INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 13 (2024), No. 1, 92 - 105

GAUSS' DIVERGENCE THEOREM ON BOUNDED DOMAINS IN MINKOWSKI SPACES WITH APPLICATIONS TO HYPERBOLIC SIMPLICES

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Abstract. For bounded domains of Euclidean spaces with piecewise smooth boundary, the integral of outward unit normal vectors of the boundary is zero. In this paper we consider a similar theorem on Minkowski spaces (Minkowski spaces does not mean finite dimensional Banach spaces but finite dimensional vector spaces with pseudo-inner products). We also consider Gauss' divergence theorem on Minkowski spaces, which implies above. Remark that this theorem implies easily the equation to calculate a kind of centroids of hyperbolic simplices.

1. Introduction

Let K be a bounded domain of Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary ∂K. Then we have the following theorem (it seems to be due to Minkowski, but the author is not sure).

Theorem 1.1 (The integral of unit normal vectors of bounded domains of Euclidean space)**.** *It holds that*

$$\int_{\partial K} d(\boldsymbol{\partial K}) = \mathbf{0},$$

where $d(\boldsymbol{\partial K})$ is Euclidean $(n-1)$-vector element of ∂K (see (3) and Figure 2).

This theorem is shown by substituting constant vectors $\mathbf{e}_0 = (1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}), \ldots, \mathbf{e}_{n-1} = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, 1)$ for $\mathbf{A}$ in the following theorem.

Keywords and phrases: bounded domain, Euclidean space, Gauss' divergence theorem, Minkowski space, pseudo-unit pseudo-normal vector, unit normal vector
(2020)Mathematics Subject Classification: 51M09, 51M10, 53A07, 53A35
Received: 31.08.2023. In revised form: 12.12.2023. Accepted: 20.11.2023.

Theorem 1.2 (Gauss' divergence theorem on Euclidean space (see e.g. [1, §4.9], [3, ch. 13, Thm. 3.2], [5, ch. 2, Thm. 5.11], [10, ch. 9, Problems, 13 (d), p. 352], or [11, ch. 7, G, Addendum 1, 57, p. 132]))**.** *For a vector field* $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$*, it holds that*

$$\int_{\partial K} \mathbf{A} \,.\, d(\boldsymbol{\partial K}) = \int_K \operatorname{div} \mathbf{A} d(\mathbb{R}^n),$$

where $d(\mathbb{R}^n)$ *is the* n*-volume element of Euclidean space* $\mathbb{R}^n$*.*

Theorem 1.1 implies the following corollary, which is useful to define a kind of centroids of spherical simplices $S \subseteq \mathbb{S}^{n-1}$. Let $\mathbf{p}_0^*,\ldots, \mathbf{p}_{n-1}^*$ be vertices of S; let S_k be the opposite facet of $\mathbf{p}_k^*$; and let $\mathbf{p}_0,\ldots, \mathbf{p}_{n-1} \in \mathbb{R}^n$ be such that $\mathbf{p}_k \,.\, \mathbf{p}_k = 1$, $\mathbf{p}_k \,.\, \mathbf{p}_k^* > 0$, $\mathbf{p}_k \,.\, \mathbf{p}_\ell^* = 0$ for all $\ell \neq k$ (see [6] and [7]); i.e.,

$$S = \{\mathbf{x} \in \mathbb{S}^{n-1} : \mathbf{x} \,.\, \mathbf{p}_0 \geq 0, \ldots, \mathbf{x} \,.\, \mathbf{p}_{n-1} \geq 0\}, \quad S_k = \{\mathbf{x} \in S : \mathbf{x} \,.\, \mathbf{p}_k = 0\}.$$

Then the following corollary is shown by simple calculations.

Corollary 1.3 (see e.g. [9, Rems. 1.1 & 1.2])**.** *It holds that*

$$\int_{\mathbf{x} \in S} \mathbf{x} d(\mathbb{S}^{n-1}) = \frac{1}{n-1} \sum_{k=0}^{n-1} |S_k| \mathbf{p}_k,$$

where $d(\mathbb{S}^{n-1})$ *is the* $(n-1)$*-volume element of the unit sphere* $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ *and* $|S_k|$ *is the* $(n-2)$*-volume of induced metric on* S_k *of* $\mathbb{R}^n$*.*

Proof. Let

$$K = \{t\mathbf{x} : \mathbf{x} \in S, 0 \leq t \leq 1\}. \tag{1}$$

Then, ∂K consists of S and

$$K_k = \{t\mathbf{x} : \mathbf{x} \in S_k, 0 \leq t \leq 1\} \quad (\text{for } k = 0, \ldots, n-1), \tag{2}$$

whose $(n-1)$-volume is $|S_k|/(n-1)$ from Lemma 3.3 and whose outward unit normal vector is $-\mathbf{p}_k$ (see Figure 1). So, from Theorem 1.1, we have

$$\int_{\mathbf{x} \in S} \mathbf{x} d(\mathbb{S}^{n-1}) + \sum_{k=0}^{n-1} \frac{|S_k|}{n-1} (-\mathbf{p}_k) = \mathbf{0}. \quad \square$$

Remark 1.1. *A kind of centroids of the spherical simplex* S *can be defined by the both sides of*

$$\frac{\int_{\mathbf{x} \in S} \mathbf{x} d(\mathbb{S}^{n-1})}{\| \int_{\mathbf{x} \in S} \mathbf{x} d(\mathbb{S}^{n-1}) \|} = \frac{\sum_{k=0}^{n-1} |S_k| \mathbf{p}_k}{\| \sum_{k=0}^{n-1} |S_k| \mathbf{p}_k \|}.$$

Euclidean space $\mathbb{R}^n$ (with the inner product) can be considered Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$ (with the pseudo-inner product $\langle \mathbf{x} \,|\, \mathbf{y} \rangle = \iota(\mathbf{x}) \,.\, \mathbf{y}$ where $\iota(x_0, \ldots, x_{n-2}, x_{n-1}) = (x_0, \ldots, x_{n-2}, -x_{n-1})$). So K can be considered a bounded domain of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$ with piecewise smooth boundary ∂K. Then we have the following theorem.

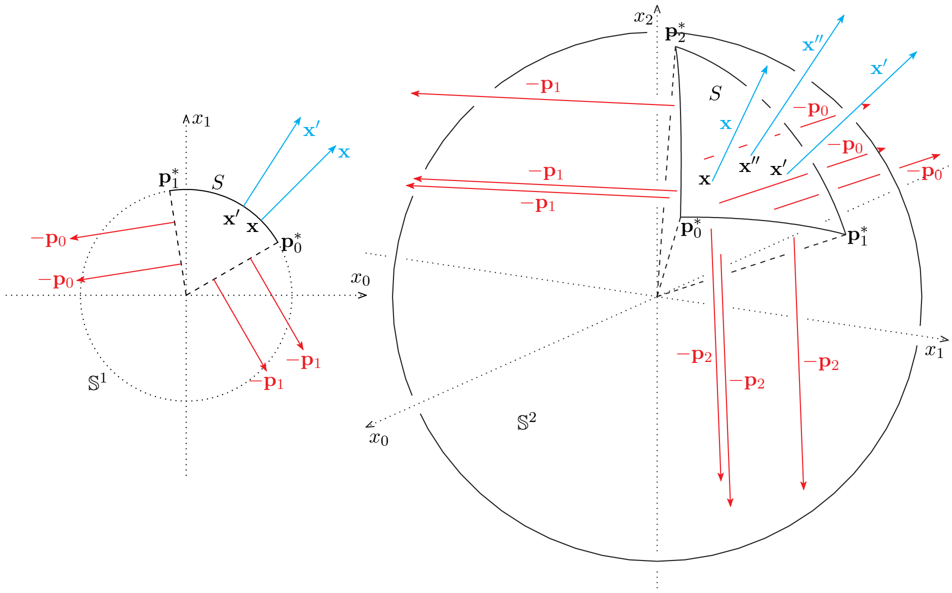

FIGURE 1. Corollary 1.3 for $n = 2$ (left) and $n = 3$ (right)

Theorem 1.4 (The integral of pseudo-unit pseudo-normal vectors of bounded domains of Minkowski space)**.** *It holds that*

$$\int_{\partial K} d'(\boldsymbol{\partial K}) = \mathbf{0},$$

where $d'(\boldsymbol{\partial K})$ is Minkowski $(n-1)$-vector element of ∂K (see (5), (11), and Figure 3).

This theorem is shown by substituting constant vectors $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ for $\mathbf{A}$ in the following theorem (see Another proof of Theorem 1.4), which is also shown later.

Theorem 1.5 (Gauss' divergence theorem on Minkowski space)**.** *For a vector field $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$, it holds that*

$$\int_{\partial K} \langle \mathbf{A} | d'(\boldsymbol{\partial K}) \rangle = -\int_K \operatorname{div} \mathbf{A} d'(\mathbb{R}^{n-1} \times \mathbb{R}),$$

where $d'(\mathbb{R}^{n-1} \times \mathbb{R})$ is the n-volume element of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$ (see (6)).

Theorem 1.4 implies the following corollary, which is useful to define a kind of centroids of hyperbolic simplices $S \subseteq \mathbb{H}^{n-1}$. Let $\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*$ be vertices of S; let S_k be the opposite facet of $\mathbf{p}_k^*$; and let $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1} \in \mathbb{R}^{n-1} \times \mathbb{R}$ be such that $\langle \mathbf{p}_k | \mathbf{p}_k \rangle = 1$, $\langle \mathbf{p}_k | \mathbf{p}_k^* \rangle > 0$, $\langle \mathbf{p}_k | \mathbf{p}_\ell^* \rangle = 0$ for all $\ell \neq k$ (see [8]); i.e.,

$$S = \{\mathbf{x} \in \mathbb{H}^{n-1} : \langle \mathbf{x} | \mathbf{p}_0 \rangle \geq 0, \ldots, \langle \mathbf{x} | \mathbf{p}_{n-1} \rangle \geq 0\}, \quad S_k = \{\mathbf{x} \in S : \langle \mathbf{x} | \mathbf{p}_k \rangle = 0\}.$$

Then the following corollary was shown by complicated computations in [9], but in this paper it is shown by simple calculations like Corollary 1.3.

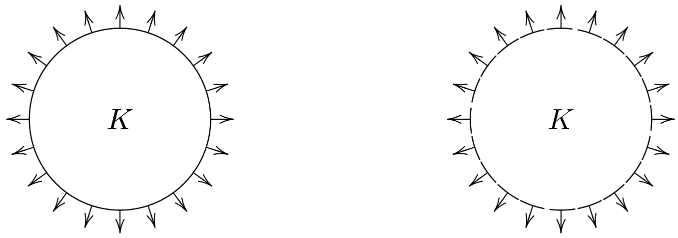

FIGURE 2. unit normal vectors $\tilde{\mathbf{n}}$ (left) and vector elements $d(\boldsymbol{\partial K})$ (right) (the volumes of domains in the right figure are constant, so the ratio of the lengths of the vectors in the left and right figures are constant)

Corollary 1.6 ([9, Thm. 1.2]). *It holds that*

$$\int_{\mathbf{x}\in S} \mathbf{x} d'(\mathbb{H}^{n-1}) = -\frac{1}{n-1}\sum_{k=0}^{n-1} |S_k|' \mathbf{p}_k,$$

where $d'(\mathbb{H}^{n-1})$ is the $(n-1)$-volume element of the hyperbolic space $\mathbb{H}^{n-1} \subseteq \mathbb{R}^{n-1}\times\mathbb{R}$ and $|S_k|'$ is the $(n-2)$-volume of induced metric on S_k of $\mathbb{R}^{n-1}\times\mathbb{R}$ (notice that S_k's are Riemannian because S is Riemannian).

Remark 1.2. *A kind of centroids of the hyperbolic simplex S can be defined by the both sides of*

$$\frac{\int_{\mathbf{x}\in S} \mathbf{x} d'(\mathbb{H}^{n-1})}{\sqrt{-\left\langle \int_{\mathbf{x}\in S} \mathbf{x} d'(\mathbb{H}^{n-1}) \middle| \int_{\mathbf{x}\in S} \mathbf{x} d'(\mathbb{H}^{n-1})\right\rangle}} = \frac{-\sum_{k=0}^{n-1} |S_k|'\mathbf{p}_k}{\sqrt{-\left\langle -\sum_{k=0}^{n-1} |S_k|'\mathbf{p}_k \middle| -\sum_{k=0}^{n-1} |S_k|'\mathbf{p}_k\right\rangle}}.$$

For prior works of hyperbolic simplices, see, e.g., [12]. For details of n-dimensional Minkowski spaces and pseudo-Riemannian manifolds, see [4].

2. Preliminaries.

For a bounded domain K of Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary ∂K, let $\mathbf{n}$ be an outward normal vector at a smooth point $\mathbf{x} \in \partial K$, let $\tilde{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$, and let $d(\boldsymbol{\partial K})$ be the vector element of ∂K in Euclidean space $\mathbb{R}^n$, i.e.,

$$d(\boldsymbol{\partial K}) = \tilde{\mathbf{n}} d(\partial K), \tag{3}$$

where $d(\partial K)$ is the $(n-1)$-volume element of induced metric on ∂K of Euclidean space $\mathbb{R}^n$, i.e., $d(\partial K) = \sqrt{\det g}\, dt_0 \cdots dt_{n-2}$ with the metric tensor

$$g = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial t_0}\cdot\frac{\partial \mathbf{x}}{\partial t_0} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial t_0}\cdot\frac{\partial \mathbf{x}}{\partial t_{n-2}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial \mathbf{x}}{\partial t_{n-2}}\cdot\frac{\partial \mathbf{x}}{\partial t_0} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial t_{n-2}}\cdot\frac{\partial \mathbf{x}}{\partial t_{n-2}} \end{pmatrix}$$

of a local coordinate system $(t_0, \ldots, t_{n-2})$ of ∂K. See Figure 2.

Consider K as a bounded domain of Minkowski space $\mathbb{R}^{n-1}\times\mathbb{R}$. Let $\mathbf{n}'$ be a pseudo-normal vector at $\mathbf{x}$ which is outward (resp. inward) if $\mathbf{x} \in \partial K$

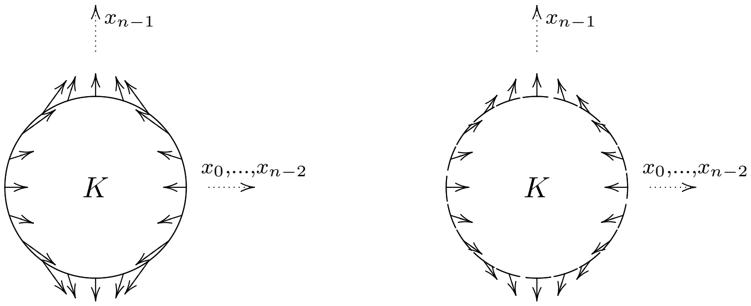

FIGURE 3. pseudo-unit pseudo-normal vectors $\tilde{\mathbf{n}}'$ (left) and vector elements $d'(\boldsymbol{\partial K})$ (right) (the pseudo-volumes of domains in the right figure are not constant, so the ratio of the lengths of the vectors in the left and right figures are not constant)

is Riemannian (resp. pseudo-Riemannian), i.e., if $T_{\mathbf{x}}(\partial K)$ is spacelike (resp. timelike), that is, if $\det g' > 0$ (resp. $\det g' < 0$) with the pseudo-metric tensor

$$g' = \begin{pmatrix} \langle \frac{\partial \mathbf{x}}{\partial t_0} | \frac{\partial \mathbf{x}}{\partial t_0} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial t_0} | \frac{\partial \mathbf{x}}{\partial t_{n-2}} \rangle \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \langle \frac{\partial \mathbf{x}}{\partial t_{n-2}} | \frac{\partial \mathbf{x}}{\partial t_0} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial t_{n-2}} | \frac{\partial \mathbf{x}}{\partial t_{n-2}} \rangle \end{pmatrix}.$$

For $\mathbf{x}$ with lightlike $T_{\mathbf{x}}(\partial K)$, i.e., with $\det g' = 0$, let $\mathbf{n}' = (\nu'_0, \dots, \nu'_{n-1})$ be in $T_{\mathbf{x}}(\partial K)$ with $\langle \mathbf{n}' | \mathbf{n}' \rangle = 0$ whose direction is determined naturally, i.e., if K is over (resp. under) ∂K, ν'_{n-1} is negative (resp. positive). To summarize,

$$\mathbf{n}' = t \cdot \iota(\mathbf{n}) \quad \text{for some } t < 0, \tag{4}$$

if $T_{\mathbf{x}}(\partial K)$ is spacelike, timelike, or lightlike. Let $\tilde{\mathbf{n}}' = \frac{\mathbf{n}'}{\sqrt{-\langle \mathbf{n}' | \mathbf{n}' \rangle}}$ (resp. $\frac{\mathbf{n}'}{\sqrt{\langle \mathbf{n}' | \mathbf{n}' \rangle}}$), and let $d'(\boldsymbol{\partial K})$ be the vector element of ∂K in Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$, i.e.,

$$d'(\boldsymbol{\partial K}) = \tilde{\mathbf{n}}' d'(\partial K), \tag{5}$$

where $d'(\partial K)$ is the $(n-1)$-volume element of induced metric on ∂K of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$, i.e., $d'(\partial K) = \sqrt{\det g'}\, dt_0 \cdots dt_{n-2}$ (resp. $\sqrt{-\det g'}\, dt_0 \cdots dt_{n-2}$) if $T_{\mathbf{x}}(\partial K)$ is spacelike (resp. timelike). See Figure 3.

The n-volume element of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$ is equal to the n-volume element of Euclidean space $\mathbb{R}^n$, i.e.,

(6)

$$d'(\mathbb{R}^{n-1} \times \mathbb{R}) = \sqrt{-\det \begin{pmatrix} \langle \mathbf{e}_0 | \mathbf{e}_0 \rangle & \cdots\cdots\cdots & \langle \mathbf{e}_0 | \mathbf{e}_{n-1} \rangle \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \langle \mathbf{e}_{n-1} | \mathbf{e}_0 \rangle & \cdots\cdots\cdots & \langle \mathbf{e}_{n-1} | \mathbf{e}_{n-1} \rangle \end{pmatrix}}\, dx_0 \cdots dx_{n-1} =$$

$$= \sqrt{-\det \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}} dx_0 \cdots dx_{n-1} = dx_0 \cdots dx_{n-1} = d(\mathbb{R}^n).$$

3. Main results.

If $T_{\mathbf{x}}(\partial K)$ is lightlike, we can define neither $\tilde{\mathbf{n}}'$ nor $d'(\partial K)$ but $d'(\boldsymbol{\partial K})$ from the following lemma.

Lemma 3.1. *For the local coordinate system* $(x_0, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_{n-1})$ *of a smooth neighborhood* U *of* ∂K ($\ell = 0, \dots, n-1$)*, which means that* $\mathbf{x} \in U$ *is represented by*

$$(x_0, \dots, x_{\ell-1}, \kappa_\ell(x_0, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_{n-1}), x_{\ell+1}, \dots, x_{n-1})$$

with a function κ_ℓ*, we have* $\nu_\ell \neq 0$, $\nu'_\ell \neq 0$,

$$d(\boldsymbol{\partial K}) = \frac{\mathbf{n}}{|\nu_\ell|} dx_0 \cdots dx_{\ell-1} dx_{\ell+1} \cdots dx_{n-1},$$

and

$$d'(\boldsymbol{\partial K}) = \frac{\mathbf{n}'}{|\nu'_\ell|} dx_0 \cdots dx_{\ell-1} dx_{\ell+1} \cdots dx_{n-1} \quad \text{if } T_{\mathbf{x}}(\partial K) \text{ is not lightlike},$$

where $\mathbf{n} = (\nu_0, \dots, \nu_{n-1})$ *and* $\mathbf{n}' = (\nu'_0, \dots, \nu'_{n-1})$.

Proof. Because $(x_0, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_{n-1})$ is a local coordinate system, $T_{\mathbf{x}}(\partial K)$ does not include the vector $\mathbf{e}_\ell = (\underbrace{0, \dots, 0}_{\ell \text{ times}}, 1, \underbrace{0, \dots, 0}_{n-1-\ell \text{ times}})$, so $\nu_\ell \neq 0$ and $\nu'_\ell \neq 0$ hold. The calculations for $\ell = 0, \dots, n-2$ are essentially same, so it is enough to show the equations of $d(\boldsymbol{\partial K})$ and $d'(\boldsymbol{\partial K})$ for $\ell = 0$ and $\ell = n-1$.

For $\ell = n-1$, i.e., for $\mathbf{x} = (x_0, \dots, x_{n-2}, \kappa_{n-1}(x_0, \dots, x_{n-2}))$, tangent vectors

$$\frac{\partial \mathbf{x}}{\partial x_i} = (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, \underbrace{0, \dots, 0}_{n-2-i \text{ times}}, \frac{\partial \kappa_{n-1}}{\partial x_i}) \quad (i = 0, \dots, n-2)$$

are perpendicular (resp. pseudo-perpendicular) to

$$(\mathbf{grad}\, \kappa_{n-1}, -1) = (\frac{\partial \kappa_{n-1}}{\partial x_0}, \dots, \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}, -1) \tag{7}$$

$$(\text{resp. } \iota(\mathbf{grad}\, \kappa_{n-1}, -1) = (\frac{\partial \kappa_{n-1}}{\partial x_0}, \dots, \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}, +1)), \tag{8}$$

which is parallel to $\mathbf{n}$ (resp. $\mathbf{n}'$). From

$$g_{(n-1)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial x_0} \cdot \frac{\partial \mathbf{x}}{\partial x_0} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial x_0} \cdot \frac{\partial \mathbf{x}}{\partial x_{n-2}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial \mathbf{x}}{\partial x_{n-2}} \cdot \frac{\partial \mathbf{x}}{\partial x_0} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial x_{n-2}} \cdot \frac{\partial \mathbf{x}}{\partial x_{n-2}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + \frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_0} & \frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_1} & \cdots & \frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \\ \frac{\partial \kappa_{n-1}}{\partial x_1}\frac{\partial \kappa_{n-1}}{\partial x_0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\partial \kappa_{n-1}}{\partial x_{n-3}}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \\ \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_0} & \cdots & \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_{n-3}} & 1 + \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \end{pmatrix},$$

$$g'_{(n-1)} = \begin{pmatrix} \langle \frac{\partial \mathbf{x}}{\partial x_0} | \frac{\partial \mathbf{x}}{\partial x_0} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial x_0} | \frac{\partial \mathbf{x}}{\partial x_{n-2}} \rangle \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \langle \frac{\partial \mathbf{x}}{\partial x_{n-2}} | \frac{\partial \mathbf{x}}{\partial x_0} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial x_{n-2}} | \frac{\partial \mathbf{x}}{\partial x_{n-2}} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_0} & -\frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_1} & \cdots & -\frac{\partial \kappa_{n-1}}{\partial x_0}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \\ -\frac{\partial \kappa_{n-1}}{\partial x_1}\frac{\partial \kappa_{n-1}}{\partial x_0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\partial \kappa_{n-1}}{\partial x_{n-3}}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \\ -\frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_0} & \cdots & -\frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_{n-3}} & 1 - \frac{\partial \kappa_{n-1}}{\partial x_{n-2}}\frac{\partial \kappa_{n-1}}{\partial x_{n-2}} \end{pmatrix},$$

and the matrix determinant lemma [2, Cor. 18.1.3],

$$\det g_{(n-1)} = 1 + \| \mathbf{grad}\, \kappa_{n-1} \|^2 \ \text{ and } \ \det g'_{(n-1)} = 1 - \| \mathbf{grad}\, \kappa_{n-1} \|^2$$

hold. Hence we have

$$d(\boldsymbol{\partial K}) = \tilde{\mathbf{n}} \cdot d(\partial K) = \frac{\mathbf{n}}{\|\mathbf{n}\|} \cdot \sqrt{\det g_{(n-1)}}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu_0, \ldots, \nu_{n-2}, \nu_{n-1})}{\sqrt{\|(\nu_0, \ldots, \nu_{n-2})\|^2 + \nu_{n-1}^2}} \cdot \sqrt{\| \mathbf{grad}\, \kappa_{n-1}\|^2 + 1}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu_0, \ldots, \nu_{n-2}, \nu_{n-1})}{|\nu_{n-1}|}dx_0 \cdots dx_{n-2},$$

$$d'(\boldsymbol{\partial K}) = \tilde{\mathbf{n}}' \cdot d'(\partial K) = \frac{\mathbf{n}'}{\sqrt{-\langle \mathbf{n}' | \mathbf{n}' \rangle}} \cdot \sqrt{+\det g'_{(n-1)}}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu'_0, \ldots, \nu'_{n-2}, \nu'_{n-1})}{\sqrt{-\|(\nu'_0, \ldots, \nu'_{n-2})\|^2 + {\nu'_{n-1}}^2}} \cdot$$
$$\cdot \sqrt{-\| \mathbf{grad}\, \kappa_{n-1}\|^2 + 1}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu'_0, \ldots, \nu'_{n-2}, \nu'_{n-1})}{|\nu'_{n-1}|}dx_0 \cdots dx_{n-2} \quad \text{if } T_{\mathbf{x}}(\partial K) \text{ is spacelike},$$

$$d'(\boldsymbol{\partial K}) = \tilde{\mathbf{n}}' \cdot d'(\partial K) = \frac{\mathbf{n}'}{\sqrt{+\langle \mathbf{n}' | \mathbf{n}' \rangle}} \cdot \sqrt{-\det g'_{(n-1)}}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu'_0, \ldots, \nu'_{n-2}, \nu'_{n-1})}{\sqrt{+\|(\nu'_0, \ldots, \nu'_{n-2})\|^2 - {\nu'_{n-1}}^2}} \cdot$$
$$\cdot \sqrt{+\| \mathbf{grad}\, \kappa_{n-1}\|^2 - 1}dx_0 \cdots dx_{n-2} =$$
$$= \frac{(\nu'_0, \ldots, \nu'_{n-2}, \nu'_{n-1})}{|\nu'_{n-1}|}dx_0 \cdots dx_{n-2} \quad \text{if } T_{\mathbf{x}}(\partial K) \text{ is timelike},$$

where the last equalities of 3 calculations above come from parallelisms of (7) and $\mathbf{n}$, (8) and $\mathbf{n}'$, and (8) and $\mathbf{n}'$, respectively.

For $\ell = 0$, i.e., for $\mathbf{x} = (\kappa_0(x_1, \ldots, x_{n-1}), x_1, \ldots, x_{n-1})$, tangent vectors

$$\frac{\partial \mathbf{x}}{\partial x_i} = (\frac{\partial \kappa_0}{\partial x_i}, \underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{n-1-i \text{ times}}) \quad (i = 1, \ldots, n-1)$$

are perpendicular (resp. pseudo-perpendicular) to

$$(-1, \mathbf{grad}\, \kappa_0) = (-1, \frac{\partial \kappa_0}{\partial x_1}, \ldots\ldots\ldots\ldots, \frac{\partial \kappa_0}{\partial x_{n-1}}) \tag{9}$$

$$(\text{resp. } \iota(-1, \mathbf{grad}\, \kappa_0) = (-1, \frac{\partial \kappa_0}{\partial x_1}, \ldots, \frac{\partial \kappa_0}{\partial x_{n-2}}, -\frac{\partial \kappa_0}{\partial x_{n-1}})), \tag{10}$$

which is parallel to $\mathbf{n}$ (resp. $\mathbf{n}'$). From

$$g_{(0)} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial x_1} \cdot \frac{\partial \mathbf{x}}{\partial x_1} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial x_1} \cdot \frac{\partial \mathbf{x}}{\partial x_{n-1}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial \mathbf{x}}{\partial x_{n-1}} \cdot \frac{\partial \mathbf{x}}{\partial x_1} & \cdots & \cdots & \frac{\partial \mathbf{x}}{\partial x_{n-1}} \cdot \frac{\partial \mathbf{x}}{\partial x_{n-1}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_1} + 1 & \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_2} & \cdots & \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_{n-1}} \\ \frac{\partial \kappa_0}{\partial x_2}\frac{\partial \kappa_0}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\partial \kappa_0}{\partial x_{n-2}}\frac{\partial \kappa_0}{\partial x_{n-1}} \\ \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_1} & \cdots & \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_{n-2}} & \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_{n-1}} + 1 \end{pmatrix},$$

$$g'_{(0)} = \begin{pmatrix} \langle \frac{\partial \mathbf{x}}{\partial x_1} | \frac{\partial \mathbf{x}}{\partial x_1} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial x_1} | \frac{\partial \mathbf{x}}{\partial x_{n-1}} \rangle \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \langle \frac{\partial \mathbf{x}}{\partial x_{n-1}} | \frac{\partial \mathbf{x}}{\partial x_1} \rangle & \cdots & \cdots & \langle \frac{\partial \mathbf{x}}{\partial x_{n-1}} | \frac{\partial \mathbf{x}}{\partial x_{n-1}} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_1} + 1 & \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_2} & \cdots & \frac{\partial \kappa_0}{\partial x_1}\frac{\partial \kappa_0}{\partial x_{n-1}} \\ \frac{\partial \kappa_0}{\partial x_2}\frac{\partial \kappa_0}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{\partial \kappa_0}{\partial x_{n-2}}\frac{\partial \kappa_0}{\partial x_{n-2}} + 1 & \frac{\partial \kappa_0}{\partial x_{n-2}}\frac{\partial \kappa_0}{\partial x_{n-1}} \\ \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_1} & \cdots & \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_{n-2}} & \frac{\partial \kappa_0}{\partial x_{n-1}}\frac{\partial \kappa_0}{\partial x_{n-1}} - 1 \end{pmatrix},$$

and the matrix determinant lemma ([2, Cor. 18.1.3] and [2, Thm. 18.1.1] for

$$R = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}$$

and $T = 1$, respectively),

$$\det g_{(0)} = 1 + \| \mathbf{grad}\, \kappa_0 \|^2 \quad \text{and} \quad \det g'_{(0)} = -(1 + \langle \mathbf{grad}\, \kappa_0 | \mathbf{grad}\, \kappa_0 \rangle)$$

hold. Hence we have

$$\begin{aligned}
d(\boldsymbol{\partial K}) &= \tilde{\mathbf{n}} \cdot d(\partial K) = \frac{\mathbf{n}}{\|\mathbf{n}\|} \cdot \sqrt{\det g_{(0)}} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu_0, \nu_1, \ldots, \nu_{n-1})}{\sqrt{\nu_0^2 + \|(\nu_1, \ldots, \nu_{n-1})\|^2}} \cdot \sqrt{1 + \|\,\mathbf{grad}\,\kappa_0\|^2} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu_0, \nu_1, \ldots, \nu_{n-1})}{|\nu_0|} dx_1 \cdots dx_{n-1}, \\
d'(\boldsymbol{\partial K}) &= \frac{\mathbf{n}'}{\sqrt{-\langle \mathbf{n}' | \mathbf{n}' \rangle}} \cdot \sqrt{+\det g'_{(0)}} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu'_0, \nu'_1, \ldots, \nu'_{n-1})}{\sqrt{-\nu_0'^2 - \langle (\nu'_1, \ldots, \nu'_{n-1}) | (\nu'_1, \ldots, \nu'_{n-1}) \rangle}} \cdot \\
&\quad \cdot \sqrt{-1 - \langle \mathbf{grad}\,\kappa_0 | \mathbf{grad}\,\kappa_0 \rangle} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu'_0, \nu'_1, \ldots, \nu'_{n-1})}{|\nu'_0|} dx_1 \cdots dx_{n-1} \quad \text{if } T_{\mathbf{x}}(\partial K) \text{ is spacelike}, \\
d'(\boldsymbol{\partial K}) &= \frac{\mathbf{n}'}{\sqrt{+\langle \mathbf{n}' | \mathbf{n}' \rangle}} \cdot \sqrt{-\det g'_{(0)}} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu'_0, \nu'_1, \ldots, \nu'_{n-1})}{\sqrt{+\nu_0'^2 + \langle (\nu'_1, \ldots, \nu'_{n-1}) | (\nu'_1, \ldots, \nu'_{n-1}) \rangle}} \cdot \\
&\quad \cdot \sqrt{+1 + \langle \mathbf{grad}\,\kappa_0 | \mathbf{grad}\,\kappa_0 \rangle} dx_1 \cdots dx_{n-1} = \\
&= \frac{(\nu'_0, \nu'_1, \ldots, \nu'_{n-1})}{|\nu'_0|} dx_1 \cdots dx_{n-1} \quad \text{if } T_{\mathbf{x}}(\partial K) \text{ is timelike},
\end{aligned}$$

where the last equalities of 3 calculations above come from parallelisms of (9) and $\mathbf{n}$, (10) and $\mathbf{n}'$, and (10) and $\mathbf{n}'$, respectively. □

$\mathbf{n}' = (\nu'_0, \ldots, \nu'_{n-1})$ is non-zero, so, one of ν'_ℓ's is non-zero. Hence we can expand $d'(\boldsymbol{\partial K})$ to all smooth points of ∂K,

$$d'(\boldsymbol{\partial K}) = \frac{\mathbf{n}'}{|\nu'_\ell|} dx_0 \cdots dx_{\ell-1} dx_{\ell+1} \cdots dx_{n-1}, \tag{11}$$

where $\ell = 0, \ldots, n-1$ such that $\nu'_\ell \neq 0$. The following lemma is essential in this paper.

Lemma 3.2. *It holds that*

$$d'(\boldsymbol{\partial K}) = -\iota(d(\boldsymbol{\partial K})).$$

Proof. Let ℓ be such that $(x_0, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_{n-1})$ is a local coordinate of ∂K around $\mathbf{x}$. Then neither ν_ℓ nor ν'_ℓ is zero, so we have

$$\begin{aligned}
d'(\boldsymbol{\partial K}) &= \frac{\mathbf{n}'}{|\nu'_\ell|} dx_0 \cdots dx_{\ell-1} dx_{\ell+1} \cdots dx_{n-1} = \\
&= -\iota\Big(\frac{\mathbf{n}}{|\nu_\ell|} dx_0 \cdots dx_{\ell-1} dx_{\ell+1} \cdots dx_{n-1}\Big) = -\iota(d(\boldsymbol{\partial K})),
\end{aligned}$$

where $\mathbf{n} = (\nu_0, \ldots, \nu_{n-1})$, $\mathbf{n}' = (\nu'_0, \ldots, \nu'_{n-1})$, and the first, second, and last equalities come from (11), (4), and Lemma 3.1. □

This lemma implies Theorems 1.5 and 1.4.

Proof of Theorem 1.5. We have

$$\int_{\partial K} \langle \mathbf{A} | d'(\boldsymbol{\partial K}) \rangle = -\int_{\partial K} \mathbf{A} \,.\, d(\boldsymbol{\partial K}) = -\int_K \operatorname{div} \mathbf{A} d(\mathbb{R}^n) = \\ = -\int_K \operatorname{div} \mathbf{A} d'(\mathbb{R}^{n-1} \times \mathbb{R}),$$

where the first, second, and last equalities come from Lemma 3.2, Theorem 1.2, and (6), respectively. □

Proof of Theorem 1.4. We have

$$\int_{\partial K} d'(\boldsymbol{\partial K}) = -\iota\left(\int_{\partial K} d(\boldsymbol{\partial K})\right) = -\iota(\mathbf{0}) = \mathbf{0},$$

where the first and second equalities come from Lemma 3.2 and Theorem 1.1, respectively. □

Theorem 1.4 can be also shown by Theorem 1.5 (see Introduction).

Another proof of Theorem 1.4. The conclusion comes from

$$\int_{\partial K} \langle \mathbf{e}_\ell | d'(\boldsymbol{\partial K}) \rangle = -\int_K \operatorname{div} \mathbf{e}_\ell d'(\mathbb{R}^{n-1} \times \mathbb{R}) = -\int_K 0 d'(\mathbb{R}^{n-1} \times \mathbb{R}) = 0,$$

for $\ell = 0, \ldots, n-1$, where the first equality comes from Theorem 1.5. □

To prove Corollaries 1.3 and 1.6, we need following Lemmas 3.3 and 3.4, respectively. Let $S \subseteq \mathbb{S}^{n-1}$ be a spherical simplex of Euclidean space $\mathbb{R}^n$ with vertices $\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*$; let S_k be the opposite facet of $\mathbf{p}_k^*$; and let K and K_k be as in (1) and (2), respectively. Then we have the following lemma.

Lemma 3.3. *It holds that*

$$|K_k| = \frac{|S_k|}{n-1},$$

where $|K_k|$ is the $(n-1)$-volume of induced metric on K_k of Euclidean space $\mathbb{R}^n$.

Proof. Let $(s_0, \ldots, s_{n-3})$ be a local coordinate of S_k, i.e., $\mathbf{y} \in S_k$ is represented by

$$(\lambda_{k,0}(s_0, \ldots, s_{n-3}), \ldots, \lambda_{k,n-1}(s_0, \ldots, s_{n-3})) \tag{12}$$

for some functions $\lambda_{k,0}, \ldots, \lambda_{k,n-1}$. Then it induces naturally a local coordinate $(s_0, \ldots, s_{n-3}, t)$ of K_k, i.e., $\mathbf{x} \in K_k$ is represented by

$$t\mathbf{y} = (t\lambda_{k,0}(s_0, \ldots, s_{n-3}), \ldots, t\lambda_{k,n-1}(s_0, \ldots, s_{n-3})). \tag{13}$$

Metric tensors of S_k and K_k are

$$h_k = \begin{pmatrix} \frac{\partial \mathbf{y}}{\partial s_0} \cdot \frac{\partial \mathbf{y}}{\partial s_0} & \cdots & \frac{\partial \mathbf{y}}{\partial s_0} \cdot \frac{\partial \mathbf{y}}{\partial s_{n-3}} \\ \vdots & & \vdots \\ \frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{y}}{\partial s_0} & \cdots & \frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{y}}{\partial s_{n-3}} \end{pmatrix}$$

and

$$g_k = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial s_0} \cdot \frac{\partial \mathbf{x}}{\partial s_0} & \cdots & \frac{\partial \mathbf{x}}{\partial s_0} \cdot \frac{\partial \mathbf{x}}{\partial s_{n-3}} & \frac{\partial \mathbf{x}}{\partial s_0} \cdot \frac{\partial \mathbf{x}}{\partial t} \\ \vdots & & \vdots & \vdots \\ \frac{\partial \mathbf{x}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{x}}{\partial s_0} & \cdots & \frac{\partial \mathbf{x}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{x}}{\partial s_{n-3}} & \frac{\partial \mathbf{x}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{x}}{\partial t} \\ \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial s_0} & \cdots & \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial s_{n-3}} & \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial t} \end{pmatrix} =$$

$$= \begin{pmatrix} t\frac{\partial \mathbf{y}}{\partial s_0} \cdot t\frac{\partial \mathbf{y}}{\partial s_0} & \cdots & t\frac{\partial \mathbf{y}}{\partial s_0} \cdot t\frac{\partial \mathbf{y}}{\partial s_{n-3}} & t\frac{\partial \mathbf{y}}{\partial s_0} \cdot \mathbf{y} \\ \vdots & & \vdots & \vdots \\ t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot t\frac{\partial \mathbf{y}}{\partial s_0} & \cdots & t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot t\frac{\partial \mathbf{y}}{\partial s_{n-3}} & t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot \mathbf{y} \\ \mathbf{y} \cdot t\frac{\partial \mathbf{y}}{\partial s_0} & \cdots & \mathbf{y} \cdot t\frac{\partial \mathbf{y}}{\partial s_{n-3}} & \mathbf{y} \cdot \mathbf{y} \end{pmatrix} =$$

$$= \begin{pmatrix} t^2\frac{\partial \mathbf{y}}{\partial s_0} \cdot \frac{\partial \mathbf{y}}{\partial s_0} & \cdots & t^2\frac{\partial \mathbf{y}}{\partial s_0} \cdot \frac{\partial \mathbf{y}}{\partial s_{n-3}} & 0 \\ \vdots & & \vdots & \vdots \\ t^2\frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{y}}{\partial s_0} & \cdots & t^2\frac{\partial \mathbf{y}}{\partial s_{n-3}} \cdot \frac{\partial \mathbf{y}}{\partial s_{n-3}} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

respectively. So we have

$$\sqrt{\det g_k} = t^{n-2}\sqrt{\det h_k}.$$

It implies that

$$\begin{aligned} |K_k| &= \int_{K_k} d(K_k) = \int_{K_k} \sqrt{\det g_k} ds_0 \cdots ds_{n-3} dt = \\ &= \int_0^1 t^{n-2} dt \int_{S_k} \sqrt{\det h_k} ds_0 \cdots ds_{n-3} = \\ &= \frac{1}{n-1} \int_{S_k} d(S_k) = \frac{1}{n-1} |S_k|. \quad \square \end{aligned}$$

Let $S \subseteq \mathbb{H}^{n-1}$ be a hyperbolic simplex of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$ with vertices $\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*$; let S_k be the opposite facet of $\mathbf{p}_k^*$; and let K and K_k be as in (1) and (2), respectively. Then we have the following lemma, which implies Corollary 1.6.

Lemma 3.4. *It holds that*

$$|K_k|' = \frac{|S_k|'}{n-1},$$

where $|K_k|'$ is the $(n-1)$-volume of induced pseudo-metric on K_k of Minkowski space $\mathbb{R}^{n-1} \times \mathbb{R}$. (notice that S_k is Riemannian and K_k is pseudo-Riemannian, precisely, a timelike hyperplane).

Proof. Let $(s_0, \ldots, s_{n-3})$ be a local coordinate of S_k. Then it induces naturally a local coordinate $(s_0, \ldots, s_{n-3}, t)$ of K_k (see (12) and (13)). Pseudo-metric tensors S_k and K_k are

$$h'_k = \begin{pmatrix} \langle \frac{\partial \mathbf{y}}{\partial s_0} | \frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & \langle \frac{\partial \mathbf{y}}{\partial s_0} | \frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle \\ \vdots & & \vdots \\ \langle \frac{\partial \mathbf{y}}{\partial s_{n-3}} | \frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & \langle \frac{\partial \mathbf{y}}{\partial s_{n-3}} | \frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle \end{pmatrix}$$

and

$$g'_k = \begin{pmatrix} \langle \frac{\partial \mathbf{x}}{\partial s_0} | \frac{\partial \mathbf{x}}{\partial s_0} \rangle & \cdots & \langle \frac{\partial \mathbf{x}}{\partial s_0} | \frac{\partial \mathbf{x}}{\partial s_{n-3}} \rangle & \langle \frac{\partial \mathbf{x}}{\partial s_0} | \frac{\partial \mathbf{x}}{\partial t} \rangle \\ \vdots & & \vdots & \vdots \\ \langle \frac{\partial \mathbf{x}}{\partial s_{n-3}} | \frac{\partial \mathbf{x}}{\partial s_0} \rangle & \cdots & \langle \frac{\partial \mathbf{x}}{\partial s_{n-3}} | \frac{\partial \mathbf{x}}{\partial s_{n-3}} \rangle & \langle \frac{\partial \mathbf{x}}{\partial s_{n-3}} | \frac{\partial \mathbf{x}}{\partial t} \rangle \\ \langle \frac{\partial \mathbf{x}}{\partial t} | \frac{\partial \mathbf{x}}{\partial s_0} \rangle & \cdots & \langle \frac{\partial \mathbf{x}}{\partial t} | \frac{\partial \mathbf{x}}{\partial s_{n-3}} \rangle & \langle \frac{\partial \mathbf{x}}{\partial t} | \frac{\partial \mathbf{x}}{\partial t} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} \langle t\frac{\partial \mathbf{y}}{\partial s_0} | t\frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & \langle t\frac{\partial \mathbf{y}}{\partial s_0} | t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle & \langle t\frac{\partial \mathbf{y}}{\partial s_0} | \mathbf{y} \rangle \\ \vdots & & \vdots & \vdots \\ \langle t\frac{\partial \mathbf{y}}{\partial s_{n-3}} | t\frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & \langle t\frac{\partial \mathbf{y}}{\partial s_{n-3}} | t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle & \langle t\frac{\partial \mathbf{y}}{\partial s_{n-3}} | \mathbf{y} \rangle \\ \langle \mathbf{y} | t\frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & \langle \mathbf{y} | t\frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle & \langle \mathbf{y} | \mathbf{y} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} t^2\langle \frac{\partial \mathbf{y}}{\partial s_0} | \frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & t^2\langle \frac{\partial \mathbf{y}}{\partial s_0} | \frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle & 0 \\ \vdots & & \vdots & \vdots \\ t^2\langle \frac{\partial \mathbf{y}}{\partial s_{n-3}} | \frac{\partial \mathbf{y}}{\partial s_0} \rangle & \cdots & t^2\langle \frac{\partial \mathbf{y}}{\partial s_{n-3}} | \frac{\partial \mathbf{y}}{\partial s_{n-3}} \rangle & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix},$$

respectively. So we have

$$\sqrt{-\det g'_k} = t^{n-2}\sqrt{\det h'_k}.$$

It implies that

$$\begin{aligned} |K_k|' &= \int_{K_k} d'(K_k) = \int_{K_k} \sqrt{-\det g'_k}\, ds_0 \cdots ds_{n-3} dt = \\ &= \int_0^1 t^{n-2} dt \int_{S_k} \sqrt{\det h'_k}\, ds_0 \cdots ds_{n-3} = \\ &= \frac{1}{n-1}\int_{S_k} d'(S_k) = \frac{1}{n-1}|S_k|'. \quad \square \end{aligned}$$

Proof of Corollary 1.6. Let K be as in (1). Then, ∂K consists of S and K_k (for $k = 0,\ldots, n-1$) as in (2) whose $(n-1)$-volume is $|S_k|'/(n-1)$ from Lemma 3.4 and whose outward pseudo-unit pseudo-normal vector is $-\mathbf{p}_k$ (see Figures 4 and 5). So, from Theorem 1.4 (remark that K_k is a timelike hyperplane), we have

$$\int_{\mathbf{x}\in S} \mathbf{x} d'(\mathbb{H}^{n-1}) + \sum_{k=0}^{n-1} \frac{|S_k|'}{n-1}\mathbf{p}_k = \mathbf{0}. \quad \square$$

Acknowledgment.

The author is thankful to the referee who gave many helpful comments which makes the paper easier to read.

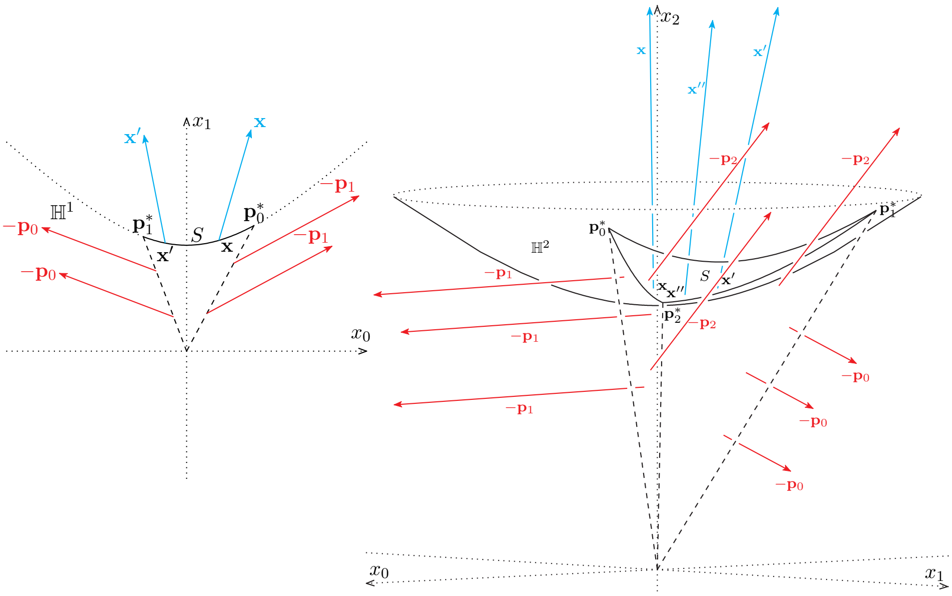

FIGURE 4. Corollary 1.6 for $n = 2$ (left) and $n = 3$ (right)

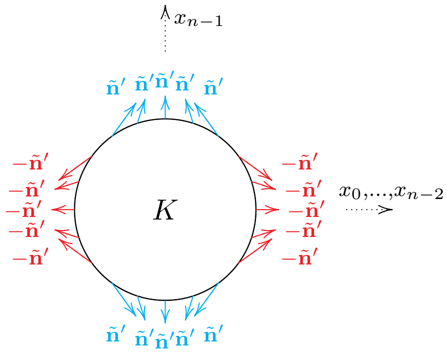

FIGURE 5. outward pseudo-unit pseudo-normal vectors

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