2),\ Z_9 = \xi = \frac{1}{x^1 z}\partial z.$$

Hence, $\mathrm{Rad}TM = \mathrm{span}\{Z_1, Z_2, Z_3\}$. Moreover $\phi Z_1 = e^{2\sigma} Z_2$ and $\phi Z_3 = e^{-2\sigma} Z_4 \in \Gamma(S(TM))$. Thus $D_1 = \mathrm{span}\{Z_1, Z_2\}$, $D_2 = \mathrm{span}\{Z_3\}$. Hence, (I) holds. Next, $\phi Z_5 = -e^{2\sigma} Z_6$, which implies that $D_0 = \mathrm{span}\{Z_5, Z_6\}$ is invariant with respect to ϕ. By direct calculations, we get

$$S(TM^{\perp}) = \mathrm{span}\{W = x^5 \partial x^5 - y^5 \partial y^5\}$$

such that $\phi(W) = -e^{2\sigma} Z_7$. Hence $L_1 = S(TM^{\perp})$ and

$$ltr(TM) = \mathrm{span}\{N_1, N_2, N_3\}$$

where $N_1 = e^{-2z}(-\partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4), N_2 = e^{-2z}(-\sin\theta \partial x_4 - \partial y_1 + \cos\theta \partial y_4), N_3 = e^{-4z}(-\partial x_2 + \partial y_3)$ such that $\phi N_1 = -e^{-2\sigma} N_2$ and $\phi N_3 = -e^{2\sigma} Z_8$.

5. Lightlike submersions

Further details for this section see [17]. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) a r-lightlike manifold, i.e., (M_2, g_2) the nullity degree of g_2 which is the rank of the radical subspace $\mathrm{Rad}\, T_x M$ of $T_x M$ defined by

$$\mathrm{Rad}\, T_p M = \{V \in T_p M : g(V, X) = 0,\ X \in T_p M\}. \tag{41}$$

We think about a smooth submersion. If $f : M_1 \longrightarrow M_2$, it is known that $f^{-1}(p)$ is a submanifold of dimension $\dim M_1 - \dim M_2$ for any $p \in M_2$. Given that M_1 is a semi-Riemannian manifold, we will then express the differential of f by f_*. According to the notion of submersion, the sum

$T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p$ is not orthogonal because we can have (or not) a non-zero intersection between $\ker f_*$ and $(\ker f_*)^\perp$. We may find four occurrences of submersions by comparing the dimension r with regard to one of the kernels of f_* and $(\ker f_*)^\perp$. Denote the intersection by $\Delta = \ker f_* \cap (\ker f_*)^\perp$

(1) $r = \dim\Delta < \min\{\dim(\ker f_*), \dim(\ker f_*)^\perp\}$, then $\mathcal{V} = \ker f_*$ and $\mathcal{H} = tr(\ker f_*) = ltr(\ker f_*) \perp S(\ker f_*)^\perp$. And f is said to be r-lightlike submersion.

(2) $r = \dim(\ker f_*) < \dim(\ker f_*)^\perp$. Then $\mathcal{V} = \Delta$ and $\mathcal{H} = S(\ker f_*)^\perp \perp ltr(\ker f_*)$. And f is an isotropic submersion.

(3) $r = \dim(\ker f_*)^\perp < \dim(\ker f_*)$. Then $\mathcal{V} = S(\ker f_*) \perp \Delta$ and $\mathcal{H} = ltr(\ker f_*)$. And f is a co-isotropic submersion.

(4) $r = \dim(\ker f_*)^\perp = \dim(\ker f_*)$. Then $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(\ker f_*)$. We call f a totally lightlike submersion.

Let's assume now that M and M' are endowed with contact structures, respectively (ϕ, ξ, η, g) and (ϕ', ξ', η', g'). Define the submersion f that satisfies

$$\text{(a) } f_*\xi = \xi', \tag{42}$$

$$\text{(b) } f_* \circ \phi = \phi' \circ f_*. \tag{43}$$

Theorem 5.1. *Define $f : M_1 \longrightarrow M_2$ to be a smooth submersion that satisfies (43), $(M_1, \phi, \xi, \eta, g)$ to be an indefinite almost contact manifold, and M_2 to be a lightlike submanifold of an indefinite almost contact manifold. Then, we have*

(a) *If f is a r-lightlike (or isotropic) submersion then, either $\xi \in \mathcal{V}$ and M_2 is of even dimensional or ξ has no component along $ltr(\ker f_*)$ and M_2 is of odd dimensional.*

(b) *If f is a co-isotropic submersion then, either $\xi \in \mathcal{V}$ and M_2 is of even dimensional or ξ is transversal (i.e., it belongs to $\Delta \oplus ltr(\ker f_*)$) and M_2 is of odd dimensional.*

(c) *If f is a totally lightlike submersion then, ξ is transversal and M_2 is of odd dimensional.*

Lemma 5.1. *Let $f : M_1 \longrightarrow M_2$ be a smooth submersion, $(M_1, \phi, \xi, \eta, g)$ an indefinite almost contact manifold and M_2 a lightlike submanifold endowed with an indefinite almost contact structure $(\phi', \xi', \eta', g_2)$, f satisfying (43) if M_1 is a locally conformal cosymplectic manifold, then the Lee vector field B is a horizontal vector field.*

Proof. Indeed, for any U a vertical vector field, we have

$$0 = \mathrm{d}\eta(U, \xi) = \eta \wedge \omega(U, \xi) = -\frac{1}{2}\omega(U)$$

which completes the proof.

Theorem 5.2. *Let $f : M_1 \longrightarrow M_2$ be a smooth submersion, $(M_1, \phi, \xi, \eta, g)$ an indefinite almost contact manifold and M_2 a lightlike submanifold of an indefinite almost contact manifold $(M', \phi', \xi', \eta', g')$, f satisfying (43). Then*

(i) *If the structure vector field ξ is vertical and M_1 is endowed with a locally conformal cosymplectic structure then the base space is indefinite locally conformal Kählerian manifold.*

(ii) *If the structure vector field ξ is horizontal and M_1 is locally conformal cosymplectic manifold then, the base space is locally conformal cosymplectic.*

Proof. Form lemma 5.1, the Lee vector field B is horizontal then it is f-related to B', i.e., $df(B) = B'$. Then for any X horizontal vector field and using the definition of submersions, one has $\omega(X) = g_1(X, B) = g_2(dfX, dfB) = g_2(X', B') \circ f = \omega'(X') \circ f$. Since $\mathrm{d}\Phi = 2\omega \wedge \Phi$, then by using (43) we may have $\Phi_1(X, Y) = g_1(X, \phi Y) = g_2(\mathrm{d}fX, \mathrm{d}f\phi Y) \circ f = g_2(\mathrm{d}fX, \phi'\mathrm{d}fY) \circ f = g_2(X', \phi'Y') \circ f = g_2(X', Y') \circ f = \Phi_2(X', Y') \circ f$ that implies $d\Phi_2 = 2\omega' \wedge \Phi_2$ for X and Y horizontal vector fields. This is the proof of the assertion (i). Let us suppose that ξ is a horizontal vector field, since B is horizontal vector field then for any E, F horizontal vector fields f-related to E', F' respectively, we have $2\mathrm{d}\eta_1(E, F) = \omega(E)\eta_1(F) - \omega(F)\eta_1(E) = (\omega'(E')\eta_2(F') - \omega'(F')\eta_2(E')) \circ f = 2\mathrm{d}\eta_2(E', F') \circ f$ that implies $\mathrm{d}\eta_2 = \omega_2 \wedge \eta_2$ and by following the calculations above we have $\mathrm{d}\Phi_2 = 2\omega_2 \wedge \Phi_2$ where $\omega_2(X') = g_2(X, B')$. This proves (ii).

Remark 5.1. *It is known that a basic vector field is an horizontal vector that is projectable, and when M_2 is odd-dimensional, the horizontal lift of ξ' is lying in the co-screen $S(\ker f_*)^{\perp}$ for (a), in lightlike transversal distribution for (b) and (c), hence we may have*

$$f_*\xi = \xi'. \tag{44}$$

And when M_2 is even-dimensional, ξ' is lying in the distribution $S(TM_2)^{\perp} \perp ltr(TM_2)$.

It is obvious that the horizontal distribution has the same structure as the base space. In the next section, we are going to consider the locally conformal cosymplectic structure on the ambiant manifold of M_2, and according to each type of submersion, the vertical distribution might inherit the almost contact structure of M_1.

Definition 5.1. *Let $f : (M_1, \phi, \xi, \eta, g_1) \longrightarrow (M_2, \phi', \xi', \eta', g_2)$ be a smooth submersion, from a contact semi-Riemannian manifold M_1 onto a lightlike almost contact manifold M_2, we call f a CR-submersion if the vertical space $\mathcal{V}$ splits into two subbundles $\mathcal{V}_1$ and $\mathcal{V}_2$ such that*

$$\begin{aligned} \phi\mathcal{V}_1 &\subset \mathcal{V}, \\ \phi\mathcal{V}_2 &\subset \mathcal{H}. \end{aligned} \tag{45}$$

It is easy to see that if $\mathcal{V}_2 = \{0\}$, then the verical and horizontal distributions are invariant with respect to ϕ and if $\mathcal{V}_1 = \{0\}$, then f is said to be anti-invariant submersion with respect to ϕ.

6. Vertical, horizontal and complete lifts

Let M be a semi-Riemannian manifold endowed with an almost contact structure (ϕ, ξ, η, g). It is known that in the triplet (TM, π, M), the map $\pi : TM \longrightarrow M$ is a submersion, and its differential is the smooth map

$d\pi : TTM \longrightarrow TM$. For any $(p, u) \in TM$, the vertical and horizontal subspaces are given, respectively, by the following equations:

(46) $$\mathcal{V}_{(p,u)} = \mathrm{Ker}(d\pi|_{(p,u)}),$$

(47) $$\mathcal{H}_{(p,u)} = \mathrm{Ker}(K_{(p,u)}),$$

where $K_{(p,u)} : T_{(p,u)}TM \longrightarrow T_pM$ is the connection map, (for more details see [10]). Hence we may express the following sum

(48) $$T_{(p,u)}TM = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)}.$$

Now, we define a natural metric denoted by $\bar{g}$ in TM with respect to the metric g of M. The natural metrics are the ones that make π a Riemannian submersion while preserving the natural splitting on TTM.
We call natural metric with respect to g, the metric in TM denoted $\bar{g}$ and satisfying

(49) $$\bar{g}(X^h, Y^h) = g(X, Y), \ \bar{g}(X^h, Y^v) = 0.$$

A natural metric is said Sasaki metric and denoted by $\hat{g}$ or g_S if it satisfies the following relations

(50) $$\hat{g}(X^v, Y^v) = g(X, Y) = \hat{g}(X^h, Y^h), \ \hat{g}(X^h, Y^v) = 0.$$

A natural metric is called Cheeger-Gromol if it satisfies

(51) $$\bar{g}_{(p,u)}(X^v, Y^v) = \frac{1}{1 + r^2}(g_p(X, Y) + g_p(X, u) \cdot g_p(Y, u))$$

where $X, Y \in \Gamma(TM)$ and $r : TM \to \mathbb{R}, (p, u) \mapsto r(p, u) = \sqrt{g_p(u, u)}$.
Suppose that M is a lightlike manifold, then it is known that it has a radical subspace defined by

(52) $$\mathrm{Rad}T_xM = \{Y \in T_xM : g(Y, X) = 0, \forall X \in T_xM\}.$$

If TM is equipped with a Sasaki metric $\hat{g}$, then it is obvious that the causal character of a vector field on M induces the one on TM in another world, the horizontal or vertical lift of a vector field has the same causal character as their basic vector field. Hence, in the same way, we may define the radical subspace of TM as

(53)
$$\mathrm{Rad}T_{(p,u)}TM = \{Y = Y^v + Y^h \in T_{(p,u)}TM : g(Y, X) = 0, \forall X \in T_{(p,u)}TM\}.$$

Therefore, there exists a screen distribution in TM denoted by $S(TTM)$ such that TTM may be written as

$$TTM = S(TTM) \perp \mathrm{Rad}TTM.$$

Hence, in view of the decomposition (48) we have

$$S(TTM) = S^{\mathcal{V}}(TTM) \oplus S^{\mathcal{H}}(TTM)$$

and

$$\mathrm{Rad}TTM = \mathrm{Rad}^{\mathcal{V}}TTM \oplus \mathrm{Rad}^{\mathcal{H}}TTM.$$

Suppose TM is equipped with the natural metric $\bar{g}$, the kernel of π_* at (p, u) is the vertical subspace of TTM defined by

(54) $$\mathrm{Ker}\,\pi_* = \{X \in T_{(p,u)}TM : \pi_*(X) = 0\}$$

and its orthogonal is defined as follows:

(55) $$(\mathrm{Ker}\,\pi_*)^{\perp} = \{Y \in T_{(p,u)}TM : \bar{g}(Y,X) = 0, \forall X \in \mathrm{Ker}\pi_*\}.$$

Since TM is a semi-Riemannian vector space, then by taking into account the relation (49) it follows that TM is a lightlike manifold of radical distribution the vertical subspace $(\mathrm{Ker}\,\pi_*)^{\perp}$. Suppose $\Delta = \mathrm{Ker}\pi_* \cap (\mathrm{Ker}\pi_*)^{\perp} \neq 0$. In [8], it is well known that there exists a lightlike transversal bundle $ltr(\mathrm{Ker}\pi_*)$ spanned by lightlike transversal vectors $\{N_i\}_{i\in I}$ such that for any W_i and V_i vector fields in horizontal and vertical subspaces, respectively, we have:

$$\bar{g}(N_i, V_j) = \delta_{ij},\ \bar{g}(N_i, N_j) = 0,\ \bar{g}(N_i, W_j) = 0$$

Lemma 6.1. *Let (M,g) be a lightlike manifold and TM equipped with the natural metric $\bar{g}$ then the submersion $\pi : TM \to M$ is a lightlike submersion in which the vertical subspace is the orthogonal decomposition sum*

$$\mathcal{V} = S(\mathrm{Ker}\pi_*) \perp \Delta$$

and the horizontal subspace

$$\mathcal{H} = ltr(\mathrm{Ker}\pi_*) \perp S(\mathrm{Ker}\pi_*)^{\perp}.$$

Vertical lifts

If f is a function in M, we write f^V for the function in TM obtained, by forming the composition of $\pi : TM \to M$ and $f : M \to \mathbb{R}$, so that

(56) $$f^V = f \circ \pi,$$

any point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

(57) $$f^V(\tilde{p}) = f^V(x,y) = f \circ \pi(\tilde{p}) = f(p) = f(x).$$

We call f^V the vertical lift of the function f. Thus the vertical lift $\tilde{X} \in \mathcal{C}^{\infty}(TM)$ of $X \in \Gamma(TM)$ is such that $\tilde{X}(f^V) = 0$. Therefore, $\tilde{X}$ is vertical if and only if its components $(\tilde{X}^h, \tilde{X}^{\bar{h}})$ in $\pi^{-1}(U)$ satisfy

$$\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{X}^{\bar{h}} \end{bmatrix}$$

and if ω is a 1-form on M and $\tilde{\omega}$ the 1-form on TM the vertical lift is such that $\tilde{\omega}(X^V) = 0$ defined by

(58) $$\omega^V = (\omega_i)^V (\mathrm{d}x^i)^V.$$

In local expression, we may write $\omega = \omega_i \mathrm{d}x^i$, then the component of ω^V is given as

(59) $$\omega^V = (\omega^i, 0)$$

and if F is a tensor of type $(1,1)$ in M its vertical lift is given by

$$F^V = \begin{bmatrix} 0 & 0 \\ F_i^h & 0 \end{bmatrix}.$$

Complete lifts

If f is a smooth function on M, the complete lift f^C of the function on TM is defined by

(60) $$f^C = \iota \mathrm{d}f,$$

its local expression is

$$f^C = y^i \partial_i f = \partial f. \tag{61}$$

For any vector field X the complete lift X^C on TM is given such that

$$X^C(f^C) = (Xf)^C. \tag{62}$$

In terms of local components, we might conclude

$$X^C = \begin{bmatrix} X^h \\ \partial X^h \end{bmatrix}.$$

For any 1-form ω on M the complete lift is defined by

$$\omega^C(X^C) = (\omega(X))^C, \tag{63}$$

then the following is its local formulation given with respect to the induced coordinates in TM:

$$\omega^C : (\partial\omega_i, \omega_i). \tag{64}$$

For any tensor F of type $(1,1)$, the complete lift is given by

$$F^V = \begin{bmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{bmatrix}.$$

Horizontal lifts

The horizontal lift for a function f of M is given by

$$f^H = f^C - \nabla_\gamma f, \tag{65}$$

the horizontal lift of a vector field X of M is given by

$$X^H = X^C - \nabla_\gamma X, \tag{66}$$

in terms of local components induced in TM,

$$X^H = \begin{bmatrix} x^h \\ -\Gamma_i^h x^i \end{bmatrix}$$

where $\Gamma_i^h = y^i \Gamma_{ji}^h$.

For any 1-form ω on M, the horizontal lift ω^H is given by

$$\omega^H = \omega^C - \nabla_\gamma \omega \tag{67}$$

using the induced coordinates in TM the component of ω^H is given by

$$\omega^H : (\Gamma_i^h \omega_h, \omega_i). \tag{68}$$

6.1. **Horizontal lifts of an indefinite a.l.c Cosymplectic manifold.** It is generally known from Theorems 2.1 and 3.1 in [5] that if M has a contact structure, then TM has an almost complex structure. The following lemma follows:

Lemma 6.2. *Let M be a lightlike manifold endowed with an almost contact metric structure (ϕ, ξ, g). The $(1,1)$ tensor metric $\bar{J}$ in TM defined by*

$$\bar{J} = \phi^H + \eta^V \otimes \xi^V - \eta^H \otimes \xi^H \tag{69}$$

is an almost complex structure. Moreover, the natural metric $\bar{g}$ given in (49) is an Hermitian metric on TM.

Proof. Indeed, it is clear to see that $\phi^H(X^H) = \phi(X)$. From (49) we may have

$$\begin{aligned}\bar{g}(\phi^H(X^H), Y^H) &= \bar{g}(\phi(X)^H, Y^H) = g(\phi X, Y) = -g(X, \phi Y)\\ &= -\bar{g}((X^H), \phi(Y)^H) = -\bar{g}((X^H), \phi^H(Y^H))\end{aligned} \tag{70}$$

and using (69) together with the observation

$$\eta^H(\xi^H) = 0,\ \eta^H(\xi^V) = \eta^V(\xi^H) = 1,\ \eta^H(X^H) = \eta^V(X^H) = 0 \tag{71}$$

then we obtain

$$\bar{g}(\bar{J}X^H, \bar{J}Y^H) = g(X, Y)\ \text{and}\ \bar{J}^2 = -I,$$

as required.

Theorem 6.1. *If M is an indefinite almost l.c. cosymplectic manifold, then TM is an almost locally conformal Kählerian manifold by virtue of the structure $(\phi^H, \bar{J}, \bar{g})$.*

Proof. From the equations (69) and (71), we may have

$$\begin{aligned}\bar{g}(X^H, \bar{J}Y^H) &= \bar{g}(X^H, \phi^H Y^H + \eta^V(Y^H)\xi^V - \eta^H(Y^H)\xi^H)\\ &= \bar{g}(X^H, \phi^H Y^H) = \bar{g}(X^H, \phi(Y)^H) = g(X, \phi Y)\\ &= \Phi(X, Y).\end{aligned} \tag{72}$$

It follows then $\Phi^H(X^H, Y^H) = \Phi(X, Y)$, since $\mathrm{d}\Phi = 2\omega \wedge \Phi$. Thus we have

$$\mathrm{d}\Phi^H = 2\omega^H \wedge \Phi^H,$$

as desired.

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