INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 13 (2024), No. 1, 35 - 38

ON VAN LAMOEN'S THEOREM

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Abstract. In this article we give an alternative aspect and proof of a lemma, on the construction of a symmedian, proved by Nguyen and supporting a proof of van Lamoen's theorem, later stating, that the circumcenters of the six triangles defined by the medians of a triangle are concyclic.

1. Introduction

We start with a simple property of the medians $\{AD, BE, CF\}$ of the triangle ABC leading immediately to a proof of van Lamoen's theorem. Figure 1 illustrates the property: Point G is the centroid and point H is the second intersection of the circumcircles (AGE) and (BDG) of the

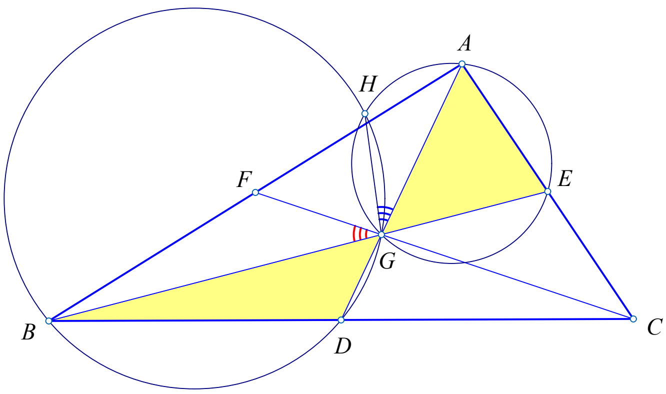

Figure 1. A basic property of the medians $\widehat{FGB} = \widehat{AGH}$

corresponding triangles formed by the medians of the triangle of reference ABC. The property states that the angles shown are equal: $\widehat{FGB} = \widehat{AGH}$, i.e. GH is the symmedian from G of the triangle AGB. The aim of this note is to supply an alternative aspect and proof (§ 4) of this property, which I encountered in the paper by Nguyen [1, Lemma 3]. The property supports a simple proof of van Lamoen's theorem. For the history of the subject I re-

Keywords and phrases: Symmedians, Triangle Geometry
(2020)Mathematics Subject Classification: 51-02; 51M15, 51N15
Received: 15.08.2023. In revised form: 26.10.2023. Accepted: 11.09.2023.

fer to Nguyen's paper. Next section motivates the use of this property for a proof of van Lamoen's theorem.

2. Van Lamoen's theorem

Theorem 2.1. *The circumcenters of the six triangles* $\{AGE, FGA, BGF\}$, $\{DGB, CGD, EGC\}$ *formed by the medians of the triangle are concyclic.*

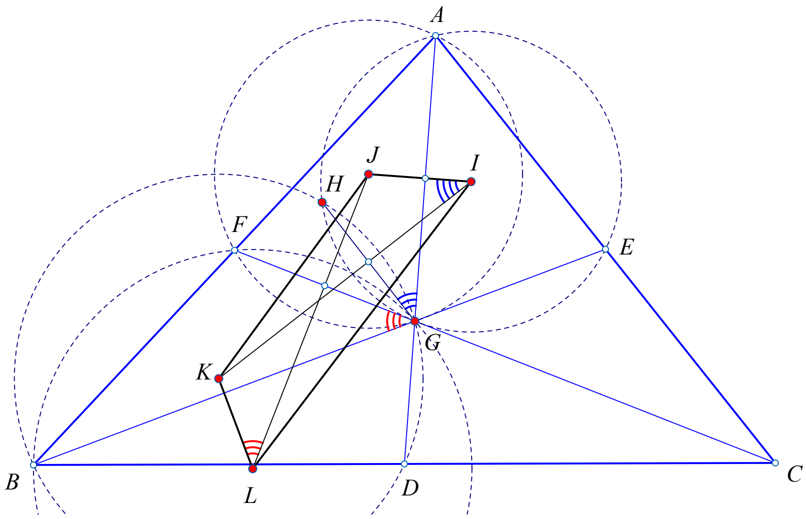

Figure 2. Concyclicity of four successive circumcenters $\{I, J, K, L\}$

Proof. We show that the circumcenters $\{I, J, K, L\}$ of the first four triangles listed above are concyclic (see Figure 2). This follows immediately from the assumed property of figure 1 and a simple angle chasing argument. In fact, angles $\{\widehat{JLK}, \widehat{FGB}\}$ are equal or supplementary since they have corresponding sides orthogonal. Analogous, property holds also for the angles $\{\widehat{JIK}, \widehat{AGH}\}$. In any case, this and the assumed property, imply easily that $IJKL$ is a cyclic quadrangle ([1]). Analogously is seen that the circumcenters of the successive four triangles of the list, starting from FGA are concyclic, and also the circumcenters of the four successive triangles, starting from BGF are concyclic. Since the three resulting quadrangles have by two three common vertices, the concyclicity of the six circumcenters follows at once.

3. An isogonal conjugate Cevian construction

Lemma 3.1. *Consider the Cevian* AD *of the triangle* ABC*. Draw the circles* $\kappa_B = (ABD)$ *and* $\kappa_C = (ACD)$ *and their tangents respectively at* $\{B, C\}$*. These intersect at a point* K *of the circumcircle of* ABC *and the line* AK *is the isogonal of* AD*. In particular, if* D *is the middle of* BC*, then* AK *is the symmedian from* A *and* $KB/KC = AB/AC$*.*

Proof. The first claim results by a simple angle chasing argument shown in figure 3. The second claim, for the symmedian, follows from the characterization of its points, according to which the ratio of the distances of K from the

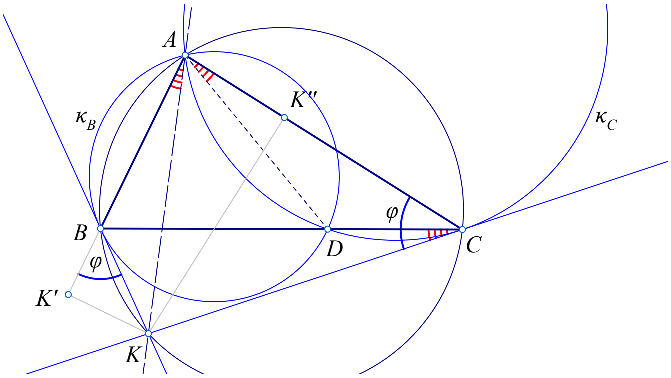

FIGURE 3. Isogonal AK of AD

sides: $KK'/KK'' = AB/AC$ and $\{KK' = KB\sin(\phi)\ ,\ KK'' = KC\sin(\phi)\}$, as seen in the figure.

4. Construction of a symmedian

Theorem 4.1. *Extend side AB of the triangle ABC to $BB' = \frac{3}{2}BA$ and the side AC to the segment $CC' = \frac{3}{2}CA$. The radical axis of the circles $\{\lambda_1 = (BAC'), \lambda_2 = (CAB')\}$ is the symmedian from A of the triangle (see Figure 4).*

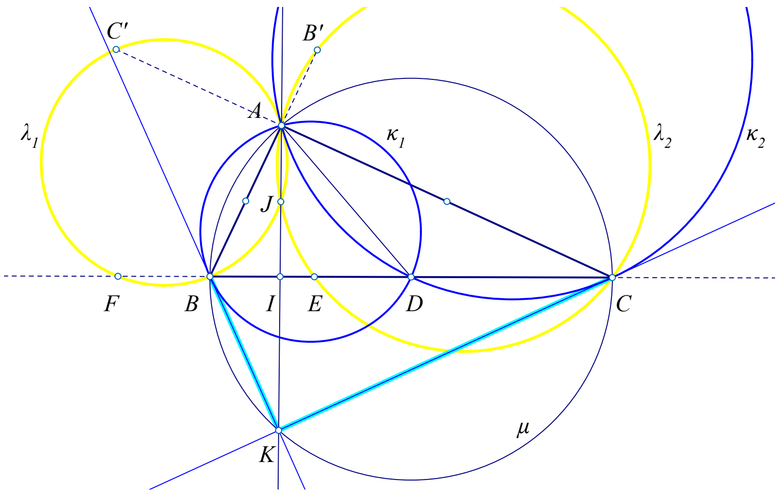

FIGURE 4. The symmedian AK of ABC from A

Hint: Consider the circles $\{\kappa_1 = (ABD), \kappa_2 = (ADC)\}$. The circumcircle μ of ABC is the common member of the circle-pencil generated by $\{\lambda_1, \kappa_1\}$ and the circle-pencil generated by $\{\lambda_2, \kappa_2\}$. Hence its points have constant ratios of powers $\frac{(\lambda_1)}{(\kappa_1)}$ and also constant ratio of powers $\frac{(\lambda_2)}{(\kappa_2)}$. Thus, the ratio of powers

$$\frac{(\lambda_1)}{(\lambda_2)} = \frac{(\lambda_1)/(\kappa_1)}{(\lambda_2)/(\kappa_2)} \cdot \frac{(\kappa_1)}{(\kappa_2)}\ . \qquad (1)$$

Computing the powers $\{(\lambda_1),(\kappa_1)\}$ at C and $\{(\lambda_2),(\kappa_2)\}$ at B we have

$$(\kappa_1)(C) = CD \cdot CB = BC^2/2 \quad , \quad (\lambda_1)(C) = CC' \cdot CA = 3CA^2/2 \ ,$$

$$(\kappa_2)(B) = BD \cdot BC = BC^2/2 \quad , \quad (\lambda_2)(B) = BB' \cdot BA = 3AB^2/2 \ .$$

Thus, the constant ratio $\frac{(\lambda_1)/(\kappa_1)}{(\lambda_2)/(\kappa_2)}$ along μ is CA^2/AB^2. For the intersection point K of the tangent to κ_1 at B and the tangent to κ_2 at C we have, according to lemma 3.1: $((\kappa_1)(K))/((\kappa_2)(K)) = KB^2/KC^2 = AB^2/CA^2$. Replacing these expressions into equation (1) we see that the ratio of powers $((\lambda_1)(K))/((\lambda_2)(K)) = 1$. Hence K is on the radical axis of $\{\lambda_1,\lambda_2\}$, which therefore coincides with the symmedian of the triangle.

References

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